ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

On the theorem of Meusnier in Weyl spaces

by A. Szybiak (Rzeszów) and Trân dinh Viên (Vinh)

Abstract. Given a regular surface in a Weyl manifold then this surface is also provided with a Weylian structure which is induced by an immersion. The paper contains some intrinsic constructions of osculating subspaces of this surface and the generalized equalities of Frenet are deduced. Then the theorem of Meusnier on the projection of a tangent vector to an $(j-1)$th osculating plane is obtained.

In this note we present another point of view and another method than those given in [4] and we extend the former result onto manifolds with a Weyl structure. A modern view upon these structures has been given in B. G. Folland's paper [2] (cf. also [3]). We present it briefly. Let W be a differentiable manifold of dimension $n \geqslant 3$. Assume that there is given a family G of Riemannian scalar products on W such that (i) if g and $\bar{g}$ are in G, then there exists a positive scalar function, σ, on W such that $g_p = \sigma(p) g_p$ for each $p \in W$, and (ii) if τ is a positive scalar on W and $g \in G$, then $\tau g \in G$. In order to treat G as a unique geometric object we have to consider some boundles over W. The first of them is a bundle $R \times W \to W$, where R denotes an additive group of real numbers. The second bundle which we introduce is the product $R \times L(W)$, where $L(W)$ denotes the common bundle of linear frames. Thus G may be viewed as a cross-section in a certain bundle which is associated with the bundle $R \times L(W)$. In fact, let

$$[\theta(p), e_1(p), \ldots, e_n(p)] \quad \text{and} \quad [\bar{\theta}(p), \bar{e}_1(p), \ldots, \bar{e}_n(p)]$$

be the two elements of $R \times L(W)$ which are bound by a relation

$$(1) \qquad \bar{\theta} = \theta + t, \quad \bar{e}_k = \Sigma A_k^l e_l, \quad \text{where} \quad [A_k^l] \in GL(R^n).$$

We consider the matrices of the form $[a_{ij}]$, $i, j = 1, \ldots, n$. We identify the two pairs, $([a_{ij}], [\theta, e_1, \ldots, e_n])$ and $([\bar{a}_{ij}], [\bar{\theta}, \bar{e}_1, \ldots, \bar{e}_n])$ iff relations (1) imply

$$(2) \qquad \bar{a}_{kh} = \sum A_h^i A_h^j \exp(2t) a_{ij}.$$

Thus we obtain a bundle over W which is associated with $R \times L(W)$. Evidently G is a cross-section of this bundle.

DEFINITION. A *Weyl structure on M* is given by a couple (W, G, γ), where W and G are as described above and γ is a connection in the bundle $R\times W\to W$.

Let Γ be a linear connection. We denote by ω its connection form and by ∇ the related covariant differentiation. Thus $\omega\oplus\gamma$ is a connection form on the bundle $R\times L(W)\to W$.

DEFINITION. A connection form $\omega\oplus\gamma$ is called *compatible* if the following conditions are satisfied: 1° ω is torsionless, 2° for any $g\in G$ and any vector fields X, Y, Z on W we have

$$(3)\qquad \nabla_Z g(X, Y) = g(\nabla_Z X, Y)+g(X, \nabla_Z Y)+2\gamma(Z)g(X, Y).$$

Remark. Compatible connections and the related covariant differentiations have been investigated by V. Hlavatý. Let us notice that for any vector fields X and Y the field $g(X, Y)$ is a geometrical object field asosociated with the bundle $R\times W\to W$. Its transformation rule is $g(X, Y)\mapsto(\exp 2t)g(X, Y)$. Thus its covariant derivative is the following:

$$\nabla_Z g(X, Y) = \partial_Z g(X, Y)+2\gamma(Z)g(X, Y).$$

Then equality (3) is analogical to the condition that some linear connection is a Levi-Civita one.

We shall find some intrinsic expressions for compatible connections. The result of the following considerations will include a proof of the existence and uniqueness of the compatible connection if the component γ is given.

We have $\nabla_X Y-\nabla_Y X = [X; Y]$ (the Poisson brackets) because the torsion of ω vanishes. Then we have

$$g(\nabla_X Y, Z) = [X, Y; Z]_g-\gamma(X)g(Y, Z)-\gamma(Y)g(X, Z)+\gamma(Z)g(X, Y).$$

We introduce the invariant Christoffels; namely, we put

$$[X, Y; Z]_g := \tfrac{1}{2}\big(\partial_X g(Y, Z)+\partial_Y g(X, Z)-\partial_Z g(X, Y)+g(X, [Y, Z])+ \\ +g(Y, [X, Z])-g(Z, [X, Y])\big).$$

Equality (3) and its cyclic variants yield

$$\begin{aligned}&\partial_X g(Y, Z)+\partial_Y g(X, Z)-\partial_Z g(X, Y)\\ &= g(Z, \nabla_X Y)+g(Z, \nabla_Y X)+g(X, \nabla_Y Z-\nabla_Z Y)+g(Y, \nabla_X Z-\nabla_Z X)+\\ &\quad+g(Y, \nabla_X Z-\nabla_Z Y)+2\big(\gamma(X)g(Y, Z)+\gamma(Y)g(Z, X)-\gamma(Z)g(X, Y)\big).\end{aligned}$$

Hence we see that the mapping $Z\mapsto[X, Y; Z]_g$ is linear and its values depend only on the point values of Z. Thus $\nabla_X Y$ is uniquely determined because g is non-degenerated. If we fix g, then we may write

$$(4)\qquad \nabla_X Y = g^{\check{}}\,[X, Y; -]_g-\gamma(X)Y-\gamma(Y)X-g(X, Y)g^{\check{}}\gamma,$$

where $g^{\check{}}$ denotes the isomorphism of the cotangent bundle onto the tangent bundle over W, this isomorphism being determined by g. In local coordinates $g^{\check{}}$ is just the lifting of indices of covectors. Let us notice that the left-hand member of (4) does not depend on the choice of g, while every right hand member does.

(*) **Construction of osculating spaces** (cf. [1]). We consider an r-dimensional manifold V which is a submanifold of the given Weyl manifold W, $r < n$. We assume that the identity mapping is just the immersion of V into W. This immersion immediately induces a Weyl structure on V. If $a \in V$, then we denote by $\overline{V}_a$ and $\overline{W}_a$ the spaces which are tangent to V and W, respectively, at the point a. We take into consideration vector fields $X, X_1, \ldots, Y, Z$ etc. in a neighbourhood of the fixed point a. Then we define a sequence of mappings and of vector spaces as follows: We set

$$\varphi_1(X) = X \quad \text{and} \quad \nabla_Y X = \varphi_2(X, Y) + \psi_1(X, Y),$$

where $\psi_1(X, Y)$ is in $\overline{V}_a$ and $\varphi_2(X, Y)$ is orthogonal to V. Evidently, all the values of φ_2 form a certain subspace $\subset \overline{W}_a$ which we denote by P_a^2. We put $P_a^1 = \overline{V}_a$. We prolong this process by induction. If $\varphi_1, \ldots, \varphi_{k-1}$, $\psi_1, \ldots, \psi_{k-2}$ and $P_a^1, \ldots, P_a^{k-1}$ are defined, then we write the decomposition

$$\nabla_Y \varphi_{k-1}(X_1, \ldots, X_{k-1}) = \varphi_k(X_1, \ldots, X_{k-1}, Y) + \psi_{k-1}(X_1, \ldots, X_k, Y),$$

where $\psi_{k-1}(X_1, \ldots, X_{k-1}, Y)$ is defined as the orthogonal projection, $\nabla_Y \varphi_{k-1}(X_1, \ldots, X_{k-1})$ onto the space

$$Q_a^{k-1} := \bigoplus_{i=1}^{k-1} P_a^i$$

We call Q_a^p the pth osculating space of the immersion $V \to W$ at the point a. Thus P_a^k is spanned by values of φ_k.

LEMMA 1. *If $k > 2$, then we have $\psi_k(X_1, \ldots, X_k, X_{k+1}) \in P_a^{k-1} \oplus P_a^k$.*

Proof. We have $g(\varphi_k(X_1, \ldots, X_k), \varphi_l(Y_1, \ldots, Y_l)) = 0$ for arbitrary vectors and $k \neq l$, g being an arbitrary element of G. If $k \leqslant l-2$, then if follows that.

$$\begin{aligned}
0 &= \nabla_Z g(\varphi_k(X_1, \ldots, X_k), \varphi_l(Y_1, \ldots, Y_l)) \\
&= g(\nabla_Z \varphi_k(X_1, \ldots, X_k), \varphi_l(Y_1, \ldots, Y_l)) + \\
&\quad + g(\varphi_{k+1}(X_1, \ldots, X_k), \varphi_{l+1}(Y_1, \ldots, Y_l, Z) + \psi_l(Y_1, \ldots, Y_l, Z)) + 2\gamma(Z)\cdot 0 \\
&= 0 + 0 + g(\varphi_k(X_1, \ldots, X_k), \psi_l(Y_1, \ldots, Y_l, Z)).
\end{aligned}$$

Thus $\psi_l(X_1, \ldots, X_l, Z)$ is orthogonal to any value of $\varphi_{k_{l-2}}$ if $k \leqslant l-2$. In consequence this value of ψ_l is always orthogonal to $\bigoplus_{i=1}^{l-2} P_a^i$.

LEMMA 2. *If $s \geqslant 2$, then the following identity holds:*

$$\nabla_{X_s}\varphi_{s-1}(\ldots, X_{s-1}) - \nabla_{X_{s-1}}\varphi_{s-1}(\ldots, X_{s-2}, X_s) = R_{X_s, X_{s-1}}\varphi_{s-2}(\ldots, X_{s-2}) + \text{some vector from } Q_a^{s-1},$$

where R denotes a Riemannian curvature tensor of ω, $R_{X,Y}(\cdot) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

Proof. The proof is easy and may be performed by a direct computation.

LEMMA 3. *The mapping*

$$(X_1, \ldots, X_k; Y_1, \ldots, Y_k) \to g_k(X_1, \ldots, X_k; Y_1, \ldots, Y_k) := g\big(\varphi_k(X_1, \ldots, X_k), \varphi_k(Y_1, \ldots, Y_k)\big)$$

is a symmetric $(2k)$-linear form.

Proof by induction. If $k = 1$, then the theorem is trivial. If $k = 2$, then we have $\varphi_2(X_1, X_2) = \nabla_{X_2} X_1 - \psi_1(X_1, X_2)$ and we see that $X_2 \mapsto g\big(\varphi_2(X_1, X_2), \varphi_2(Y_1, Y_2)\big)$ is linear. We have

$$\varphi_2(X_1, X_2) - \varphi_2(X_2, X_1) = -[X_1, X_2] - \psi_1(X_1, X_2) + \psi_1(X_2, X_1)$$

and

$$g_2(X_1, X_2; Y_1, Y_2) - g_2(X_2, X_1; Y_1, Y_2) = 0.$$

This implies that the lemma is valid for $k = 2$. Assume that it is valid for $k = 1, \ldots, s-1$. If we make use of Lemma 2, then we obtain the identities

$$\begin{aligned} g_s(\ldots, X_{s-1}, X_s; Y_1, \ldots, Y_s) - g_s(\ldots, X_s, X_{s-1}; \ldots, Y_s) &= g\big(\nabla_{X_{s-1}}\varphi_{s-1}(\ldots, X_{s-1}) - \nabla_{X_{s-1}}\varphi_{s-1}(\ldots, X_s), \varphi_s(\ldots, Y_s)\big) \\ &= g\big(R_{X_s, X_{s-1}}\varphi_{s-2}(\ldots, X_{s-2}), \varphi_s(\ldots, Y_s)\big), \end{aligned}$$

which imply our lemma.

LEMMA 4. *Let C be a C^∞-differentiable arc in W. Then there exists a subclass $G_C \subset G$ such that:* (i) *if $g \in G_C$, then the corresponding component γ of the connection form is zero on C,* (ii) *if $g \in G_C$ and $h \in G_C$, then there exists a positive number a such that we have $g = ah$ on C.* (Cf. [2].)

Proof. We start with some parametrization $\varphi\colon]-1, 1[\to W$ of C. We choose some $\mathring{g} \in G$ and the corresponding j and solve the following ordinary differential equation:

$$\mathring{\gamma} \circ d\varphi + dt \circ d\varphi = 0. \tag{5}$$

A C^∞ solution t satisfying an initial condition $t(0) = 0$ exists and is unique. We assume $g = e^{-2t}\,\mathring{g}$ on C. If γ corresponds to g, then we have $\gamma = \mathring{\varphi} + dt = 0$. Then we prolong g from C onto W so that it wholly remains in G. Then we define G_C to be a class of g which are constructed in the above way. We see that a change of the initial value of a solution of (5) implies a multiplication of g by some function which is constant on C.

Let us remark that Lemma 4 yields a possibility of parallel transport of angles along a curve and a possiblity to compare the lengths of vectors along C.

Frenet's rigging and curvatures. Let C and G_C be as above. We fix some $g \in G_C$. We check a field C^∞ of vectors X on C such that $g(X, X) = 1$. We set $I_1 = X$ and $\nabla_X X = \varkappa_1 I_2$, where $\varkappa_1$ is real and I_2 is such that $g(I_2, I_2) = 1$. $g(I_1, I_2) = 0$ and $g \in G_C$ imply that I_2 is orthogonal to I_1. We prolong this process in construction (*).

If $I_1, \ldots, I_l$ and $\varkappa_1, \ldots, \varkappa_{l-1}$ are defined, then we write the decomposition

$$\nabla_x I_l = \sum_{j=1}^{l} c_j I_j + \varkappa_e I_{l+1}.$$

By the same argument as in Lemma 1 we have $c_j = 0$ for $j < s-1$ and from $g(I_j, I_k) = \delta_{jk}$ we deduce that $c_j = 0$ and $c_{l-1} = -\varkappa_{l-2}$. Thus we obtain a sequence of osculating vector spaces along $C, L_1 \subset \ldots \subset L_q \ldots$, each L_q being spanned by vectors $I_1, \ldots, I_q$. If at some point of our arc we have $L_q = L_{q+1}$, then $L_{q+1} = L_{q+2} = \ldots$ and $\varkappa_s = 0$ for $s \geqslant q$. Then we may write

$$\nabla_{I_1} I_1 = \varkappa_1 I_2, \quad \nabla_{I_1} I_2 = -\varkappa_1 I_1 + \varkappa_2 I_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\nabla_{I_1} I_p = -\varkappa_{p-1} I_{p-1} + \varkappa_p I_{p+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\nabla_{I_1} I_q = -\varkappa_{q-1} I_{q-1}.$$

The above system of equalities has not any invariant sense because all I_p and all $\varkappa_p$ depend on the choice of g. If we check any other $g \in G_C$ and we fix $\bar{I}_1$ so that $\bar{g}(\bar{I}_1, \bar{I}_1) = 1$, then we have $\bar{I}_1 = a I_1$, where $a = \text{const.}$ Thus we have

$$\nabla_{\bar{I}_1} \bar{I}_1 = \bar{\varkappa}_1 \bar{I}_2 = a^2 \nabla_{I_1} I_1 = a\varkappa_1 \cdot a I_2$$

and in consequence $\bar{I}_2 = a I_2$ and $\bar{\varkappa}_1 = a\varkappa_1$. If we prolong this process, then we obtain $\bar{I}_p = a I_p$ and $\bar{\varkappa}_p = a\varkappa_p$. We may formulate the following:

PROPOSITION. *Let us fix a point $a \in C$. Then there exist $a \neq 0$ and vectors $J_1 = aI_1, \ldots, J_q = aI_q$ such that*

$$\nabla_{J_1} J_1 = k_1 J_2, \qquad \nabla_{J_1} J_2 = -k_1 J_1 + k_2 J_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{6}$$
$$\nabla_{J_1} J_{q-1} = -k_{q-2} J_{q-2} + J_q, \qquad \nabla_{J_1} J_q = -J_{q-1}$$

The numbers $k_1, \ldots, k_{q-2}$ have a geometrical meaning because they do not depend on the choice of g from G_C. Thus they may be considered as curvatures of C.

GENERALIZED THEOREM OF MEUSNIER. *We consider the following situation: C is an arc in the manifold V and V is a submanifold in a Weyl manifold W. Let $a \in C$ be fixed. We construct a sequence of osculating subspaces $Q_a^1, Q_a^2, \ldots$ of V and a sequence of vectors $J_1, J_2, \ldots$ which satisfy* (6). *We denote the angle between J_k and Q_a^{k-1} by ν_k. Let $g \in G_C$ be such that $g(J_1, J_1) = 1$. Then we have at a*

$$g_{p+1}(J_1, \ldots, J_1; J_1, \ldots, J_1) = (k_1 \ldots k_p \sin \nu_{p+1})^2$$

for $p = 1, \ldots, q-1$.

Proof. By easy induction we obtain from the Proposition

$$\underbrace{\nabla_{J_1} \ldots \nabla_{J_1}}_{p} J_1 = \sum_{i=1}^{p} C_i^{(p)} J_i + k_1 \ldots k_p J_{p+1}$$

for $p = 1, \ldots, q-1$. If we split $\nabla_{J_1} \ldots \nabla_{J_1} J_1$ into $\varphi_{p+1}(J_1, \ldots, J_1)$ and $\psi_p(J_1, \ldots, J_1)$ according to construction $(*)$, then we see that $\sum_{i=1}^{p} c_i^{(p)} J_i$ enters into the ψ-component. By the definition of ν_{p+1} we have

$$\varphi_{p+1}(J_1, \ldots, J_1) = k_1 \ldots k_p \quad \text{(orthogonal projection of } J_{p+1} \text{ to } P_a^{p+1}\text{)}.$$

Hence

$$g_{p+1}(J_1, \ldots, J_1; J_1, \ldots, J_1) = g\big(\varphi_{p+1}(J_1, \ldots, J_1), \varphi_{p+1}(J_1, \ldots, J_1)\big)(k_1 \ldots k_p \sin \nu_{p+1})^2.$$

References

[1] L. Boček, *Globaldifferentialgeometrie der Untermannigfaltigkeiten in E_n und S_n*, Czechoslovak Math. J. 17 (92) (1967), p. 36–44.

[2] B. G. Folland, *Weyl manifolds*, J. of Diff. Geometry 4 (2) (1970), p. 145–154.

[3] A. Szybiak, *A contribution to the connection of V. Hlavatý*, ce volume, p. 141–151.

[4] Trần dinh Viên, *Sur les généralisations du théorème de Meusnier*, Ann. Polon. Math. 27 (1972), p. 231–245.

Reçu par la Rédaction le 25. 1. 1972