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NUMBER OF POLYNOMIALS IN DEPENDENCE PRESERVING ALGEBRAS

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By $S(\mathfrak{A})$ we denote the set of all non-negative integers n such that the algebra $\mathfrak{A}$ contains an essentially n-ary polynomial, i.e., a polynomial depending on its all n variables. M. Sekanina ([1]) described the set $S(\mathfrak{A})$ in bidirected algebras. In particular, he proved that in bidirected algebras any algebraic operation f has the following property:

(*) If $f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$ is a polynomial in $\mathfrak{A}$ depending on x_k $(1 \leqslant k \leqslant n)$, then after identification $x_j = x_k = x$ $(1 \leqslant j \leqslant n)$ the resulting polynomial depends on x.

E. Marczewski conjectured that in algebras having property (*) results of Sekanina on $S(\mathfrak{A})$ remain true. Examining Sekanina's paper we have checked that this conjecture is true, i.e., if (*) is assumed, then we obtain a description of the sets $S(\mathfrak{A})$ the same as in that paper.

Moreover, in the case where $2 \notin S(\mathfrak{A})$ and $3 \in S(\mathfrak{A})$ we give a simpler proof and show essentially n-ary polynomials explicitly, while Sekanina has proved only the existence of such polynomials. The algebras satisfying (*) will be called *dependence preserving.*

THEOREM 1. *If $\mathfrak{A}$ is a dependence preserving algebra, $2 \notin S(\mathfrak{A})$ and $3 \in S(\mathfrak{A})$, then $S(\mathfrak{A}) = \{1, 3, 4, \ldots\}$.*

Proof. If $s = s(x_1, x_2, x_3)$ is an essentially ternary polynomial (which exists by assumption $3 \in S(\mathfrak{A})$), then, since $\mathfrak{A}$ is dependence preserving algebra with $2 \notin S(\mathfrak{A})$, we have

$$s(x, x, y) = s(x, y, x) = s(y, x, x),$$

where $\hat{s}(x) = s(x, x, y)$ depends on x.

Write

$$s_3 = s_3(x_1, x_2, x_3) = s(x_1, x_2, x_3),$$

$$s_{2n+1} = s_{2n+1}(x_1, x_2, \ldots, x_{2n+1}) = s_{2n-1}\big(s(x_1, x_2, x_3), x_4, \ldots, x_{2n+1}\big),$$

([1]) M. Sekanina, *Number of polynomials in ordered algebras*, this fascicle, p. 181–192.

and

$$s_{2n+1}^* = s_{2n+1}^*(x_1, x_2, \ldots, x_{2n+1}) = s_{2n+1}\big(\hat{s}(x_1), x_2, \ldots, x_{2n+1}\big).$$

Observe first that $s_3^*(x_1, x_2, x_2) = \hat{s}(x_2)$, hence that s_3^* depends on at least one of the variables x_1, x_2. If, for example, s_3^* does not depend on x_3, then it does depend on x_2, because we then have

$$s_3^*(x_1, x_2, x_3) = s\big(\hat{s}(x_1), x_2, x_3\big) = s\big(\hat{s}(x_1), x_2, \hat{s}(x_1)\big) = \hat{s}\big(\hat{s}(x_1)\big).$$

Thus s_3^* depends on x_2, x_3, and by $2 \notin S(\mathfrak{A})$ it is essentially ternary. Hence and from $(*)$ we infer that $\hat{s}(x) = s\big(\hat{s}(x), x, y\big) = s\big(\hat{s}(x), x, \hat{s}(x)\big) = \hat{s}\ (\hat{s}(x))$.

Suppose that s_{2n-1}^* is essentially $(2n-1)$-ary. Consider s_{2n+1}^*. It depends on the variables $x_4, x_5, \ldots, x_{2n+1}$, because

$$s_{2n+1}^*(x_3, x_3, x_3, x_4, \ldots, x_{2n+1}) = s_{2n-1}^*(x_3, x_4, \ldots, x_{2n+1}).$$

If s_{2n+1}^* does not depend on x_1, then

$$\begin{aligned} s_{2n+1}^*(x_1, x_2, \ldots, x_{2n+1}) &= s_{2n+1}\big(\hat{s}(x_1), x_2, \ldots, x_{2n+1}\big) \\ &= s_{2n-1}\Big(s\big(\hat{s}(x_1), x_2, x_3\big), x_4, \ldots, x_{2n+1}\Big) \\ &= s_{2n-1}\Big(s\big(\hat{s}(x_2), x_2, x_3\big), x_4, \ldots, x_{2n+1}\Big) = s_{2n-1}\Big(s\big(\hat{s}(x_3), x_2, x_3\big), x_4, \ldots, x_{2n+1}\Big) \\ &= s_{2n-1}^*(x_2, x_3, x_4, \ldots, x_{2n+1}) = s_{2n-1}^*(x_3, x_4, \ldots, x_{2n+1}), \end{aligned}$$

where the right-hand side of the last formula does not depend on x_2, a contradiction. To prove that s_{2n+1}^* depends on x_2, x_3 we proceed as in the case of s_3^*. Consider operations $s_{2n+1}^*(x_1, x_2, x_3, x_3, x_5, \ldots, x_{2n+1})$. By

$$\begin{aligned} s_{2n+1}(x_2, x_2, x_3, x_3, x_5, \ldots, x_{2n+1}) &= s_{2n-1}\Big(s\big(\hat{s}(x_2), x_2, x_3\big), x_3, x_5, \ldots, x_{2n+1}\Big) \\ &= s_{2n-1}\big(\hat{s}(x_2), x_3, x_5, \ldots, x_{2n+1}\big) = s_{2n-1}^*(x_2, x_3, \ldots, x_{2n+1}) \end{aligned}$$

we infer that $s_{2n+1}^*(x_1, x_2, x_3, x_3, x_5, \ldots, x_{2n+1})$ depends on $x_3, x_5, x_6, \ldots, x_{2n+1}$. Analogously as for s_{2n+1}^* we prove that $s_{2n+1}^*(x_1, x_2, x_3, x_3, x_5, \ldots, x_{2n+1})$ depends on x_1, x_2. Thus $n \in S(\mathfrak{A})$ for $n = 1, 3, 4, \ldots$

We have yet to prove that $0 \notin S(\mathfrak{A})$. It is easy to check that if a is an element of $\mathfrak{A}$ and $f(x_1, x_2, x_3)$ is an essentially ternary polynomial, then $f(x, y, a)$ is an essentially binary function because $\mathfrak{A}$ is dependence preserving. Thus if a were an algebraic constant we would get an essentially binary polynomial, a contradiction.

THEOREM 2. *If $\mathfrak{A}$ is a dependence preserving algebra, then $S(\mathfrak{A})$ is of one of the following forms*: $\{0, 1, 2, \ldots, n\}$, $\{1, 2, 3, \ldots, n\}$, $\{0, 1, 2, 3, \ldots\}$, $\{1, 2, 3, \ldots\}$, $\{1, 3, 4, 5, \ldots\}$.

Proof can be deduced from that of Sekanina (1).

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