

M. M. SYSŁO (Wrocław)

## GENERALIZATIONS OF THE STANDARD TRAVELLING SALESMAN PROBLEM

The Standard Travelling Salesman Problem (STSP) is to find a minimal closed tour of a weighted graph. A number of procedures have been developed for solving the STSP but there is no known one which grows less than exponentially with the number of nodes of the graph. The STSP has many applications to scheduling theory, vehicle routing, the implementation of algorithms in complex computing machines, the construction of information systems, etc., and to meet additional requirements of some of these applications the STSP has been generalized in many ways. Most of the problems which can be considered as generalizations of the STSP are to find the optimal routes for a group of salesmen provided the routes satisfy a set of requirements. Then a given generalization is obtained by specifying meanings of the terms "optimal", "a group of salesmen", and "a set of requirements imposed on routes".

One of the aims of this note\* is to present a family of generalizations of the STSP which correspond to all meanings of the above-mentioned terms introduced so far and known to the author. Some problems are reducible to others, in particular, to the STSP, but there are only a few simple generalizations which are not NP-complete. The STSP is closely related to the assignment problem (AP) which has also been generalized in many ways, for instance as the multi-index problem or as the independent assignment problem. Thus the following question arises: how the generalizations of the STSP are related to those of the AP.

1. In the Travelling Salesman Problem we are given a non-negative integer  $n$  and an  $n$ -dimensional square distance matrix  $D = (d_{ij})$  or, equivalently, we are given an  $n$ -node complete network. Any sequence  $t = (i_1, i_2, \dots, i_p, i_1)$  of integers from  $\{1, 2, \dots, n\}$ , which contains each of the  $n$  integers at least once, and the first and last integers of which are identical, is called a *tour*. A tour is a feasible solution to the travelling

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salesman problem and an optimal solution is a tour such that

$$(1) \quad z(t) = \sum_{(i,j) \in t} d_{ij}$$

is minimized, where

$$t = [(i_1, i_2), (i_2, i_3), \dots, (i_{p-1}, i_p), (i_p, i_1)].$$

This formulation of the problem can be found in many papers and the problem will be referred to hereafter as the *Standard Travelling Salesman Problem* (STSP) since it is closely related to the application which is given in the name: find the shortest closed path of a salesman who must visit all cities of a network.

Let us suppose that the matrix  $D$  is feasible for the STSP, i.e., it does not produce any closed path of negative length.

Most generalizations of the STSP, which are very often called *multi-salesmen problems*, are concerned with the minimization of the length of a tour which is required to satisfy some additional requirements.

Let us review some properties of optimal tours for the STSP. The following theorem is well known:

**THEOREM 1.** *If  $D$  satisfies the triangle inequality, then there is an optimal tour which visits each node once and only once.*

If  $D$  is an arbitrary matrix, we can apply this theorem to the matrix  $D^* = (d_{ij}^*)$ , where  $d_{ij}^*$  is the length of the shortest path from  $i$  to  $j$ , and then an arc  $(i_p, i_q)$  of an optimal tour under  $D^*$  represents the shortest path from  $i_p$  to  $i_q$  under  $D$ . Hence

**COROLLARY 1.** *An optimal tour contains no arc  $(i_p, i_q)$  for which there exists an  $i_r$  such that  $d_{i_p i_r} + d_{i_r i_q} < d_{i_p i_q}$ .*

Therefore, we can eliminate from considerations those arcs which do not satisfy the triangle inequality and, furthermore, in certain cases the STSP may be treated as the *General Routing Problem* (GRP) which is to find a minimal tour traversing each arc in a required subset of arcs of the network and visiting each node in a required subset of nodes of the network. The STSP and the *Chinese Postman Problem* (CPP) are special cases of the GRP. Any feasible tour of the STSP on a graph can be considered as an Euler tour on a multigraph derived from a subgraph of the graph. Euler tours are simply characterized and easily identified, while no efficient and elegant characterization of Hamiltonian cycles has been obtained. It has been pointed out by Orloff [8]-[10] that the GRP approach can be more efficient than that of the STSP, since the GRP takes more advantage of the special structure of a network.

We can easily prove the existence of an optimal tour of the STSP which has the following properties:

PROPERTY 1. *There is an optimal tour such that each its elementary subtour (i.e., a closed path without multiple arcs) contains a node of degree 2.*

COROLLARY 2. *There is an optimal tour which contains a node of degree 2.*

PROPERTY 2. *There is an optimal tour which contains less than  $2n$  arcs.*

Thus, in fact, an optimal solution to the STSP is that to the following multisalesmen problem:

PROBLEM (P1). A salesman is required to visit each of  $n - 1$  "customer" nodes from a "base" node (which is not fixed a priori). During his travel he is allowed to return to the base. It is required to find such a base node which minimizes the number of the base visits and the total distance travelled by the salesman.

By Corollary 2, there are an optimal solution and a base which is visited exactly once.

Note that for a fixed node a tour can be considered as the union of elementary subtours. Notwithstanding, there is no need to introduce different salesmen for different elementary subtours in the case of the STSP with the optimality criterion of form (1). Simply, we may allow the salesman to visit the base node during his travel. However, if  $D$  is a time matrix, then we can state the following

PROBLEM (P2). There are  $m$  salesmen available at a "base" node (which is not fixed a priori) who are required to visit  $n - 1$  "customer" nodes and return to the base node. The problem is how many of the  $m$  salesmen should be used to minimize the time of travelling, i.e., to minimize

$$(2) \quad y(t) = \min_{1 \leq k \leq n} \max_{\{v_s\}_k} \sum_{(i,j) \in v_s} d_{ij},$$

where  $\{v_s\}_k$  is the set of elementary subtours of  $t$  with respect to node  $k$  ( $k = 1, 2, \dots, n$ ).

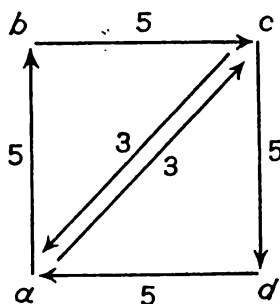


Fig. 1

Fig. 1 shows the network for which

$$z(t_1^*) = z((abcd a)) = 20 \quad \text{and} \quad y(t_2^*) = y((abcacda)) = 13.$$

2. The STSP has found application to many scheduling and sequencing problems, in some cases, however, there is a need to introduce more than one salesman.

Let us define the class of generalizations of the STSP as follows. We are given  $n$  nodes  $N = \{1, 2, \dots, n\}$ ,  $m$   $n$ -dimensional square distance matrices  $D^k = (d_{ij}^k)$  ( $k = 1, 2, \dots, m$ ) and the cost vector  $C = (c_1, c_2, \dots, c_m)^T$ . Let  $\{t_k\}$  denote a family of subtours (i.e., closed paths) covering all  $n$  nodes and let  $N_k$  denote the set of nodes of  $t_k$  ( $k = 1, 2, \dots, m$ ). We have

$$\bigcup_k N_k = N.$$

The problem is to determine subtours  $\{t_k\}$  (i.e., also subsets  $\{N_k\}$ ) such that one of the following optimality criteria is minimized:

$$(3) \quad Z(\{t_k\}) = \sum_k L_k + \sum_{\{k: N_k \neq \emptyset\}} c_k,$$

$$(4) \quad Y(\{t_k\}) = \max_k L_k + \sum_{\{k: N_k \neq \emptyset\}} c_k,$$

where  $L_k$  is the length of  $t_k$ , and  $c_k$  is the cost incurred if a salesman  $k$  is actively used. The restrictions on subtours  $\{t_k\}$  and/or subsets  $\{N_k\}$  can be of the form

$$(5) \quad |N_k| > 1,$$

$$(6) \quad N_{k_1} \cap N_{k_2} = \emptyset \quad (k_1 \neq k_2),$$

$$(7) \quad \bigcap_{\{k: N_k \neq \emptyset\}} N_k \ni i^*,$$

$$(8) \quad t_k \text{ is the Hamiltonian cycle in } N_k.$$

Basing on restrictions imposed on subtours  $\{t_k\}$  and subsets  $\{N_k\}$ , and the form of the criterion we can formulate the following problems:

PROBLEMS 1, 2. For a number of subtours less than or equal to  $m$  determine the feasible subtours  $\{t_k\}$  satisfying (3), (5), (6) or (4)-(6).

PROBLEMS 3, 4. For a number of subtours less than or equal to  $m$  determine the point  $i^* \in N$  and the feasible subtours  $\{t_k\}$  satisfying (3), (5), (7) or (4), (5), (7).

PROBLEMS 5, 6. For a fixed node  $i^* \in N$  and a number of subtours less than or equal to  $m$  determine the feasible subtours  $\{t_k\}$  satisfying (3), (5), (7) or (4), (5), (7).

As particular cases of Problems 1-6 we can consider Problems 1'-6' in which the number of subtours  $m$  is fixed.

In further Problems  $1_H-6_H$  and  $1'_H-6'_H$  restriction (8) is additionally imposed on the subtours  $\{t_k\}$ .

In addition to the above problems we can formulate the whole series of other problems with many practical applications. For instance, we introduce the following additional requirements:

1° Assume that the  $k$ -th salesman can visit no more than (or exactly)  $p_k$  nodes, i.e.

$$(9) \quad |N_k| \leq p_k \text{ (or } = p_k) \quad (k = 1, 2, \dots, m).$$

2° We are given matrices  $V^k = (v_{ij}^k)$  of the times of travelling. Let  $T_k$  denote the length of  $t_k$  under  $V^k$ . Then  $T_k \leq s_k$  ( $k = 1, 2, \dots, m$ ), i.e., the time of travelling the  $k$ -th salesman is bounded by  $s_k$ .

For the sake of simplicity, the class of generalizations of the STSP can be characterized in terms of a five-tuple  $\text{GTSP}(\cdot/\cdot/\cdot/\cdot/\cdot)$ , where the first parameter is the number of nodes, the second term is used to define the number of the subtours, the third term defines properties of subtours  $\{t_k\}$ , the fourth term describes properties of matrices  $\{D^k\}$ ,  $\{V^k\}$  and of the vector  $C$ , and the last one defines the optimality criterion. For instance, the STSP can be characterized by  $\text{GTSP}(n/1/a \text{ tour}/D/\Sigma)$ , and, in virtue of Theorem 1,

$$\text{GTSP}(n/1/a \text{ tour}/D \text{ satisfies the triangle inequality}/\Sigma)$$

is equivalent to

$$\text{GTSP}(n/1/a \text{ Hamiltonian cycle}/D/\Sigma),$$

etc. It is easy to see that  $\text{GTSP}(n/\leq n/(7)/D/\Sigma)$  is equivalent to P1 which, in turn, is equivalent to the STSP, and that  $\text{GTSP}(n/\leq n/(7)/T/\text{minmax})$  is equivalent to P2.

The STSP is an NP-complete problem, and so seem to be nearly all generalizations of it. It is interesting to note that  $\text{GTSP}(n/\leq n/(6) \text{ and } (8)/D/\Sigma)$  is equivalent to the assignment problem which has an algorithm that runs in polynomial time, but a slight modification of the form

$$\text{GTSP}(n/\leq n/(6) \text{ and } (8)/D, C/\Sigma + \Sigma)$$

has no polynomial time algorithm, since the STSP is a special case of it, namely the STSP is equivalent to

$$\text{GTSP}(n/\leq n/(6) \text{ and } (8)/D, C = (e_1, \infty, \dots, \infty)/\Sigma + \Sigma).$$

There are some papers presenting the polynomial time transformations of some generalizations to the STSP for which many algorithms have been published. Unfortunately, these papers deal with problems which assume the same distance matrix for all salesmen.

The most wide class of generalizations of the STSP has been presented by Zak [12]. He has considered problems  $1_H-6_H$  and algorithms for solving them by a branch-and-bound method.

Some generalizations have found application to the problem of routing when there is more than one method of travelling, so that it is apparent that some generalizations are related to the multi-index problem.

Let us define one new notion to show the relations between some generalizations of the STSP and special cases of the multi-index problem. If problem P is to minimize  $c(X)$  subject to  $X \in A$  and problem R is to minimize  $c(X)$  subject to  $X \in B$ , then R is called a *relaxation* of P if  $B \supseteq A$ . It is evident that if  $X_R^*$  and  $X_P^*$  are the optimal solutions to R and P, respectively, then  $c(X_R^*) \leq c(X_P^*)$ . For instance, the AP is a relaxation of GTSP  $(n/1/(8)/D/\Sigma)$ . Problem GTSP  $(n/\leq n/(6) \text{ and } (8)/\{D^k\}/\Sigma)$  can be considered as a generalization of the AP; the distance matrices are different for each subcycle of a solution, and the following problem is its relaxation:

Minimize

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n d_{ij}^k x_{ij}^k$$

subject to

$$\begin{aligned} \sum_{k=1}^n \sum_{i=1}^n x_{ij}^k &= 1 \quad (j = 1, 2, \dots, n), & \sum_{k=1}^n \sum_{j=1}^n x_{ij}^k &= 1 \quad (i = 1, 2, \dots, n), \\ \sum_{i=1}^n x_{ij}^k &\leq 1 \quad (j, k = 1, 2, \dots, n), & \sum_{j=1}^n x_{ij}^k &\leq 1 \quad (i, k = 1, 2, \dots, n), \\ x_{ij}^k &\geq 0 \quad (i, j, k = 1, 2, \dots, n). \end{aligned}$$

Similarly, the problem to minimize

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m d_{ij}^k x_{ij}^k$$

subject to

$$\begin{aligned} \sum_{k=1}^m \sum_{i=1}^n x_{ij}^k &= 1 \quad (j = 1, 2, \dots, n; j \neq i^*), & \sum_{k=1}^m \sum_{i=1}^n x_{ii^*}^k &= (\text{or } \leq) m, \\ \sum_{k=1}^m \sum_{j=1}^n x_{ij}^k &= 1 \quad (i = 1, 2, \dots, n; i \neq i^*), & \sum_{k=1}^m \sum_{j=1}^n x_{i^*j}^k &= (\text{or } \leq) m, \\ \sum_{i=1}^n x_{ij}^k &\leq 1 \quad (j = 1, 2, \dots, n; k = 1, 2, \dots, m), \\ \sum_{j=1}^n x_{ij}^k &\leq 1 \quad (i = 1, 2, \dots, n; k = 1, 2, \dots, m), \\ x_{ij}^k &\geq 0 \quad (i, j = 1, 2, \dots, n; k = 1, 2, \dots, m) \end{aligned}$$

is a relaxation of

GTSP ( $n/m$  (or  $\leq m$ )/(5), (7) and (8),  $i^*$  is fixed/ $\{D^k\}/\Sigma$ ).

These relaxations are special forms of the problem of transportation by different methods, which, in turn, can be transformed to the multi-index problem.

**3. Conclusion.** Some comments on the STSP have been presented and the family of generalizations of the STSP has been described in this paper. It is rather unlikely that there exists a great number of generalizations which are not NP-complete. It would be of interest to find all "easy" generalizations and to find polynomial time transformations of generalizations to the problems for which some algorithms have already been developed.

It seems likely that the multi-index-like relaxations of the generalizations can be used in methods of solving in the way the AP is applied to solve the STSP. This approach needs an answer to the following question: how to eliminate unfeasible (for a given generalization) solutions of the corresponding relaxation, i.e., solutions which do not satisfy requirements imposed on the subtours.

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INSTITUTE OF INFORMATICS  
UNIVERSITY OF WROCLAW  
50-384 WROCLAW

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M. M. SYSŁO (Wrocław)

## UOGÓLNIENIA STANDARDOWEGO PROBLEMU KOMIWOJAŻERA

### STRESZCZENIE

Klasyczny problem komiwojażera (STSP) polega na wyznaczeniu najkrótszej drogi zamkniętej, przechodzącej przez wszystkie wierzchołki grafu, w którym każdemu połączeniu została przyporządkowana liczba — interpretowana jako długość połączenia. Jest to jeden z najtrudniejszych problemów optymalizacji kombinatorycznej, dla którego każda ze znanych metod rozwiązywania jest jedynie pewną odmianą ogólnego schematu postępowania, polegającego na niejawnym przeglądzie całego zbioru rozwiązań dopuszczalnych. Problem komiwojażera ma wiele zastosowań do szeregowania czynności, do wyznaczania optymalnych tras dla pojazdów czy wreszcie do konstrukcji systemów informacyjnych i został uogólniony na wiele różnych spo-



sobów, tak aby mógł być stosowany do rozwiązywania rzeczywistych problemów z dodatkowymi ograniczeniami. Większość problemów, uważanych za uogólnienia klasycznego problemu komiwojażera, polega na wyznaczeniu optymalnych tras dla grupy komiwojażerów przy założeniu, że trasy spełniają pewne dodatkowe ograniczenia.

Jednym z celów tej pracy jest zdefiniowanie dość obszernej rodziny uogólnień klasycznego problemu komiwojażera, która zawiera wszystkie dotychczas rozpatrywane modyfikacje STSP. Pewne uogólnienia są redukowalne do innych, w szczególności do STSP, ale tylko dla niektórych istnieją efektywne metody rozwiązywania.

Końcowa część pracy zawiera relaksacje niektórych uogólnień STSP, będące wieloindeksowymi problemami transportowymi.