ZASTOSOWANIA MATEMATYKI
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Z. JACKIEWICZ (Gdańsk)

GLOBAL ERROR ESTIMATION IN THE NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS BY EULER'S METHOD

This paper deals with the estimation of the global error propagated in the numerical solution of Volterra integro-differential equations by Euler's method. Zadunaisky's estimation technique is used. The main result is that the global error committed in the numerical solution of the pseudoproblem can be used as an estimate of the global error propagated in the numerical integration of the original problem. The proof of this result relies on checking if Euler's method has the "property (E)" defined by Stetter. Some computational aspects are also discussed and numerical examples are given which confirm the main theorem of this paper.

1. Introduction. Let an interval $I: = [t_0, t_0+a]$ and the functions

$$f: I\times R^2 \to R \quad \text{and} \quad k: \Delta \times R \to R, \quad \Delta: = \{(t,s): t_0 \leqslant s \leqslant t \leqslant t_0+a\}$$

be given and consider the initial-value problem for Volterra integro-differential equation

$$\begin{aligned} y'(t) &= f\big(t, y(t), Iy(t)\big), \quad t \in I, \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

where for any continuous function $g: I\to R$

$$Ig(t) = \int_{t_0}^{t} k\big(t, s, g(s)\big)\,ds.$$

For any function $g: D\to R$, where $D \subset R^3$, we use the notation $\|g\|: = \sup\{|g(t)|: t\in D\}$. If the function g is sufficiently smooth, we denote by $D_i g$, $i = 1,2,3$, the partial derivative with respect to the i-th argument and by D_{ij}^2, $i,j = 1,2,3$, the partial derivative of the second order with respect to the i-th and j-th arguments. We write D_i^2 instead of D_{ii}^2, $i = 1,2,3$.

We assume the following:

(H_1) The function f is of class C^1 with respect to the first argument and of class C^2 with respect to the second and third arguments and there exists a constant $M < \infty$ such that $\|f\| \leqslant M$, $\|D_i f\| \leqslant M$, $i = 1, 2, 3$, $\|D_2^2 f\| \leqslant M$,

$\|D_3^2 f\| \leqslant M$, and $\|D_{23}^2 f\| \leqslant M$. Moreover, there exists a constant $L < \infty$ such that

$$|D_i f(t_1, y_1, z_1) - D_i f(t_2, y_2, z_2)| \leqslant L[|t_1 - t_2| + |y_1 - y_2| + |z_1 - z_2|], \quad i = 1, 2, 3,$$

for any $t_1, t_2 \in I$ and $y_1, y_2, z_1, z_2 \in R$.

(H_2) The function k is of class C^1 with respect to the first argument and of class C^2 with respect to the second and third arguments and there exists a constant $P < \infty$ such that $\|k\| \leqslant P$, $\|D_i k\| \leqslant P$, $i = 1, 2, 3$, $\|D_2^2 k\| \leqslant P$, $\|D_3^2 k\| \leqslant P$, and $\|D_{23}^2 k\| \leqslant P$. Moreover, there exists a constant $Q < \infty$ such that the Lipschitz conditions

$$|D_1 k(t_1, s, y_1) - D_1 k(t_2, s, y_2)| \leqslant Q[|t_1 - t_2| + |y_1 - y_2|],$$

$$|D_3 k(t, s, y_1) - D_3 k(t, s, y_2)| \leqslant Q|y_1 - y_2|$$

hold for all (t, s) (t_1, s), $(t_2, s) \in \Delta$ and $y_1, y_2 \in R$.

Remark. It follows from (H_1) and (H_2), respectively, that

$$|f(t_1, y_1, z_1) - f(t_2, y_2, z_2)| \leqslant M[|t_1 - t_2| + |y_1 - y_2| + |z_1 - z_2|],$$

$$|k(t_1, s_1, y_1) - k(t_2, s_2, y_2)| \leqslant P[|t_1 - t_2| + |s_1 - s_2| + |y_1 - y_2|].$$

Let $h \in (0, h_0]$, $h_0 > 0$, be the constant step size and put $t_n = t_0 + nh$, $n = 0, 1, \ldots, N$, where $Nh = a$. In order to solve (1) numerically consider Euler's method defined by

$$y_h(t_n + rh) = y_h(t_n) + rhf_h\big(t_n, y_h(t_n), Iy_h(t_n)\big), \quad n = 0, 1, \ldots, N-1 \text{ and } r \in [0, 1], \tag{2}$$

where $y_h(t_0) = y_0$ and

$$f_h\big(t, g(t), Ig(t)\big) = f\Big(t, g(t), h \sum_{j=0}^{q}{}'' k\big(t_q, t_j, g(t_j)\big)\Big), \qquad q = \operatorname{entier}(t/h).$$

We use the notation

$$\sum_{j=0}^{q}{}'' a_j = \tfrac{1}{2}a_0 + a_1 + \ldots + a_{q-1} + \tfrac{1}{2}a_q, \qquad \sum_{j=0}^{0}{}'' a_j = 0.$$

Define the global error function of the method (2) by $e_h(t) := y_h(t) - y(t)$, where y is the solution of (1). It follows from (H_1) and (H_2) that this solution exists and is unique. To obtain an estimate of the global error we use the method of Zadunaisky [12] (see also [5] and [10]). This method consists in the following:

Consider the pseudoproblem

$$(3)\qquad \begin{aligned} u'(t) &= f\big(t, u(t), Iu(t)\big)+d_h(t), \qquad t\in I,\\ u(t_0) &= y_0, \end{aligned}$$

whose exact solution u is known in advance and, moreover, the defect d_h is "small". Denote by e_h^* the global error committed in the numerical solution of (3) by (2). If we denote by u_h an approximate solution which we obtain applying (2) to (3), e_h^* is given by the difference $e_h^*(t) = u_h(t)-u(t)$. It turns out that under certain conditions we may take e_h^* as an approximation of e_h.

In Section 2 we describe the construction of the pseudoproblem (3). An example is also given of the situation when Zadunaisky's approach to the estimation of the global error cannot be used. In Section 3 we prove the theorem concerning the global error estimate. The proof of this theorem relies on checking if Euler's method (2) has the so-called "property (E)" defined by Stetter [10] (see also [5]). In Section 4 some computational aspects are discussed, and in Section 5 numerical examples are given.

2. Construction of the pseudoproblem. To construct the pseudoproblem we may take advantage of the fact that the method (2) generates an approximate solution y_h defined on the whole interval I. For $n = 0,1,\dots$ $\dots,N-1$ we define the pseudoproblem by

$$(4)\qquad \begin{aligned} U'(t) &= f\big(t, U(t), IU(t)\big)+D_h(t), \qquad t\in [t_n, t_{n+1}),\\ U_h(t_n) &= y_h(t_n), \end{aligned}$$

where

$$D_h(t) = y_h'(t)-f\big(t, y_h(t), Iy_h(t)\big), \qquad t\in [t_n, t_{n+1}).$$

By $y_h'(t_n)$ we mean the right hand derivative. It is easy to see that the function y_h is the exact solution of (4). The method (2) applied to (4) takes the form

$$(5)\qquad U_h(t_n+rh) = U_h(t_n +rh\left[f_h\big(t_n, U_h(t_n), IU_h(t_n)\big)+D_h(t_n)\right],$$
$$n = 0,1,\dots,N-1 \text{ and } r\in[0,1],$$

where $U_h(t_0) = y_0$. Denote the global error function by $E_h^*(t) := U_h(t)-$ $-y_h(t)$. We have the following

THEOREM 1. *Assume that* (H_1) *and* (H_2) *hold. Then* $E_h^*(t) = O(h^2)$ *as* $h\to 0$ *uniformly in* t.

To prove this theorem we need the following lemma:

LEMMA 1. *Assume that* (H_1) *and* (H_2) *hold and let* y_h *be the function defined by* (2). *Then*

$$Iy_h(t_n) - h\sum_{j=0}^{n}{}'' k(t_n, t_j, y_h(t_j)) = O(h^2) \quad \text{as } h\to 0$$

for any grid point t_n, $n = 0, 1, \ldots, N$.

Proof. We have

$$\Big|Iy_h(t_n) - h\sum_{j=0}^{n}{}'' k(t_n, t_j, y_h(t_j))\Big|$$

$$\leqslant \sum_{i=0}^{n-1}\Big|\int_{t_i}^{t_{i+1}} k(t_n, s, y_h(s))\,ds - \frac{h}{2}\big[k(t_n, t_i, y_h(t_i)) + k(t_n, t_{i+1}, y_h(t_{i+1}))\big]\Big|$$

$$\leqslant \sum_{i=0}^{n-1}\frac{h^3}{12}\Big|\frac{d^2k}{ds^2}(t_n, s, y_h(s))\big|_{s=\xi_i}\Big|.$$

It is easy to see using (H_1) and (H_2) that

$$\Big|\frac{d^2k}{ds^2}(t_n, s, y_h(s))\big|_{s=\xi_i}\Big| \leqslant M(P+1)$$

for any n and i. Consequently,

$$\Big|Iy_h(t_n) - h\sum_{j=0}^{n}{}'' k(t_n, t_j, y_h(t_j))\Big| \leqslant aM(P+1)h^2/12$$

and the lemma follows.

Proof of Theorem 1. Denote the local error of the method (5) at the point t_n+rh ($n = 0, 1, \ldots, N-1$ and $r\in[0,1]$) by

$$\xi(t_n, r, h) := y_h(t_n+rh) - y_h(t_n) - rh\big[f_h\big(t_n, y_h(t_n), Iy_h(t_n)\big) + D_h(t_n)\big].$$

From Lemma 1 it follows that

$$f_h\big(t_n, y_h(t_n), Iy_h(t_n)\big) = f\big(t_n, y_h(t_n), Iy_h(t_n)\big) + O(h^2) \quad \text{as } h\to 0.$$

Therefore, after simple calculations we obtain $\xi(t_n, r, h) = O(h^3)$ as $h\to 0$. Now the result follows from the general Theorem 3 of [7].

It follows from Theorem 1 that although Euler's method is, in general, only of order one, it converges with order two when applied to the special problem (4). As a consequence, E_h^* cannot be used as an estimate of e_h because both these quantities have different orders.

To obtain a reliable estimate of the global error e_h we must use at least a quadratic interpolation of the numerical solutions $\{y_h(t_n)\}_{n=0}^{N}$. Assume that N is even and interpolate the values $y_h(t_{2m})$, $y_h(t_{2m+1})$, $y_h(t_{2m+2})$, $m = 0, 1, \ldots, N/2-1$, using the Newton polynomial of degree two. This polynomial takes the form

$$P_m(t) = a_0^m + (t-t_{2m})\big(a_1^m + (t-t_{2m+1})a_2^m\big), \qquad t \in [t_{2m}, t_{2m+2}],$$

where a_i^m $(i = 0, 1, 2)$ are divided differences given by

$$a_0^m = [t_{2m}; y_h] = y_h(t_{2m}),$$

$$a_1^m = [t_{2m}, t_{2m+1}; y_h] = f_h\big(t_{2m}, y_h(t_{2m}), Iy_h(t_{2m})\big),$$

$$a_2^m = [t_{2m}, t_{2m+1}, t_{2m+2}; y_h] = \frac{1}{2h}\big[f_h\big(t_{2m+1}, y_h(t_{2m+1}), Iy_h(t_{2m+1})\big) - $$

$$-f_h\big(t_{2m}, y_h(t_{2m}), Iy_h(t_{2m})\big)\big].$$

Define the function $P: I \to R$ by $P(t) = P_m(t)$ for $t \in [t_{2m}, t_{2m+2}]$, and for any $h = a/N$ (N even) consider the pseudoproblem

$$\text{(6)} \qquad \begin{aligned} u'(t) &= f\big(t, u(t), Iu(t)\big) + d_h(t), \qquad t \in [t_{2m}, t_{2m+2}), \\ u(t_{2m}) &= y_h(t_{2m}), \end{aligned}$$

$m = 0, 1, \ldots, N/2-1$, where

$$d_h(t) = P'(t) - f\big(t, P(t), IP(t)\big), \qquad t \in [t_{2m}, t_{2m+2}).$$

By $P'(t_{2m})$ we mean the right hand derivative. The problem (6) has the exact solution P. The method (2) applied to (6) takes the form

$$\text{(7)} \qquad u_h(t_n + rh) = u_h(t_n) + rh\big[f_h\big(t_n, u_h(t_n), Iu_h(t_n)\big) + d_h(t_n)\big],$$

$n = 0, 1, \ldots, N-1$ and $r \in [0, 1]$. It will be shown in Section 3 that the global error $e_h^*(t) = u_h(t) - P(t)$ committed in the numerical solution of (6) by (7) is a good estimate of $e_h(t) = y_h(t) - y(t)$.

3. Global error estimate theorem. Throughout this section we assume that (H_1) and (H_2) hold and that N is even. This section is devoted to the proof of the following

THEOREM 2. *We have* $e_h(t) = e_h^*(t) + O(h^2)$ *as* $h \to 0$ *for* $t \in I$.

We divide the proof into a sequence of lemmas.

LEMMA 2. *There exists a constant* $A < \infty$ *independent of* m *and* h *such that* $|a_i^m| \leqslant A$ *for* $m = 0, 1, \ldots, N/2-1$ *and* $i = 0, 1, 2$.

Proof. The proof for $i = 0$ and $i = 1$ is obvious. Set $i = 2$. Using Lemma 1 and assumption (H_1) we obtain

$$|a_2^m| \leqslant \frac{1}{2h}\left|f\big(t_{2m+1}, y_h(t_{2m+1}), Iy_h(t_{2m+1})\big) - f\big(t_{2m}, y_h(t_{2m}), Iy_h(t_{2m})\big)\right| + O(h)$$

$$\leqslant \frac{M}{2h}[h + |y_h(t_{2m+1}) - y_h(t_{2m})| + |Iy_h(t_{2m+1}) - Iy_h(t_{2m})|] + O(h).$$

It follows that

$$|Iy_h(t_{2m+1}) - Iy_h(t_{2m})|$$
$$\leqslant \left|\int_{t_0}^{t_{2m+1}} k\big(t_{2m+1}, s, y_h(s)\big)ds - \int_{t_0}^{t_{2m+1}} k\big(t_{2m}, s, y_h(s)\big)ds\right| +$$
$$+\left|\int_{t_0}^{t_{2m+1}} k\big(t_{2m}, s, y_h(s)\big)ds - \int_{t_0}^{t_{2m}} k\big(t_{2m}, s, y_h(s)\big)ds\right| \leqslant P(a+1)h.$$

Finally,

$$|a_2^m| \leqslant \tfrac{1}{2}M\big(1 + M + P(a+1)\big) + O(h)$$

and the proof is complete.

LEMMA 3. $|y_h(s) - P(s)| = O(h^2)$ *as* $h \to 0$ *for* $s \in I$.

Proof. Let $s \in [t_{2j}, t_{2j+2})$, $j = 0, 1, \ldots, N/2-1$. For $s \in [t_{2j}, t_{2j+1})$ we have

$$y_h(s) = y_h(t_{2j}) + rhf_h\big(t_{2j}, y_h(t_{2j}), Iy_h(t_{2j})\big),$$
$$P(s) = y_h(t_{2j}) + (s - t_{2j})\big(a_1^j + (s - t_{2j})a_2^j\big),$$

where $r = (s - t_{2j})/h$. Subtracting these equations and using Lemma 2 we get

$$|y_h(s) - P(s)| = |rh(s - t_{2j})a_2^j| \leqslant Ah^2.$$

Assume now that $s \in [t_{2j+1}, t_{2j+2})$. Then

$$y_h(s) - P(s) = y_h(t_{2j+1}) + rhf_h\big(t_{2j+1}, y_h(t_{2j+1}), Iy_h(t_{2j+1})\big) -$$
$$- y_h(t_{2j}) - (s - t_{2j})f_h\big(t_{2j}, y_h(t_{2j}), Iy_h(t_{2j})\big) + O(h^2),$$

where $r = (s - t_{2j+1})/h$. After simple calculations we obtain

$$|y_h(s) - P(s)| = \big|hf_h\big(t_{2j}, y_h(t_{2j}), Iy_h(t_{2j})\big) + rhf_h\big(t_{2j+1}, y_h(t_{2j+1}), Iy_h(t_{2j+1})\big) -$$
$$- (r+1)hf_h\big(t_{2j}, y_h(t_{2j}), Iy_h(t_{2j})\big)\big| + O(h^2)$$
$$\leqslant hM\big(h + hM + hP(a+1)\big) + O(h^2),$$

which is the desired conclusion.

LEMMA 4. *$|d_h(t)| = O(h)$ as $h\to 0$ for $t\in I$.*

Proof. For $t\in[t_{2m}, t_{2m+2})$ we get

$$d_h(t) = f\big(t_{2m}, y_h(t_{2m}), Iy_h(t_{2m})\big)+a_2^m[(t-t_{2m})+(t-t_{2m+1})]- \\ -f\big(t, P(t), IP(t)\big)+O(h^2).$$

By (H_1) and Lemma 2 we have

$$|d_h(t)|\leqslant 3Ah+M\big(|t_{2m}-t|+|y_h(t_{2m})-P(t)|+|Iy_h(t_{2m})-IP(t)|\big)+O(h^2).$$

It is easy to see that $|y_h(t_{2m})-P(t)| = O(h)$ for $t\in[t_{2m}, t_{2m+2})$. Proceeding similarly as in the proof of Lemma 2 and using Lemma 3 we obtain

$$|Iy_h(t_{2m})-IP(t)|\leqslant\Big|\int_{t_0}^{t_{2m}} k\big(t_{2m}, s, y_h(s)\big)ds-\int_{t_0}^{t_{2m}} k\big(t, s, P(s)\big)ds\Big|+ \\ +\Big|\int_{t_0}^{t_{2m}} k\big(t, s, P(s)\big)ds-\int_{t_0}^{t} k\big(t, s, P(s)\big)ds\Big|\leqslant 2aPh+2Ph+O(h^2).$$

Consequently, $|d_h(t)| = O(h)$ as $h\to 0$ for any $t\in I$.

LEMMA 5. *Let P be the function defined in Section 2. Then*

$$IP(t_n)-h\sum_{j=0}^{n}{}'' k\big(t_n, t_j, P(t_j)\big) = O(h^2) \quad \text{as } h\to 0$$

for any grid point t_n, $n = 0, 1, \ldots, N$.

Proof. Proceeding similarly as in the proof of Lemma 1 we obtain

$$\Big|IP(t_n)-h\sum_{j=0}^{n}{}'' k\big(t_n, t_j, P(t_j)\big)\Big|\leqslant aBh^2/12$$

where $B := P[1+2A(1+2h_0)+A^2(1+2h_0)^2+2A]$. This is our claim.

The next two lemmas are concerned with the asymptotic expansions for the error functions e_h and e_h^*. Results of this kind were obtained by Feldstain and Sopka [3] for a class of p-th order perturbed Taylor's algorithms. Euler's method is a special case of the above-mentioned algorithm but we derive the asymptotic expansions not only for $t = t_n$ but for all $t\in I$.

LEMMA 6. *Let e be the solution of the problem*

$$(8)\qquad \begin{aligned} e'(t) &= D_2f\big(t, y(t), Iy(t)\big)e(t)+D_3f\big(t, y(t), Iy(t)\big)\int_{t_0}^{t} D_3k\big(t, s, y(s)\big)e(s)\,ds- \\ &\quad -\tfrac{1}{2}y''(t), \qquad t\in I, \\ e(t_0) &= 0. \end{aligned}$$

Then $e_h(t_n+rh) = he(t_n+rh)+O(h^2)$ *as* $h\to 0$ *for* $n = 0,1,\dots,N-1$ *and* $r\in[0,1]$.

Proof. Define the local error $\mu(t_n, r, h)$ of the method (2) at the point t_n+rh by

$$y(t_n+rh) = y(t_n)+rhf_h\big(t_n, y(t_n), Iy(t_n)\big)+\mu(t_n, r, h), \tag{9}$$

$$n = 0,1,\dots,N-1 \text{ and } r\in[0,1],$$

where y is the solution of (1). Routine calculations yield

$$\mu(t_n, r, h) = y''(t_n)r^2h^2/2+O(h^3) \quad \text{as } h\to 0.$$

Subtracting (9) from (2) and taking into account that $f_h = f+O(h^2)$ we get

$$e_h(t_n+rh) = e_h(t_n)+rh\big[f\big(t_n, y_h(t_n), Iy_h(t_n)\big)-f\big(t_n, y(t_n), Iy(t_n)\big)\big]-$$
$$-y''(t_n)r^2h^2/2+O(h^3).$$

After simple calculations we obtain

$$\begin{aligned}e_h(t_n+rh) = e_h(t_n)+rh\big[&D_2f\big(t_n, y(t_n), Iy(t_n)\big)e_h(t_n)+\\ &+D_3f\big(t_n, y(t_n), Iy(t_n)\big)\big(Iy_h(t_n)-Iy(t_n)\big)+\\ &+\tfrac{1}{2}D_2^2f(t_n, y^*, z^*)e_h^2(t_n)+\\ &+\tfrac{1}{2}D_{23}^2f(t_n, y^*, z^*)e_h(t_n)\big(Iy_h(t_n)-Iy(t_n)\big)+\\ &+\tfrac{1}{2}D_3^2f(t_n, y^*, z^*)\big(Iy_h(t_n)-Iy(t_n)\big)^2\big]-y''(t_n)r^2h^2/2+O(h^3)\end{aligned}$$

with some $y^*\in J\big(y_h(t_n), y(t_n)\big)$ and $z^*\in J\big(Iy_h(t_n), Iy(t_n)\big)$. Here, for any $a, b\in R$ we mean $J(a, b) = (a, b)$ if $a<b$ and $J(a, b) = (b, a)$ if $b<a$. Taking into account that $|e_h(s)| = O(h)$ for $s\in I$ and noting that

$$Iy_h(t_n)-Iy(t_n) = \int_{t_0}^{t_n} D_3k\big(t_n, s, y(s)\big)e_h(s)\,ds+O(h^2),$$

by (H_1) we get

$$\begin{aligned}e_h(t_n+rh) = e_h(t_n)+rh\Big[&D_2f\big(t_n, y(t_n), Iy(t_n)\big)e_h(t_n)+\\ &+D_3f\big(t_n, y(t_n), Iy(t_n)\big)\int_{t_0}^{t_n} D_3k\big(t_n, s, y(s)\big)e_h(s)\,ds\Big]-\\ &-y''(t_n)r^2h^2/2+O(h^3).\end{aligned}$$

Write $\bar{e}_h(t_n+rh) = e_h(t_n+rh)/h$. Then

$$(10)\quad \bar{e}_h(t_n+rh) = \bar{e}_h(t_n)+rh\Big[D_2f\big(t_n, y(t_n), Iy(t_n)\big)\bar{e}_h(t_n)+ + D_3f\big(t_n, y(t_n), Iy(t_n)\big)\int_{t_0}^{t_n} D_3k\big(t_n, s, y(s)\big)\bar{e}_h(s)\,ds - -ry''(t_n)/2\Big]+O(h^2).$$

Putting $\bar{e}_h(t_0) = 0$ we can regard (10) as the result of applying to equation (8) some numerical method with additional error of order two. This method is close to Euler's method although not identical with it. We now investigate the order of the method (10). Denoting the local error of the method (10) at the point t_n+rh by $\nu(t_n, r, h)$ we have

$$\nu(t_n, r, h) = \tfrac{1}{2}(r^2-r)hy''(t_n)+O(h^2) \quad \text{as } h\to 0.$$

Consequently, $\nu(t_n, r, h) = O(h)$ and $\nu(t_n, 1, h) = O(h^2)$ as $h\to 0$, and the method (10) is of order one (cf. [8] and [11]). It follows from Theorem 5 of [7] that

$$|\bar{e}_h(t_n+rh)-e(t_n+rh)| = O(h) \quad \text{as } h\to 0.$$

Finally, $e_h(t_n+rh) = he(t_n+rh)+O(h^2)$ as $h\to 0$, and the lemma follows.

LEMMA 7. *Let e^* be the continuous solution of the problem*

$$(11)\quad \begin{gathered}(e^*)'(t) = D_2f\big(t, P(t), IP(t)\big)e^*(t)+ \\ +D_3f\big(t, P(t), IP(t)\big)\int_{t_0}^{t} D_3k\big(t, s, P(s)\big)e^*(s)\,ds - \tfrac{1}{2}P''(t), \qquad t\in[t_{2m}, t_{2m+2}), \\ e^*(t_0) = 0,\end{gathered}$$

$m = 0, 1, \ldots, N/2-1$, *where P is the solution of* (6). *Then*

$$e_h^*(t_n+rh) = he^*(t_n+rh)+O(h^2) \quad \textit{as } h\to 0$$

for $n = 0, 1, \ldots, N-1$ *and* $r\in[0, 1]$.

Proof. It follows from Lemma 5 that

$$f_h\big(t_n, P(t_n), IP(t_n)\big) = f\big(t_n, P(t_n), IP(t_n)\big)+O(h^2) \quad \text{as } h\to 0.$$

Therefore, after simple calculations we obtain

$$\mu^*(t_n, r, h) = P''(t_n)r^2h^2+O(h^3) \quad \text{as } h\to 0,$$

where $\mu^*(t_n, r, h)$ denotes the local error of the method (7) at the point t_n+rh. Proceeding similarly as in the proof of Lemma 6 and putting

$\bar{e}_h^*(t_n+rh) = e_h^*(t_n+rh)/h$ we get

$$\bar{e}_h^*(t_n+rh) = \bar{e}_h^*(t_n)+rh\Big[D_2f\big(t_n, P(t_n), IP(t_n)\big)\bar{e}_h^*(t_n)+ + D_3f\big(t_n, P(t_n), IP(t_n)\big)\int_{t_0}^{t_n} D_3k\big(t_n, s, P(s)\big)\bar{e}_h^*(s)ds - -rP''(t_n)/2\Big]+O(h^2).$$

The local error $\nu^*(t_n, r, h)$ of this method is equal to

$$\nu^*(t_n, r, h) = \tfrac{1}{2}(r^2-r)hP''(t_n)+O(h^2) \quad \text{as } h\to 0.$$

Hence, by Lemma 2, $\nu^*(t_n, r, h) = O(h)$, $\nu^*(t_n, 1, h) = O(h^2)$ as $h\to 0$, and the proof is complete.

Remark. In the terminology of Frank [4], Lemma 7 means that the family of pseudoproblems (6) is of type (A). As we have seen the proof of this fact is simple in our case, i.e. for Euler's method (2). It is a non-trivial task for more general methods even in the case of ordinary differential equations (see [4], p. 45-70).

The next lemma is a generalization of Gronwall's inequality.

LEMMA 8. *Assume that* $w(t)\geqslant 0$, $t\in I$, *and*

$$w(t)\leqslant W(t) := A_1\int_{t_0}^{t} w(s)ds+A_2\int_{t_0}^{t}\int_{t_0}^{s} w(\xi)d\xi ds+C,$$

where A_1, A_2, *and* C *are non-negative constants. Then*

$$w(t)\leqslant C\exp[(A_1+A_2a)(t-t_0)], \quad t\in I.$$

Proof. We have

$$W(t_0) = C \quad \text{and} \quad W'(t) = A_1w(t)+A_2\int_{t_0}^{t} w(\xi)d\xi.$$

Since $W'(t)\geqslant 0$, $t\in I$, the function W is non-decreasing. Hence

$$W'(t)\leqslant A_1W(t)+A_2\int_{t_0}^{t} W(\xi)d\xi\leqslant (A_1+A_2a)W(t).$$

It follows from the theory of differential inequalities that $W(t)\leqslant Q(t)$, $t\in I$, where Q is the solution of the problem

$$Q'(t) = (A_1+A_2a)Q(t), \quad t\in I,$$
$$Q(t_0) = C.$$

Thus $w(t)\leqslant W(t)\leqslant C\exp[(A_1+A_2a)(t-t_0)]$, which completes the proof.

LEMMA 9. *We have* $|y(t)-P(t)| = O(h)$ *and* $|y'(t)-P'(t)| = O(h)$ *as* $h\to 0$ *for* $t\in I$, *where* y *is the solution of* (1), P *is the solution of* (6), *and* $P'(t_{2m})$, $m = 0,1,\ldots,N/2-1$, *denotes the right hand derivative.*

Proof. Integrating (1) and (6) we obtain

$$y(t) = y_0+\int_{t_0}^{t} f\big(s, y(s), Iy(s)\big)ds,$$

$$P(t) = y_0+\int_{t_0}^{t} f\big(s, P(s), IP(s)\big)ds+\int_{t_0}^{t} d_h(s)ds.$$

Subtracting these equations and using (H_1) we get

$$|y(t)-P(t)|\leqslant M\int_{t_0}^{t}[|y(s)-P(s)|+|Iy(s)-IP(s)|]ds+\int_{t_0}^{t}|d_h(s)|ds, \quad t\in I.$$

We also have

$$|Iy(s)-IP(s)|\leqslant P\int_{t_0}^{t}|y(\xi)-P(\xi)|d\xi.$$

Putting $w(t) := |y(t)-P(t)|$ we get

$$w(t)\leqslant M\int_{t_0}^{t}w(s)ds+MP\int_{t_0}^{t}\int_{t_0}^{s}w(\xi)d\xi+C(h), \quad t\in I,$$

where $C(h) := a\sup\{|d_h(t)|: t\in I\}$. Observe that $C(h) = O(h)$ as $h\to 0$. Now the first part of our result follows from Lemma 8. The second part of the lemma follows from the inequality

$$|y'(t)-P'(t)|\leqslant M|y(t)-P(t)|+MP\int_{t_0}^{t}|y(\xi)-P(\xi)|d\xi+|d_h(t)|, \quad t\in I.$$

LEMMA 10. *We have* $e^*(t) = e(t)+O(h)$ *as* $h\to 0$, $t\in I$, *where* e *and* e^* *are the solutions of* (8) *and* (11), *respectively.*

Proof. Integrating (8) and (11) and subtracting the resulting equations we get

(12)

$$\begin{aligned}|e(t)-e^*(t)|\leqslant &\int_{t_0}^{t}\big|D_2f\big(s, y(s), Iy(s)\big)e(s)-D_2f\big(s, P(s), IP(s)\big)e^*(s)\big|ds+\\ &+\int_{t_0}^{t}\Big|D_3f\big(s, y(s), Iy(s)\big)\int_{t_0}^{s}D_3k\big(s, \xi, y(\xi)\big)e(\xi)d\xi-\\ &-D_3f\big(s, P(s), IP(s)\big)\int_{t_0}^{t}D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)d\xi\Big|ds+\\ &+\tfrac{1}{2}\big(|y'(t)-P'(t)|+|y'(t_0)-P'(t_0)|\big), \quad t\in I.\end{aligned}$$

Putting $E := \sup\{|e^*(s)|: s \in I\}$ we obtain

$$\begin{aligned}
&\big|D_2f\big(s, y(s), Iy(s)\big)e(s) - D_2f\big(s, P(s), IP(s)\big)e^*(s)\big| \\
&\quad \leqslant \big|D_2f\big(s, y(s), Iy(s)\big)e(s) - D_2f\big(s, y(s), Iy(s)\big)e^*(s)\big| + \\
&\quad\quad + \big|D_2f\big(s, y(s), Iy(s)\big)e^*(s) - D_2f\big(s, P(s), IP(s)\big)e^*(s)\big| \\
&\quad \leqslant M|e(s) - e^*(s)| + LE\big(|y(s) - P(s)| + |Iy(s) - IP(s)|\big) \\
&\quad = M|e(s) - e^*(s)| + O(h).
\end{aligned}$$

We also have

$$\begin{aligned}
&\Big|D_3f\big(s, y(s), Iy(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, y(\xi)\big)e(\xi)d\xi - \\
&\quad - D_3f\big(s, P(s), IP(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)d\xi\Big| \\
&\leqslant \Big|D_3f\big(s, y(s), Iy(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, y(\xi)\big)e(\xi)d\xi - \\
&\quad - D_3f\big(s, y(s), Iy(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)d\xi\Big| + \\
&\quad + \Big|D_3f\big(s, y(s), Iy(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)d\xi - \\
&\quad - D_3f\big(s, P(s), IP(s)\big)\int_{t_0}^{s} D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)d\xi\Big| \\
&\leqslant M\int_{t_0}^{s} \big|D_3k\big(s, \xi, y(\xi)\big)e(\xi) - D_3k\big(s, \xi, y(\xi)\big)e^*(\xi)\big|d\xi + \\
&\quad + M\int_{t_0}^{s} \big|D_3k\big(s, \xi, y(\xi)\big)e^*(\xi) - D_3k\big(s, \xi, P(\xi)\big)e^*(\xi)\big|d\xi + \\
&\quad + aPEL\big(|y(s) - P(s)| + |Iy(s) - IP(s)|\big) \\
&\leqslant MP\int_{t_0}^{s} |e(\xi) - e^*(\xi)|d\xi + MEQ\int_{t_0}^{s} |y(\xi) - P(\xi)|d\xi + \\
&\quad + aPEL\big(|y(s) - P(s)| + |Iy(s) - IP(s)|\big) \\
&= MP\int_{t_0}^{s} |e(\xi) - e^*(\xi)|d\xi + O(h).
\end{aligned}$$

Now we can write inequality (12) in the form

$$|e(t)-e^*(t)| \leqslant M \int_{t_0}^{t} |e(s)-e^*(s)|\,ds + MP \int_{t_0}^{t}\int_{t_0}^{s} |e(\xi)-e^*(\xi)|\,d\xi + O(h),$$

$$t \in I.$$

By Lemma 8 we get $|e(t)-e^*(t)| = O(h)$, which is the desired result.

Proof of Theorem 2. It follows from Lemmas 6, 7, and 10 that

$$\begin{aligned} e_h(t_n+rh) &= he(t_n+rh)+O(h^2) = h[e^*(t_n+rh)+O(h)]+O(h^2) \\ &= he^*(t_n+rh)+O(h^2) = e_h^*(t_n+rh)+O(h^2), \end{aligned}$$

which completes the proof.

4. Computational aspects. The method (7) is not suitable for practical computations since, in general, we are not able to compute the quantities $d_h(t_n)$, $n = 0, 1, \ldots, N$. We can overcome this difficulty replacing (7) by the method

$$\bar{u}_h(t_n+rh) = \bar{u}_h(t_n)+rh\left[f_h\big(t_n, \bar{u}_h(t_n), I\bar{u}_h(t_n)\big)+\bar{d}_h(t_n)\right],$$

$$n = 0, 1, \ldots, N-1 \text{ and } r \in [0, 1],$$

where

$$\bar{d}_h(t) = P'(t)-f_h\big(t, P(t), IP(t)\big), \qquad t \in [t_{2m}, t_{2m+2}).$$

THEOREM 3. *If the assumptions* (H_1) *and* (H_2) *are satisfied, then* $\bar{u}_h(t)-P(t) = e_h(t)+O(h^2)$ *as* $h \to 0$ *for* $t \in I$.

Proof. Taking into account that $\bar{d}_h(t) = d_h(t)+O(h^2)$ we can prove, using arguments similar as in the proof of Theorem 1, that $|u_h(t)-\bar{u}_h(t)| = O(h^2)$ as $h \to 0$ for $t \in I$. Hence

$$\bar{u}_h(t)-P(t) = u_h(t)-P(t)+O(h^2) = e_h^*(t)+O(h^2) = e_h(t)+O(h^2)$$

$$\text{as } h \to 0$$

and the theorem holds.

5. Numerical examples. In this section we present the results of numerical experiments. We have solved the following examples:

Example 1 (Day [2]). The exact solution of the problem

$$y'(t) = 1-\int_0^t y(s)\,ds, \qquad t \in [0, 1],$$

$$y(0) = 1$$

is $y(t) = \sin t$.

Example 2 (Day [2], Goldfine [6], Linz [9]). The exact solution of the problem

$$y'(t) = 1+2t-y(t)+\int_0^t t(1+2t)\exp\big(s(t-s)\big)y(s)\,ds, \qquad t\in[0,1],$$

$$y(0) = 1$$

is $y(t) = \exp(t^2)$.

Example 3 (Brunner and Lambert [1]). The exact solution of the problem

$$y'(t) = -\frac{1+t(1+t)^2}{(1+t)^2}+\ln\left(\frac{2+2t}{2+t}\right)/y(t)+\int_0^t \frac{ds}{1+(1+t)y(s)}, \qquad t\in[0,1],$$

$$y(0) = 1$$

is $y(t) = 1/(1+t)$.

Example 4 (Feldstain and Sopka [3]). The exact solution of the problem

$$y'(t) = -0.25t^3+1.25\exp\big(-y(t)\big)+\int_1^t \big[s^2\exp\big(y(s)\big)/t\big]\,ds, \qquad t\in[1,2],$$

$$y(1) = 0$$

is $y(t) = \ln t$.

The results of computations are given in Tables 1-4, where we put $E := e_h(t_0+a)-e_h^*(t_0+a)$. These results confirm Theorem 3. They suggest even more, namely that there exists a limit $\big(e_h(t)-e_h^*(t)\big)/h^2$ as $h\to 0$.

TABLE 1. Results for Example 1

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-1}	0.096 029	0.058 594	0.150
2^{-2}	0.051 100	0.043 274	0.125
2^{-3}	0.026 035	0.024 328	0.109
2^{-4}	0.013 097	0.012 704	0.101
2^{-5}	0.006 563	0.006 469	0.096
2^{-6}	0.003 284	0.003 262	0.094
2^{-7}	0.001 643	0.001 637	0.093
2^{-8}	0.000 822	0.000 820	0.093

TABLE 2. Results for Example 2

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-1}	−0.968 282	−0.609 375	−1.435
2^{-2}	−0.540 549	−0.417 811	−1.964
2^{-3}	−0.286 484	−0.253 626	−2.103
2^{-4}	−0.147 580	−0.139 241	−2.135
2^{-5}	−0.074 914	−0.072 822	−2.142
2^{-6}	−0.037 744	−0.037 220	−2.144
2^{-7}	−0.018 944	−0.018 813	−2.144
2^{-8}	−0.009 490	−0.009 457	−2.145

TABLE 3. Results for Example 3

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-1}	$-0.086\,849$	$-0.136\,907$	0.200
2^{-2}	$-0.051\,608$	$-0.065\,381$	0.220
2^{-3}	$-0.027\,268$	$-0.030\,541$	0.210
2^{-4}	$-0.013\,982$	$-0.014\,787$	0.206
2^{-5}	$-0.007\,077$	$-0.007\,277$	0.204
2^{-6}	$-0.003\,560$	$-0.003\,610$	0.204
2^{-7}	$-0.001\,785$	$-0.001\,798$	0.203
2^{-8}	$-0.000\,894$	$-0.000\,897$	0.203

TABLE 4. Results for Example 4

h	$e_h(2)$	$e_h^*(2)$	E/h^2
2^{-1}	0.410 478	0.430 303	-0.079
2^{-2}	0.128 753	0.135 798	-0.113
2^{-3}	0.046 814	0.046 776	0.002
2^{-4}	0.019 151	0.018 895	0.066
2^{-5}	0.008 515	0.008 422	0.095
2^{-6}	0.003 992	0.003 965	0.109
2^{-7}	0.001 929	0.001 922	0.116
2^{-8}	0.000 948	0.000 946	0.119

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF GDAŃSK
80-952 GDAŃSK

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