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# SOME UNION THEOREMS FOR CONFLUENT MAPPINGS

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Several theorems dealing with the notion of confluent mappings and their properties related to the procedure of forming the union of spaces have been recently established in [3] and [4]. One of them is generalized in the present paper (compare Corollary 1) and we show that a generalization of another one is impossible by providing an example. All spaces which we discuss here are compact metric spaces and all mappings are assumed to be continuous functions. A mapping is called *confluent* [1] provided each component of the inverse image of each continuum is mapped onto that continuum. The very nature of this definition seems to indicate that, in any theorem deriving the confluency of a mapping from the confluency of its restrictions to some sets whose union is the domain space, these sets ought to be the inverse images of some subsets of the range space, which actually reduces the problem to representing the latter space as the union of sets satisfying appropriate conditions. Also, probably very little can be said in a situation where one has the union of an uncountable collection of such sets. In our result, the range space is represented as the union of countably many closed subsets and the conditions are imposed on their intersections. Well-known examples of mappings [3] can be used to show that neither of these conditions can be omitted. Before stating and proving our result we have to introduce a notation and prove a lemma.

Suppose $K$ is a continuum and $B \subset K$ is a non-empty closed subset. For each $n = 1, 2, \ldots$, let $G_n(B)$ be the open neighbourhood of $B$ in $K$ defined by the formula

$$G_n(B) = \{y \in K \colon \operatorname{dist}(y, B) < n^{-1}\},$$

and, for each point $y \in K \setminus B$, let $n(y)$ be the least positive integer such that $n(y)^{-1} \leqslant \operatorname{dist}(y, B)$. Then $y \in K \setminus G_n(B)$ for $n(y) \leqslant n$, and let $C_n(B, y)$ be the component of $K \setminus G_n(B)$ which contains $y$. Observe that $G_{n+1}(B) \subset G_n(B)$ for $n = 1, 2, \ldots$, whence

$$C_n(B, y) \subset C_{n+1}(B, y) \quad \text{for } n(y) \leqslant n.$$

This means that, for each point $y \in K \setminus B$, the sets $C_n(B, y)$ constitute an increasing sequence of subcontinua of $K$. Consequently, their union

$$C(B, y) = \bigcup_{n=n(y)}^{\infty} C_n(B, y)$$

is a connected subset of $K \setminus B$ to which the point $y$ belongs. Since $K$ is a continuum, each component $C_n(B, y)$ meets the boundary of the set $G_n(B)$ in $K$ (see [2], p. 172), so that the closure $\mathrm{cl}\, C(B, y)$ of $C(B, y)$ meets $B$ for $y \in K \setminus B$.

LEMMA 1. *If $K$ is a continuum and $B_0, B_1 \subset K$ are non-empty disjoint closed subsets, then there exists a point $y \in K \setminus (B_0 \cup B_1)$ such that*

$$B_i \cap \mathrm{cl}\, C(B_0 \cup B_1, y) \neq \emptyset \quad \text{for } i = 0, 1.$$

Proof. There exists a continuum $X \subset K$ which is irreducibly connected between $B_0 \cap X$ and $B_1 \cap X$; that is $B_i \cap X \neq \emptyset$ for $i = 0, 1$ and $X$ is an irreducible continuum between each two points $p_0$ and $p_1$ such that $p_i \in B_i \cap X$ for $i = 0, 1$ (see [2], p. 219 and 222). We consider two cases as follows:

Case 1. $X$ is indecomposable. Then there exists a composant $C$ of $X$ such that $C \cap (B_0 \cup B_1) = \emptyset$ (ibidem, p. 223). Let $y \in C$ be a point. Since $C$ is the union of some continua containing $y$, we have $C \subset C(B_0 \cup B_1, y)$. But the composant $C$ being dense in $X$, we conclude (for $i = 0, 1$) that

$$\emptyset \neq B_i \cap X = B_i \cap \mathrm{cl}\, C \subset B_i \cap \mathrm{cl}\, C(B_0 \cup B_1, y).$$

Case 2. $X$ is decomposable. Then there exists a decomposition $X = X_0 \cup X_1$ of $X$ into two proper subcontinua $X_0$ and $X_1$ of $X$. Since $X$ is irreducible between each pair of points selected from $B_0 \cap X$ and $B_1 \cap X$, respectively, neither $X_0$ nor $X_1$ meets both $B_0$ and $B_1$. Without loss of generality, we can assume that $B_i \cap X \subset X_i$ for $i = 0, 1$. Thus also $B_0 \cap X_1 = \emptyset = B_1 \cap X_0$ and let $y \in X_0 \cap X_1$ be a point. Denote by $C_i$ the set $C(B_i \cap X, y)$ as applied to $K = X_i$ ($i = 0, 1$). Then the closure of $C_i$ meets $B_i \cap X$ and $C_i$ is the union of some continua which contain $y$ and are contained in $X_i \setminus B_i$, whence also in $X_i \setminus (B_0 \cup B_1)$. It follows that $C_i \subset C(B_0 \cup B_1, y)$ and we get (for $i = 0, 1$) the same needed conclusion that

$$\emptyset \neq B_i \cap X \cap \mathrm{cl}\, C_i \subset B_i \cap \mathrm{cl}\, C(B_0 \cup B_1, y).$$

THEOREM. *Suppose $X$ and $Y$ are compact metric spaces, $f: X \to Y$ is a continuous mapping of $X$ onto $Y$, and*

$$Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$$

*is a decomposition of $Y$ into closed subsets $Y_i$ such that the following three conditions are satisfied:*

(i) $f|f^{-1}(Y_i)$ *is a confluent mapping of* $f^{-1}(Y_i)$ *onto* $Y_i$ *for* $i = 0, 1, 2, \ldots,$

(ii) $Y_i \cap Y_j \subset Y_0$ *for* $i \neq j$ *and* $i, j = 1, 2, \ldots,$

(iii) $K \cap Y_0$ *has only a finite number of components for each subcontinuum* $K$ *of* $Y$.

*Then* $f$ *is confluent.*

Proof. Let $K \subset Y$ be a continuum and let $C$ be a component of $f^{-1}(K)$. To prove that $f(C) = K$ it is enough to show that, given a point $q \in K$, we have $q \in f(C)$. Let us take a point $p \in f(C)$ and assume $p \neq q$.

There exists a point $x_0 \in C$ such that $p = f(x_0)$. By (iii), the set

$$B = (K \cap Y_0) \cup \{p, q\} \tag{1}$$

has a finite number of components. If $p$ and $q$ belong to the same component, say $A$, of $B$, then, by (1), $A \subset K \cap Y_0$ and a component $C'$ of $f^{-1}(A)$ contains $x_0$. But since $A \subset Y_0$ and, by (i), the mapping $f|f^{-1}(Y_0)$ is confluent, we have $f(C') = A$, whence $q \in f(C')$, and $A \subset K$ implies $C' \subset C$, which means that $q \in f(C)$. Thus we can assume that $p$ and $q$ belong to two different components, say $A_p$ and $A_q$, of $B$, respectively. The set $B$ is a closed subset of the continuum $K$. We claim that there exist components $A_0, A_1, \ldots, A_m$ of $B$ and there exist points $y_1, y_2, \ldots \ldots, y_m$ of $K \setminus B$ such that $A_0 = A_p$, $A_m = A_q$ and

$$A_{j-i} \cap \operatorname{cl} C(B, y_j) \neq \emptyset \quad \text{for } i = 0, 1 \text{ and } j = 1, 2, \ldots, m. \tag{2}$$

In other words, the components $A_p$ and $A_q$ of $B$ can be joined together by means of a finite chain of sets composed, alternately, of some other components of $B$ and of the closures of those connected subsets of $K \setminus B$. Indeed, if it were not so, then denoting by $B_0$ the union of all components of $B$ which can be joined with $A_p$ in such a way, we would have $A_p \subset B_0$ and $A_q \subset B_1 = B \setminus B_0$. Thus $B_0, B_1$ would be non-empty closed sets in $K$ which are not joined by any continuum of type $\operatorname{cl} C(B, y)$ for $y \in K \setminus B$, contrary to Lemma 1.

Now, according to (2), let $q_{ij}$ be points such that $q_{ij} \in A_{j-i}$ and $q_{ij} \in \operatorname{cl} C(B, y_j)$ for $i = 0, 1$ and $j = 1, 2, \ldots, m$. We put $q_{00} = p$ and we are going to prove by induction on $j$ that $q_{0j} \in f(C)$ for $j = 0, 1, \ldots, m$. We know that $q_{00} \in f(C)$ and let us assume that $q_{0,j-1} \in f(C)$ for a positive integer $j \leqslant m$. First, we have to show that $q_{1j} \in f(C)$. If $q_{0,j-1} = q_{1j}$, we are done. If $q_{0,j-1} \neq q_{1j}$, then observe that $q_{0,j-1} \in A_{j-1}$ and $q_{1j} \in A_{j-1}$, where $A_{j-1}$ is a component of $B$. Consequently, by (1), we have $A_{j-1} \subset K \cap Y_0$. There exists a point $x_{j-1} \in C$ such that $q_{0,j-1} = f(x_{j-1})$, and a component $C''$ of $f^{-1}(A_{j-1})$ contains $x_{j-1}$. But since $A_{j-1} \subset Y_0$ and $f|f^{-1}(Y_0)$ is confluent, we have $f(C'') = A_{j-1}$, whence $q_{1j} \in f(C'')$, and $A_{j-1} \subset K$ implies $C'' \subset C$, which means that $q_{1j} \in f(C)$.

Let $u_j$ be a point of $C$ such that $q_{1j} = f(u_j)$. The set $C(B, y_j)$ is the union of the increasing sequence of the sets $C_n(B, y_j)$ which are continua contained in $K \setminus B$. Therefore, by (1), each of these continua is contained in $Y \setminus Y_0$, whence also in the union $Y_1 \cup Y_2 \cup \ldots$ Moreover, the sets $Y_1, Y_2, \ldots$ are mutually disjoint outside $Y_0$, according to (ii). It follows (see [2], p. 173) that none of the continua $C_n(B, y_j)$, where $n(y_j) \leqslant n$, meets two of the sets $Y_1, Y_2, \ldots$ We conclude that there exists a positive integer $k_j$ such that $C(B, y_j) \subset Y_{k_j}$. Thus $C_j = \mathrm{cl}\, C(B, y_j)$ is a continuum contained in $K \cap Y_{k_j}$ and $C_j$ contains both $q_{1j}$ and $q_{0j}$. Since, by (i), the mapping $f|f^{-1}(Y_{k_j})$ is confluent, the same argument as above shows that $q_{0j} \in f(C)$; one has only to replace $A_{j-1}, Y_0, x_{j-1}, q_{0,j-1}$ and $q_{1j}$ by $C_j$, $Y_{k_j}, u_j, q_{1j}$ and $q_{0j}$, respectively.

As a result, we obtain $q_{0m} \in f(C)$. If $q_{0m} \neq q$, the continuum $A_m$ which is a component of $B$ contains both $q_{0m}$ and $q$, whence, by (1), $A_m \subset K \cap Y_0$. Again, the same argument utilizing the confluency of $f|f^{-1}(Y_0)$ shows that, in any case, we have $q \in f(C)$.

Remarks. Examples 4.2 and 5.6 of [3], or their variants, indicate that without conditions (ii) and (iii) the mapping is not necessarily confluent. As a matter of fact, the mapping in the first of those examples is locally confluent and the mappings in both of them even are not weakly confluent ([1]). Condition (iii), however, assumes a simpler form in the case of hereditarily unicoherent spaces ([2]). This can be seen in the following corollary which generalizes Theorem 5.4 of [3]:

COROLLARY 1. *If $f$ is a mapping of a compact metric space onto a hereditarily unicoherent compact metric space $Y$ for which there exists a decomposition*

$$Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$$

*into closed subsets $Y_i$ such that $Y_0$ has only a finite number of components and conditions* (i) *and* (ii) *are satisfied, then $f$ is confluent.*

It has been proved by Read [4] that if $f$ is a locally confluent mapping of a continuum onto a hereditarily unicoherent continuum having no more than two arcwise connected components, then $f$ is weakly confluent. We are going to show that hereditary unicoherence in Read's theorem cannot be replaced by unicoherence. In our example both the domain and range spaces are arcwise connected, unicoherent, rational continua

([1]) We say that a mapping $f$ is *locally confluent* provided there exist closed subsets $Y_i$ of the range space $Y$ such that the interiors of $Y_i$ cover $Y$ and condition (i) is fulfilled. A mapping is called *weakly confluent* provided at least one component of the inverse image of each continuum is mapped onto that continuum.

([2]) A space is said to be *hereditarily unicoherent* provided the common part of each two of its subcontinua is a continuum (although it can be the empty set or a degenerate continuum).

and the mapping is locally confluent without being weakly confluent. Consequently, there seems to be an off chance to have a union theorem for confluent mappings in the case of unicoherent spaces. But, first, we need to deduce another corollary and prove some lemmas.

COROLLARY 2. *If $f$ is a mapping of a compact metric space onto a compact metric space $Y$, $F \subset Y$ is a finite set, and $Y = Y_1 \cup Y_2 \cup \ldots$, where $Y_i$ are closed in $Y$, $f|f^{-1}(Y_i)$ are confluent, and $Y_i \cap Y_j \subset F$ for $i \neq j$ and $i, j = 1, 2, \ldots$, then $f$ is confluent.*

LEMMA 2. *If $X$ is a non-degenerate continuum and $p \in X$ is a point, then there exists a continuum $U(X, p)$ and its decomposition*

$$U(X, p) = X \cup A_1 \cup A_2 \cup \ldots$$

*such that each $A_i$ is an arc with end-points $p$ and $p_i$ $(i = 1, 2, \ldots)$, and the following two conditions are satisfied:*

(I) $\{p\} = A_i \cap X = A_i \cap A_j$ *for* $i \neq j$ *and* $i, j = 1, 2, \ldots$,

(II) $X = \operatorname{Ls}_{i \to \infty} A_{k_i}$ *for each infinite sequence of integers* $1 \leqslant k_1 < k_2 < \ldots$

Proof. We denote by $H$ the Hilbert cube which we consider to be a convex metric space with unique straight-line metric segments and the metric in $H$ denoted by $\varrho$. We also think of $H$ as identical with the subset $H \times \{0\}$ of the product $H \times [0, 1]$, and assume $X$ to be a subset of $H$. Given two points $q_0$ and $q_1$ of $H \times [0, 1]$, we denote by $\overline{q_0 q_1}$ the straight-line (closed) segment having end-points $q_0$ and $q_1$ in $H \times [0, 1]$.

Let $x_1, x_2, \ldots$ be points of $X$ such that the set $\{x_1, x_2, \ldots\}$ is dense in $X$. For each $i = 1, 2, \ldots$, we define an arc $B_i$ in $H \times [0, 1]$ as follows. Since $X$ is a continuum, there exist points $x_{i0}, x_{i1}, \ldots, x_{im_i}$ of $X$ such that $x_{i0} = p$, the set $\{x_1, x_2, \ldots, x_i\}$ is contained in the set $\{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$, and

$$\varrho(x_{i,j-1}, x_{ij}) < i^{-1} \quad \text{for } j = 1, 2, \ldots, m_i.$$

Let $t_{i0}, t_{i1}, \ldots, t_{im_i}$ be real numbers such that

$$(i+1)^{-1} < t_{i0} < t_{i1} < \ldots < t_{im_i} < i^{-1}$$

and let $q_{ij} = (x_{ij}, t_{ij})$ for $j = 0, 1, \ldots, m_i$. We put

$$B_i = \bigcup_{j=1}^{m_i} \overline{q_{i,j-1} q_{ij}}.$$

Observe that $B_1, B_2, \ldots$ are mutually disjoint arcs in $H \times [0, 1]$ and each of them is disjoint with $X$. Moreover, if $q \in B_i$, then $q \in \overline{q_{i,j-1} q_{ij}}$ for some $j = 1, 2, \ldots, m_i$, whence

$$\varrho(q, q_{ij}) \leqslant \varrho(q_{i,j-1}, q_{ij}) \leqslant$$
$$\leqslant \varrho(q_{i,j-1}, x_{i,j-1}) + \varrho(x_{i,j-1}, x_{ij}) + \varrho(x_{ij}, q_{ij}) < t_{i,j-1} + i^{-1} + t_{ij} < 3i^{-1},$$

and thus

$$\varrho(q, x_{ij}) \leqslant \varrho(q, q_{ij}) + \varrho(q_{ij}, x_{ij}) < 4i^{-1}.$$

We see that the arc $B_i$ lies in the $(4i^{-1})$-neighbourhood of $X$ in $H \times [0, 1]$. It follows that

$$\operatorname*{Ls}_{i\to\infty} B_i \subset X$$

and the set $Y = X \cup B_1 \cup B_2 \cup \ldots$ is compact. On the other hand, if $h \leqslant i$ are positive integers, then $x_h = x_{ij}$ for some $j = 1, 2, \ldots, m_i$ and

$$\varrho(x_h, q_{ij}) = \varrho(x_{ij}, q_{ij}) = t_{ij} < i^{-1},$$

so that the set $\{x_1, x_2, \ldots, x_i\}$ lies in the $(i^{-1})$-neighbourhood of $B_i$. Consequently, since the set $\{x_1, x_2, \ldots\}$ is dense in $X$, we obtain

$$X \subset \operatorname*{Ls}_{i\to\infty} B_{k_i}$$

for each infinite sequence of integers $1 \leqslant k_1 < k_2 < \ldots$ As a result, condition (II) is fulfilled for $A_{k_i}$ replaced by $B_{k_i}$.

In order to satisfy also condition (I) we need to identify some points of $Y$. The end-points of the arc $B_i$ are $q_{i0} = (p, t_{i0})$ and $q_{im_i}$ $(i = 1, 2, \ldots)$, and we have

$$p = \lim_{i\to\infty} q_{i0}.$$

Let $P = \{p\} \cup \{q_{10}, q_{20}, \ldots\}$ and let us determine an upper semi-continuous decomposition of $Y$ by selecting $P$ to be the only non-degenerate element of the decomposition. We define $U(X, p)$ to be the resulting quotient space and let $\varphi$ be the natural projection of $Y$ onto $U(X, p)$. Clearly, we can assume that $\varphi$ is the identity on $X$ and now both conditions (I) and (II) are satisfied for $A_i$ defined as $A_i = \varphi(B_i)$. The set $A_i$ is an arc with end-points $\varphi(q_{i0}) = \varphi(p) = p$ and $p_i = \varphi(q_{im_i})$ $(i = 1, 2, \ldots)$, and

$$U(X, p) = \varphi(Y) = \varphi(X) \cup \varphi(B_1) \cup \varphi(B_2) \cup \ldots = X \cup A_1 \cup A_2 \cup \ldots$$

LEMMA 3. *If a continuum $U(X, p)$ has all the properties listed in Lemma 2, then $U(X, p)$ is unicoherent.*

Proof. Assume $U(X, p) = L \cup M$ is a decomposition of $U(X, p)$ into continua $L$ and $M$. First, we prove that at least one of these continua contains $X$. In fact, if $X$ is disjoint with $L$ or $M$, then $X$ is contained in $M$ or $L$, respectively; so that we can assume, for this part of the proof, that $X$ meets both $L$ and $M$. If $X$ is not contained in $L$, then there exists a point $q \in X \setminus L$ and, by (II), we have $q \in \operatorname*{Ls}_{i\to\infty} A_i$, whence there is an infinite sequence of integers $1 \leqslant k_1 < k_2 < \ldots$ and points $q_i \in A_{k_i}$ $(i = 1, 2, \ldots)$ such that

$$q = \lim_{i\to\infty} q_i.$$

Consequently, there exists an integer $i_0$ such that $q_i \notin L$ for $i_0 \leqslant i$. Since each point of the arc $A_i$ $(i = 1, 2, \ldots)$ except the end-point $p_i$ cuts the continuum $U(X, p)$ according to (I), we see that every subcontinuum of $U(X, p)$ which contains both end-points of $A_i$ contains the whole arc $A_i$. If the continuum $L$ contains $p_i$, it also contains $p$ (because of $L \cap X \neq \emptyset$), and thus $A_i \subset L$. Hence, if $L$ contained $p_{k_i}$, it would contain $q_i$ which is not true for $i_0 \leqslant i$, so that $p_{k_i} \notin L$ for $i_0 \leqslant i$. But then $p_{k_i} \in M$ for $i_0 \leqslant i$ and, by the same reason (because of $M \cap X \neq \emptyset$), we get $A_{k_i} \subset M$ for $i_0 \leqslant i$. It follows from (II) that $X \subset M$.

Without loss of generality, we can assume that $X \subset M$, and then, to complete the proof of Lemma 3, three cases are to be distinguished.

Case 1. $L \cap X \neq \emptyset$ and $p \notin L$. Since, by (I), the point $p$ cuts the continuum $U(X, p)$ between each pair of points selected from $X \setminus \{p\}$ and $A_i \setminus \{p\}$, respectively, we conclude that the continuum $L$ is disjoint with each of the arcs $A_i$ $(i = 1, 2, \ldots)$. Thus $L \subset X \subset M$, and $L \cap M = L$ is a continuum.

Case 2. $L \cap X \neq \emptyset$ and $p \in L$. We need to know that $L \cap X$ is a continuum. To show this, let us suppose, on the contrary, that $L \cap X$ is not connected, i.e. we have a decomposition $L \cap X = D_1 \cup D_2$ of $L \cap X$ into two disjoint non-empty closed sets $D_1$ and $D_2$. Since $p \in L \cap X$, we can assume that $p \in D_1$, and let us select a point $d \in D_2$. There exist open sets $G_1$ and $G_2$ in $U(X, p)$ such that $D_j \subset G_j$ for $j = 1, 2$ and $\mathrm{cl}G_1 \cap \mathrm{cl}G_2 = \emptyset$. Thus $d \in G_2$ and $p \notin \mathrm{cl}G_2$. Since $L$ is a continuum, the component $C$ of $L \cap \mathrm{cl}G_2$ which contains $d$ meets the boundary of the set $L \cap \mathrm{cl}G_2$ in $L$ (see [2], p. 172). That boundary, however, is disjoint with both $D_1$ and $D_2$, whence the continuum $C$ meets the set $L \setminus X$ which is contained in the union of the arcs $A_i$. Since $p \notin C$, it follows from (I) that the sets $C \cap A_i$ and $C \cap X$ are mutually disjoint and, at least two of them being non-empty, the decomposition of $C$ into these sets contradicts the fact that $C$ is a continuum (ibidem, p. 173). Consequently, the set $L \cap X$ is a continuum. But $X \subset M$ implies that

$$L \cap M = (L \cap X) \cup \bigcup_{i=1}^{\infty} (L \cap M \cap A_i). \tag{3}$$

Moreover, we have $p \in L \cap A_i$ and each point of $A_i$ except $p_i$ cuts $U(X, p)$ according to (I). We then see that each non-degenerate set $L \cap A_i$ is an arc having $p$ as an end-point. Because $p \in M \cap A_i$, the same is true for $M \cap A_i$, and we conclude that the sets $L \cap M \cap A_i$ are continua $(i = 1, 2, \ldots)$. Since the point $p$ belongs to each of them and to $L \cap X$ as well, it follows from (3) that $L \cap M$ is connected.

Case 3. $L \cap X = \emptyset$. In this case, by (I), there exists a positive integer $k$ such that $L \subset A_k \setminus \{p\}$. Thus $L$ is an arc or a degenerate set.

Since $X \subset M$ and $p \in M$, the set $M \cap A_k$, similarly to what was stated in Case 2, is an arc containing $p$ or it is equal to $\{p\}$. Consequently, the common part $L \cap M = L \cap M \cap A_k$, if non-degenerate, is also a subarc of $A_k$.

LEMMA 4. *If $X$ and $Y$ are non-degenerate continua, $p \in X$ is a point and $f\colon X \to Y$ is a confluent (locally confluent) mapping of $X$ onto $Y$, where $q = f(p)$, then there exist continua $U(X, p)$ and $U(Y, q)$ having all the properties listed in Lemma 2 and a confluent (locally confluent) mapping*

$$f^*\colon\ U(X, p) \to U(Y, q)$$

*of $U(X, p)$ onto $U(Y, q)$ such that $f^*(x) = f(x)$ for $x \in X$, and $f^* | U(X, p) \setminus X$ is a homeomorphism of $U(X, p) \setminus X$ onto $U(Y, q) \setminus Y$.*

Proof. Let $U(X, p)$ be the continuum described in Lemma 2 and consider the upper semi-continuous decomposition of $U(X, p)$ into the sets $f^{-1}(y)$ for $y \in Y$ and the degenerate sets $\{x\}$ for $x \in U(X, p) \setminus X$. Let $U(Y, q)$ be the resulting quotient space and let us denote by $f^*$ the natural projection. We can then treat the space $Y$ as embedded in $U(Y, q)$ in a way such that $f^*(x) = f(x)$ for $x \in X$. Clearly, $f^*$ is a homeomorphism outside $X$, and $U(Y, q)$ is decomposed into $Y$ and the arcs $f^*(A_i)$, which have all the properties indicated in Lemma 2 as applied to $Y$ and $q$ instead of $X$ and $p$, respectively. Moreover, if $f$ is confluent, then so is $f^* | X = f$ as well as each $f^* | A_i$ $(i = 1, 2, \ldots)$, and it follows from Corollary 2 that $f^*$ is confluent. If $f$ is locally confluent, we conclude from Corollary 2 that the restrictions of $f^*$ to the inverses of some closed neighbourhoods of points in $U(Y, q)$ are confluent, which means that $f^*$ is also locally confluent.

Example. We construct a locally confluent mapping $g\colon X \to Y$ of a continuum $X$ onto a continuum $Y$ such that $g$ is not weakly confluent and both $X$ and $Y$ are arcwise connected unicoherent rational curves. Our construction is based upon Example 4.2 of [3]. First, we define a subset $X_0$ of the Euclidean plane to be the set

$$X_0 = \{(x, 2)\colon\ -1 \leqslant x \leqslant 3\} \cup \left\{\left(\sin\frac{\pi}{y-2}, y\right)\colon\ 2 < y \leqslant 3\right\} \cup$$

$$\cup \{(x, 3)\colon\ 0 \leqslant x \leqslant 2\} \cup \{(2, y)\colon\ -1 \leqslant y \leqslant 3\} \cup$$

$$\cup \left\{\left(x, \sin\frac{\pi}{x-2}\right)\colon\ 2 < x \leqslant 3\right\} \cup \{(3, y)\colon\ 0 \leqslant y \leqslant 2\},$$

and let $p = (2, 2)$. Then $X_0$ is an arcwise connected (non-unicoherent) rational curve with $p \in X_0$. Let us take the upper semi-continuous decomposition of $X_0$ whose only non-degenerate elements are the two-point sets $\{(t, 2), (2, t)\}$ for $-1 \leqslant t < 2$. Denote by $Y_0$ the resulting quotient space and let $f\colon X_0 \to Y_0$ be the natural projection of $X_0$ onto $Y_0$. Put $q = f(p)$. It is not difficult to see that $f$ is locally confluent without being weakly confluent. By Lemma 4, we get an extension $g = f^*$ of $f$ which maps $X = U(X_0, p)$ onto $Y = U(Y_0, q)$. Moreover, $g$ is locally confluent and $g(X \setminus X_0) = Y \setminus Y_0$, whence $g^{-1}(B) = f^{-1}(B)$ for $B \subset Y_0$. Thus $g$ is not weakly confluent either. Now, since $U(X_0, p)$ and $U(Y_0, q)$ satisfy all conditions from Lemma 2, we conclude, by Lemma 3, that these continua are unicoherent. It is rather apparent that both of them are also arcwise connected and rational.

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