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On generalized Pascu class of functions

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Abstract. In this paper two new subclasses $M_g(\alpha; h)$ and $R_g(\alpha; h)$ of the class of univalent functions are defined and results connected with the transform under a certain integral operator, inclusion relation and convolution of functions belonging to these classes are established.

Let $E = \{z \in C: |z| < 1\}$ and $H(E)$ be the class of all functions f holomorphic in E. Let A be the subclass of $H(E)$ of functions f with Montel's normalizations $f(0) = 0 = f'(0)-1$. Further let $S = \{f \in A: f \text{ is univalent in } E\}$ and K and S^* denote the well-known subclasses of functions of S which are convex and starlike respectively. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be any two functions in $H(E)$. Then the Hadamard product or the convolution of f and g is given by $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Let $g, G \in H(E)$. Then g is said to be *subordinate to* G (written $g \prec G$) if $g(0) = G(0)$ and there exists a Schwarz function $\omega(z)$ in E such that $g(z) = G(\omega(z))$. In particular, if G is univalent then $g(E) \subset G(E)$.

Initially we state three theorems without proof which we will be using in the sequel:

THEOREM A [2]. *Let $\beta, \gamma \in C$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}(\beta h(z)+\gamma) > 0$, $z \in E$. If $p(z) = 1+p_1 z+\ldots$ is analytic in E, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

The generalized version of Theorem A established by K. S. Padmanabhan and R. Parvatham in [3] is as follows.

THEOREM B [3]. *Let $\beta, \gamma \in C$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}(\beta h(z)+\gamma) > 0$, $z \in E$, and let $q \in H(E)$ with $q(0) = 1$ and $q(z) \prec h(z)$ in E. If $p(z) = 1+p_1 z+\ldots$ is analytic in E, then*

$$p(z)+\frac{zp'(z)}{\beta q(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

THEOREM C [1]. *Let Φ and g be analytic in E with $\Phi(0) = g(0) = 0$ and $\Phi'(0)g'(0) \neq 0$. Suppose for each α ($|\alpha| = 1$) and σ ($|\sigma| = 1$) we have*

$$\left[\Phi*\left(\frac{1+\alpha\sigma z}{1-\sigma z}\right)g\right](z) \neq 0 \quad \text{on } 0<|z|<r \leqslant 1. \tag{1.1}$$

Then for each F in $H(E)$, the image of $|z| < r$ under $\Phi*Fg/\Phi*g$ is a subset of the convex hull of $F(E)$.

Remark 1. If $\Phi \in K$ and $g \in S^*$, then it was shown in [7] that (1.1) is satisfied for all z in E.

Unless stated otherwise, throughout this paper $g(z)$ stands for a function holomorphic in E with $g(0) = 0 = g'(0)-1$ and $h(z)$ stands for a holomorphic convex univalent function in E with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ in E.

First let us define the class $M_g(\alpha; h)$.

DEFINITION 1. Let $M_g(\alpha, h)$ denote the class of all functions $f \in A$ with $(g*f)'(z)(g*f)(z) \neq 0$ in $E-\{0\}$ satisfying

$$\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)} \prec h(z)$$

for $z \in E$ and $\alpha \geqslant 0$.

Remark 2. For $\alpha = 0$, the class $M_g(\alpha; h)$ coincides with the class $S_g(h)$ studied in [8]. For the choice $g(z) = k_a(z) = z/(1-z)^a$ (a real), $M_g(\alpha; h)$ is the same as the class $M_a(\alpha, h)$ in [4] which, in turn, is a generalization of the class of Pascu and Poderu [6].

DEFINITION 2. Let $R_g(\alpha; h)$ denote the class of all functions $f \in A$ with $(g*f)(z)/z \neq 0$ in E such that

$$\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*\Phi)'(z)+(1-\alpha)(g*\Phi)(z)} \prec h(z)$$

for $\Phi \in M_g(\alpha; h)$ and $\alpha \geqslant 0$.

Remark 3. For $\alpha = 0$, the class $R_g(\alpha; h)$ coincides with the class $C_g(h)$ studied in [8]. For the choice $g(z) = k_a(z) = z/(1-z)^a$ (a real), $R_g(\alpha; h)$ is the same as the class $R_a(\alpha; h)$ studied in [4] which, in turn, is a generalization of the class studied by Pascu [5].

THEOREM 1. *We have the following inclusion relation*:

$$M_g(\alpha; h) \subset M_g(0; h) = S_g(h) \quad \text{for } 0 \leqslant \alpha \leqslant 1.$$

Proof. Let $f(z) \in M_g(\alpha; h)$ and $p(z) = z(g*f)'(z)/(g*f)(z)$.

Then

$$\begin{aligned}
&\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)\\
&\qquad=\alpha z(g*f)(z)p'(z)+\alpha z p(z)(g*f)'(z)+(1-\alpha)+(1-\alpha)p(z)(g*f)(z)\\
&\qquad=\big[\alpha z p'(z)+p(z)\big(\alpha p(z)+(1-\alpha)\big)\big](g*f)(z);\\
&\qquad\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)=\big(\alpha p(z)+(1-\alpha)\big)(g*f)(z).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)}\\
&\qquad\qquad=\frac{\alpha z p'(z)+p(z)\big(\alpha p(z)+(1-\alpha)\big)}{\alpha p(z)+(1-\alpha)}=\frac{zp'(z)}{p(z)+(1/\alpha-1)}+p(z).
\end{aligned}$$

Since $f\in M_g(\alpha;\, h)$,

$$\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)}\prec h(z).$$

Thus for $0\leqslant\alpha\leqslant 1$ an application of Theorem A gives $p(z)\prec h(z)$ in E, which implies that $f\in M_g(0;\, h)=S_g(h)$.

Now, we prove that the class $M_g(\alpha;\, h)$ is closed under the transform by an integral operator.

THEOREM 2. *Let $f\in M_g(\alpha,\, h)$. Then, for $0<\alpha\leqslant 1$,*

$$F(z)=\frac{1}{\alpha z^{1/\alpha-1}}\int_0^z t^{1/\alpha-2}f(t)\,dt\in M_g(\alpha;\, h).$$

Proof. Differentiating $F(z)$ with respect to z and simplifying we get

$$\alpha zF'(z)+(1-\alpha)F(z)=f(z).$$

This on convolution with $g(z)$ gives

$$\alpha z(g*F)'(z)+(1-\alpha)(g*F)(z)=(g*f)(z)$$

($\neq 0$ in $E-\{0\}$ by Definition 1), where we used the fact that $g*\big(zF'(z)\big)=z(g*F)'(z)$. Taking logarithmic derivative with respect to z and multiplying by z we get

$$\frac{\alpha z\big(z(g*F)'(z)\big)'(z)+(1-\alpha)z(g*F)'(z)}{\alpha z(g*F)'(z)+(1-\alpha)(g*F)(z)}=\frac{z(g*f)'(z)}{(g*f)(z)}.$$

The member on the right side is subordinate to $h(z)$ since $f\in M_g(\alpha;\, h)\subset M_g(0;\, h)=S_g(h)$ for $0<\alpha\leqslant 1$ by the previous theorem. Hence,

$$\frac{\alpha z\big(z(g*F)'(z)\big)'(z)+(1-\alpha)z(g*F)'(z)}{\alpha z(g*F)'(z)+(1-\alpha)(g*F)(z)} \prec h(z) \quad \text{in } E \text{ for } 0<\alpha\leqslant 1.$$

Also $F(z)=\gamma_\alpha(z)*f(z)$ where

$$\gamma_\alpha(z)=z+\sum_{n=2}^{\infty}\frac{1/\alpha}{(1/\alpha+n-1)}z^n.$$

Since $(g*f)(z)\neq 0$ in $E-\{0\}$ and $\alpha>0$ we have $(g*F)(z)=\gamma_\alpha(z)*(g*f)(z)\neq 0$ and hence $(g*F)'(z)\neq 0$ in $E-\{0\}$. Thus $F\in M_g(\alpha;\, h)$.

THEOREM 3. *Let $\Phi\in K$. Then for every $f\in M_g(\alpha;\, h)$, $\Phi*f\in M_g(\alpha;\, h)$.*

Proof. Let

$$F(z)=\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)}.$$

If $f\in M_g(\alpha;\, h)$ then by Theorem 1, $F(z)\prec h(z)$ in E. Now consider

$$\frac{\alpha z\big(z(g*\Phi*f)'\big)'(z)+(1-\alpha)z(g*\Phi*f)'(z)}{\alpha z(g*\Phi*f)'(z)+(1-\alpha)(g*\Phi*f)(z)}$$

$$=\frac{\alpha z\big(\Phi*z(g*f)'\big)'(z)+(1-\alpha)\big(\Phi*z(g*f)'\big)(z)}{\alpha\big(\Phi*z(g*f)'\big)(z)+(1-\alpha)(\Phi*g*f)(z)}$$

$$=\frac{\Big(\Phi*\big(\alpha z\big(z(g*f)'\big)'+(1-\alpha)z(g*f)'\big)\Big)(z)}{\Big(\Phi*\big(\alpha z(g*f)'+(1-\alpha)(g*f)\big)\Big)(z)}$$

$$=\frac{\left(\Phi*\left[\dfrac{\alpha z\big(z(g*f)'\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)}\right]\big(\alpha z(g*f)'+(1-\alpha)(g*f)\big)\right)(z)}{\Big(\Phi*\big(\alpha z(g*f)'+(1-\alpha)(g*f)\big)\Big)(z)}$$

$$=\frac{(\Phi*FG)(z)}{(\Phi*G)(z)}$$

where $G(z)=\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)$. Now

$$\frac{zG'(z)}{G(z)}=\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z)+(1-\alpha)(g*f)(z)} \prec h(z)$$

since $f\in M_g(\alpha;\, h)$. Or $\mathrm{Re}[zG'(z)/G(z)]>0$, which implies that G is starlike. Further, $F(z)\prec h(z)$ and h is convex univalent. Hence from Remark 1 under Theorem C we see that the image of E under $\Phi*FG/\Phi*G$ lies in the convex hull of $F(E)\subset h(E)$, a convex set. Thus,

$$\frac{\alpha z\big(z(g*\Phi*f)'(z)\big)'+(1-\alpha)z(g*\Phi*f)'(z)}{\alpha z(g*\Phi*f)'(z)+(1-\alpha)(g*\Phi*f)(z)}=\frac{(\Phi*FG)(z)}{(\Phi*G)(z)}\prec h(z),$$

which implies that $\Phi*f\in M_g(\alpha;h)$.

THEOREM 4. *For every convex univalent function* $\Phi\in A$, $M_g(\alpha;h)\subseteq M_{\Phi*g}(\alpha;h)$.

Proof. Let $f\in M_g(\alpha;h)$. Then by the previous theorem $\Phi*f\in M_g(\alpha;h)$. That is,

$$\frac{\alpha z\big(z(g*\Phi*f)'(z)\big)'(z)+(1-\alpha)z(g*\Phi*f)'(z)}{\alpha z(g*\Phi*f)'(z)+(1-\alpha)(g*\Phi*f)(z)}\prec h(z).$$

Or equivalently $f\in M_{\Phi*g}(\alpha;h)$.

Next we prove an inclusion relation and also the fact that the class $R_g(\alpha;h)$ is closed under a certain integral operator.

THEOREM 5. *For* $0\leqslant\alpha\leqslant 1$, $R_g(\alpha;h)\subset R_g(0;h)=C_g(h)$.

Proof. Let $f\in R_g(\alpha;h)$. Then there exists a $\Phi\in M_g(\alpha;h)$ such that

$$\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*\Phi)'(z)+(1-\alpha)(g*\Phi)(z)}\prec h(z).$$

Setting

$$p(z)=\frac{z(g*f)'(z)}{(g*\Phi)(z)}\quad\text{and}\quad q(z)=\frac{z(g*\Phi)'(z)}{(g*\Phi)(z)}$$

we have

$$\begin{aligned}\frac{\alpha z\big(z(g*f)'(z)\big)'(z)+(1-\alpha)z(g*f)'(z)}{\alpha z(g*\Phi)'(z)+(1-\alpha)(g*\Phi)(z)}&\\ &=\frac{\alpha zp'(z)+p(z)\left(\dfrac{\alpha z(g*\Phi)'(z)}{(g*\Phi)(z)}+(1-\alpha)\right)}{\dfrac{\alpha z(g*\Phi)'(z)}{(g*\Phi)(z)}+(1-\alpha)}\\ &=p(z)+\frac{zp'(z)}{\dfrac{z(g*\Phi)'(z)}{(g*\Phi)(z)}+(1/\alpha-1)}\\ &=p(z)+\frac{zp'(z)}{q(z)+(1/\alpha-1)}\prec h(z),\end{aligned}$$

since $f\in R_g(\alpha;h)$. Here $q(z)\prec h(z)$ by Theorem 1. Since $0\leqslant\alpha\leqslant 1$, an application of Theorem B gives $p(z)\prec h(z)$, thereby establishing the theorem.

THEOREM 6.

$$F(z) = \frac{1}{\alpha z^{1/\alpha-1}} \int_0^z t^{1/\alpha-2} f(t)\,dt \in R_g(\alpha;\, h)$$

whenever $f \in R_g(\alpha;\, h)$ *for* $0 < \alpha \leqslant 1$.

Proof. On differentiating F with respect to z we have

$$\alpha z F'(z) + (1-\alpha) F(z) = f(z).$$

This on convolution with $g(z)$ gives

$$\alpha z (g * F)'(z) + (1-\alpha)(g * F)(z) = (g * f)(z),$$

where we used the fact that $g * (zF'(z)) = z(g * F)'(z)$. Again differentiating with respect to z and multiplying by z we get

$$\alpha z \big(z(g * F)'(z)\big)'(z) + (1-\alpha) z (g * F)'(z) = z(g * f)'(z).$$

Since $f \in R_g(\alpha;\, h)$ there exists a $\varphi \in M_g(\alpha;\, h)$ such that

$$\frac{\alpha z \big(z(g * f)'(z)\big)'(z) + (1-\alpha) z (g * f)'(z)}{\alpha z (g * \varphi)'(z) + (1-\alpha)(g * \varphi)(z)} \prec h(z) \quad \text{in } E.$$

By Theorem 2, Φ defined by

$$\Phi(z) = \frac{1}{\alpha z^{1/\alpha-1}} \int_0^z t^{1/\alpha-2} \varphi(t)\,dt \in M_g(\alpha;\, h)$$

for $0 < \alpha \leqslant 1$ whenever $\varphi \in M_g(\alpha;\, h)$. Differentiating $\Phi(z)$ with respect to z, and convoluting the result with $g(z)$ we have

$$\alpha z (g * \Phi)'(z) + (1-\alpha)(g * \Phi)(z) = (g * \varphi)(z) \neq 0 \quad \text{in } E - \{0\}.$$

Thus

$$\frac{\alpha z \big(z(g * F)'(z)\big)'(z) + (1-\alpha) z (g * F)'(z)}{\alpha z (g * \Phi)'(z) + (1-\alpha)(g * \Phi)(z)} = \frac{z(g * f)'(z)}{(g * \varphi)(z)}.$$

Since $f \in R_g(\alpha;\, h)$ and $R_g(\alpha;\, h) \subset R_g(0;\, h)$ (by Theorem 5) we have

$$\frac{z(g * f)'(z)}{(g * \varphi)(z)} \prec h(z) \quad \text{for } z \in E,\ 0 \leqslant \alpha \leqslant 1.$$

Hence

$$\frac{\alpha z \big(z(g * F)'(z)\big)'(z) + (1-\alpha) z (g * F)'(z)}{\alpha z (g * \Phi)'(z) + (1-\alpha)(g * \Phi)(z)} \prec h(z) \quad \text{in } E$$

for $0 \leqslant \alpha \leqslant 1$.

In the same way as in Theorem 2 we can show that $(g*F)(z) \neq 0$ and $(g*F)'(z) \neq 0$ in $E-\{0\}$ from the fact that $(g*f)(z) \neq 0$ and $(g*f)'(z) \neq 0$ in $E-\{0\}$ for $\alpha > 0$. Thus we get $F \in R_g(\alpha; h)$.

THEOREM 7. *Let $f \in R_g(\alpha; h)$ with respect to a function $\Psi \in M_g(\alpha; h)$. Then for every convex univalent function $\Phi \in A$, $\Phi * f \in R_g(\alpha; h)$ with respect to $\Phi * \Psi \in M_g(\alpha; h)$.*

Proof. From Theorem 3, it follows that $\Phi * \Psi \in M_g(\alpha; h)$ whenever $\Psi \in M_g(\alpha; h)$. Let

$$F(z) = \frac{\alpha z\big(z(g*f)'(z)\big)'(z) + (1-\alpha)z(g*f)'(z)}{\alpha z(g*\Psi)'(z) + (1-\alpha)(g*\Psi)(z)}.$$

Since $f \in R_g(\alpha; h)$ with respect to $\Psi \in M_g(\alpha; h)$ we have $F(z) \prec h(z)$ in E. Now,

$$\frac{\alpha z\big(z(g*\Phi*f)'(z)\big)'(z) + (1-\alpha)z(g*\Phi*f)'(z)}{\alpha z(g*\Phi*\Psi)'(z) + (1-\alpha)(g*\Phi*\Psi)(z)}$$

$$= \frac{\alpha z\big(\Phi*z(g*f)'(z)\big)'(z) + (1-\alpha)\big(\Phi*z(g*f)'(z)\big)(z)}{\alpha\big(\Phi*z(g*\Psi)'(z)\big)(z) + (1-\alpha)\big(\Phi*(g*\Psi)\big)(z)}$$

$$= \frac{\left(\Phi*\left[\dfrac{\alpha z\big(z(g*f)'(z)\big)'(z) + (1-\alpha)z(g*f)'(z)}{\alpha z(g*\Psi)'(z) + (1-\alpha)(g*\Psi)(z)}\right]\big(\alpha z(g*\Psi)' + (1-\alpha)(g*\Psi)\big)\right)(z)}{\Phi*\big(\alpha z(g*\Psi)'(z) + (1-\alpha)(g*\Psi)(z)\big)}$$

$$= \frac{(\Phi*FG)(z)}{(\Phi*G)(z)},$$

where $G(z) = \alpha z(g*\Psi)'(z) + (1-\alpha)(g*\Psi)(z)$. Now

$$\frac{zG'(z)}{G(z)} = \frac{\alpha z\big(z(g*\Psi)'(z)\big)'(z) + (1-\alpha)z(g*\Psi)'(z)}{\alpha z(g*\Psi)'(z) + (1-\alpha)(g*\Psi)(z)} \prec h(z)$$

since $\Psi \in M_g(\alpha; h)$. From this we have $\operatorname{Re}[zG'(z)/G(z)] > 0$, which implies that G is starlike. Also $F(z) \prec h(z)$ and h is a convex univalent function. Thus from Remark 1 we get $(\Phi*FG)(z)/(\Phi*G)(z) \prec h(z)$. Thus $\Phi*f \in R_g(\alpha; h)$.

THEOREM 8. *$R_g(\alpha; h) \subseteq R_{\Phi*g}(\alpha; h)$ for every convex univalent function $\Phi \in A$.*

The proof of Theorem 8 runs along the same lines as that of Theorem 4 and hence is omitted.

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