COLLOQUIUM MATHEMATICUM

VOL. XXII 1970 FASC. 1

A CHARACTERIZATION OF SEMILATTICES

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Dedicated to Professor Edward Marczewski on the 40th Anniversary of the publication of His first paper

Let $\mathfrak{A} = \langle A; \cdot \rangle$ be a semilattice (2), that is $\cdot$ is an idempotent, commutative, and associative binary operation. If $|A| > 1$, then for every $n \geqslant 2$ there is exactly one n-ary polynomial depending on all n variables, namely $x_0 \cdot x_1 \cdot \ldots \cdot x_{n-1}$. Obviously, the only unary polynomial is e_0^1 (that is, the polynomial $p(x) = x$).

For an algebra $\mathfrak{A}$ and $n \geqslant 2$ let $p_n(\mathfrak{A})$ denote the number of n-ary polynomials depending on all n variables (called *essentially n-ary polynomials*). Let $p_1(\mathfrak{A})$ denote the number of non-constant unary polynomials other than e_0^1, and let $p_0(\mathfrak{A})$ be the number of constant unary polynomials. A sequence $\langle p_0, \ldots, p_n, \ldots \rangle$ is called *representable* (3) if there exists an algebra $\mathfrak{A}$ with $p_n = p_n(\mathfrak{A})$ for all n.

Using these concepts we can say that a semilattice (with more than one element) represents the sequence $\langle 0, 0, 1, \ldots, 1, \ldots \rangle$. In this note we prove that this is the only bounded representable sequence that starts with two zeros and has no more zeros.

Since every algebra representing $\langle 0, 0, 1, \ldots, 1, \ldots \rangle$ is equivalent to a semilattice, the above property characterizes semilattices.

The theorem we prove is somewhat stronger:

THEOREM. *Let $\mathfrak{A}$ be an algebra satisfying $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0$. If $1 \leqslant p_2(\mathfrak{A})$, and for some integer k, $1 \leqslant p_n(\mathfrak{A}) \leqslant k$ for infinitely many n, then $p_n(\mathfrak{A}) = 1$ for all $n \geqslant 2$, and $\mathfrak{A}$ is equivalent to a semilattice.*

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(1) This research of both authors was supported by the National Research Council of Canada.

(2) We use the notation of [2]; for all undefined concepts see [2].

(3) See [3] and [4], of which the present paper is a continuation. It is explained in [4] how the problem of representability is connected with Problem 42 of [2].

COROLLARY. *The bounded sequence* $\langle 0, 0, p_2, p_3, \ldots\rangle$, $p_n \neq 0, n \geqslant 2$, *is representable if and only if* $p_n = 1$ *for all* $n > 1$.

The proof of the Theorem is based on a recent result of Dudek [1]:

Let $\mathfrak{A} = \langle A; \cdot\rangle$, *where* $\cdot$ *is a binary idempotent operation. Then either* $\mathfrak{A}$ *is a semilattice, or* $\mathfrak{A}$ *is a diagonal algebra (that is* $x^2 = x$ *and* $(xy)z = x(yz) = xz$), *or* $p_n(\mathfrak{A}) \geqslant n$ *for all* $n > 2$.

Henceforth let $\mathfrak{A}$ be an algebra satisfying the conditions of the Theorem. Since $1 \leqslant p_2(\mathfrak{A})$, there is a binary polynomial $\cdot$. By Dudek's theorem $\cdot$ is either a semilattice or diagonal. The following lemmas prove that $\cdot$ cannot be diagonal.

Call an n-ary polynomial $p(x_0, \ldots, x_{n-1})$ of $\mathfrak{A}$ *transitive* if for $i, j < n$, $i \neq j$, there exists a permutation a of $\{0, \ldots, n-1\}$ such that $ia = j$ and

$$(*) \qquad p(x_0, \ldots, x_{n-1}) = p(x_{0a}, \ldots, x_{(n-1)a}).$$

LEMMA 1. *Let* $1 \leqslant p_n(\mathfrak{A}) < n$, *and let* $p(x_0, \ldots, x_{n-1})$ *be an essentially* n-*ary polynomial of* $\mathfrak{A}$. *Then* p *is transitive.*

Proof. Assume that p is non-transitive. Then $\{0, \ldots, n-1\} = X_0 \cup \ldots \cup X_{t-1}$, where $t > 1$, and $i, j \in X_a$ for some $a < t$ if $ia = j$ for some permutation a and $(*)$ holds. If $|X_0| = k_0$, we can get at least $\binom{n}{k_0}$ different polynomials from p by permuting the variables, contradicting $p_n < n$.

LEMMA 2. *Let* $p(x_0, \ldots, x_{n-1})$ *be a transitive essentially* n-*ary polynomial and* $\cdot$ *an essentially binary diagonal polynomial. Then* $p_0 = p \cdot x_0, \ldots, p_{n-1} = p \cdot x_{n-1}$ *are all distinct essentially* n-*ary polynomials.*

Proof. Consider $p(x_0, \ldots, x_{n-1}) \cdot x_n = f$; f depends on x_n, otherwise after setting $x_0 = \ldots = x_{n-1}$ we would get that $x_0 \cdot x_n$ does not depend on x_n. Since p is transitive, f depends on all $x_0, \ldots, x_{n-1}$ or on none of them. In the latter case we again set $x_0 = \ldots = x_{n-1}$, and get the contradiction that $x_0 x_n$ does not depend on x_0. Thus f is essentially $(n+1)$-ary.

p_0 depends on x_0, otherwise $up_0 = ux_0$ does not depend on x_0. If p_0 does not depend on, say, x_1, then $p_0 x_n$ does not depend on x_1 either, contradicting $p_0 x_n = f$, and that f is essentially $(n+1)$-ary. Thus all p_i are essentially n-ary.

If, say, $p_0 = p_1$, then $up_0 = up_1$, that is $ux_0 = ux_1$; this contradiction completes the proof of Lemma 2.

Now if $\cdot$ is diagonal, choose an n such that $1 \leqslant p_n < n$. Let p be an essentially n-ary polynomial. Then, by Lemma 1, p is transitive. By Lemma 2 there are at least n essentially n-ary polynomials, a contradiction.

So we can assume that $\cdot$ is a semilattice operation. This case will be handled based on the following result:

LEMMA 3. *Let $p(x_0, \ldots, x_{n-1})$ be an essentially n-ary polynomial of $\mathfrak{A}$ which is not a polynomial of $\langle A; \cdot\rangle$. Let $m \geqslant 3n$. Then*

$$f(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}) = p(x_0, \ldots, x_{n-1})\cdot y_0 \cdot \ldots \cdot y_{m-1}$$

is an essentially $(n+m)$-ary polynomial and f is not transitive.

Proof. Obviously, f depends on $y_0, \ldots, y_{m-1}$ since substituting $x = x_0 = \ldots = x_{n-1}$ we get $x \cdot y_0 \cdot \ldots \cdot y_{m-1}$ which depends on $y_0, \ldots, y_{m-1}$. To show that f depends on $x_0, \ldots, x_{n-1}$, say x_0, take a substitution $a = p(a_0, \ldots, a_{n-1}) \neq p(a_0', a_2, \ldots, a_{n-1}) = b$, showing that p depends on x_0; let, say, $a \neq ab$ and set $y_0 = \ldots = y_{m-1} = a$. Then $f(a_0, \ldots, a_{n-1}, a, a, \ldots) = aa = a$ and $f(a_0', a_2, \ldots, a_{n-1}, a, a, \ldots) = ba$, $a \neq ba$, showing that f depends on x_0.

Now assume that f is transitive. Then there is a permutation α_i of $\{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}\}$ such that $f(x_0, \ldots, y_0, \ldots) = f(x_0\alpha_i, \ldots, y_0\alpha_i, \ldots)$ and $x_i\alpha_i = y_0$. Since any permutation of $\{y_0, \ldots, y_{m-1}\}$ leaves f unchanged, we can assume that $\{x_0, \ldots, x_{n-1}\}\alpha_i \subseteq \{y_0, \ldots, y_{n-1}\}$, and $\{x_0, \ldots, x_{n-1}\}\alpha_i^{-1} \subseteq \{y_0, \ldots, y_{2n-1}\}$.

First we apply α_0 to f; this takes x_0 to y_0; then we permute the variables outside of p to take all the x_i outside of p into $\{y_{2n}, \ldots, y_{3n-1}\}$. Then we apply α_1 to take x_1 to y_0, unless x_1 is already out, and move x_1 to $\{y_{2n}, \ldots, y_{3n-1}\}$ while keeping all the x_i in this set fixed. In n steps every variable within p will be a y_i and $\{x_0, \ldots, x_{n-1}\}$ takes the place of $\{y_{2n}, \ldots, y_{3n-1}\}$.

Substitute $y_0 = \ldots = y_{m-1} = y$. Then we get

$$p(x_0, \ldots, x_{n-1})y \ldots y = p(y, \ldots, y)y \ldots yx_0 \ldots x_{n-1},$$

that is

$$p(x_0, \ldots, x_{n-1})y = y \cdot x_0 \ldots x_{n-1},$$

and so

$$\begin{aligned} p(x_0, \ldots, x_{n-1}) &= p(x_0, \ldots, x_{n-1})\cdot p(x_0, \ldots, x_{n-1}) \\ &= p(x_0, \ldots, x_{n-1})\cdot x_0 \cdot \ldots \cdot x_{n-1} = x_0 \cdot \ldots \cdot x_{n-1} \cdot x_0 \cdot \ldots \cdot x_{n-1} \\ &= x_0 \cdot \ldots \cdot x_{n-1}, \end{aligned}$$

contradicting the assumption that p is not a semilattice polynomial. This completes the proof of the lemma.

To complete the proof of the Theorem let $\cdot$ be a semilattice operation, and let $\mathfrak{A}$ be not equivalent to a semilattice. Then there exists an integer n, and an essentially n-ary polynomial $p(x_0, \ldots, x_{n-1})$ which is not a semilattice polynomial. Choose a $t > n$ with $p_t < k$, $t > k$, and $t - n \geqslant 2n$. Then $f(x_0, \ldots, x_{t-1}) = p(x_0, \ldots, x_{n-1})x_n \ldots x_{t-1}$ is by Lemma 3 a non-transitive polynomial, hence $p_t \geqslant t$ by Lemma 1, contradicting $p_t < k < t$. This completes the proof of the Theorem.

The proof shows that we verified the conclusion of the Theorem under somewhat milder conditions than the ones given. Namely, it suffices to assume that $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0, 1 \leqslant p_2(\mathfrak{A})$, there exists an $m > 2$ with $1 \leqslant p_m(\mathfrak{A}) < m$, and $p_n(\mathfrak{A}) \leqslant n$ for infinitely many n.

This statement would be obvious with "$p_n(\mathfrak{A}) < n$" in the last clause. We can write "$p_n(\mathfrak{A}) \leqslant n$" because in fact Lemma 3 gives $n+m+1$ essentially $(n+m)$-ary polynomials. Indeed, we get $n+m$ from Lemma 1, and we have one more: $x_0 \ldots x_{n-1} \cdot y_0 \ldots y_{m-1}$. Since this is essentially $(n+m)$-ary, it suffices to see that $p(x_0, \ldots, x_{n-1}) y_0 \ldots y_{m-1} \neq x_0 \ldots x_{n-1} \cdot y_0 \ldots y_{m-1}$. If they were equal, then $p(x_0, \ldots, x_{n-1}) y_0 \ldots y_{m-1}$ would be transitive, contradicting Lemma 3.

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Reçu par la Rédaction le 18. 5. 1969