COLLOQUIUM MATHEMATICUM

VOL. XLIX 1984 FASC. 1

ON THE THIRD ITERATES OF THE φ- AND σ-FUNCTIONS

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1. Introduction. Let σ be the sum-of-divisors-function, and φ the Euler function. Put

$$\sigma_1(n) = \sigma(n), \quad \varphi_1(n) = \varphi(n)$$

and for $k > 1$

$$\sigma_k(n) = \sigma_1(\sigma_{k-1}(n)), \quad \varphi_k(n) = \varphi_1(\varphi_{k-1}(n)).$$

Schinzel [6] conjectured that for every k

$$\liminf_{n\to\infty} \frac{\sigma_k(n)}{n} < \infty. \tag{1}$$

The conjecture (1) is known to be true for $k = 1, 2$ (see [5] and [6]). Mąkowski [4] showed that (1) is true if another conjecture of Schinzel is true, namely the conjecture H on the existence of an infinity of certain prime k-tuples (see [7]).

The analogous relation for φ_k:

$$\limsup_{n\to\infty} \frac{\varphi_k(n)}{n} > 0$$

is trivial because $\varphi_k(2^m) = 2^{m-k}$. These differences in difficulty seem to vanish if we ask for quantitative results. Denote by $N_\sigma(k, \alpha, x)$ the number of integers $n \leqslant x$ for which $\sigma_k(n) < \alpha n$, and by $N_\varphi(k, \alpha, x)$ the number of integers $n \leqslant x$ for which $\varphi_k(n) > \alpha n$. Erdös [2] found the following results:

For arbitrarily large t and for $x > x_0(t)$ we have

$$N_\sigma(2, 2, x) > \frac{x}{\log x} (\log\log x)^t,$$

and for every $\alpha > 0$, $\varepsilon > 0$, and $x > x_0(\varepsilon, \alpha)$ we obtain

$$N_\sigma(2, \alpha, x) < \frac{x}{\log x} (\log x)^\varepsilon,$$

$$N_\sigma(3, \alpha, x) < \frac{x}{\log^2 x} (\log x)^\varepsilon.$$

For every $\alpha < 1/2$, $\varepsilon > 0$, $t > 0$, and $x > x_0(\alpha, t, \varepsilon)$ we have

$$\frac{x}{\log x}(\log\log x)^t < N_\varphi(2, \alpha, x) < \frac{x}{\log x}(\log x)^\varepsilon;$$

further, for every $\alpha > 0$, $\varepsilon > 0$, and $x > x_0(\alpha, \varepsilon)$ we get

$$N_\varphi(3, \alpha, x) < \frac{x}{\log^2 x}(\log x)^\varepsilon.$$

The purpose of this paper is to obtain lower estimates for $N_\sigma(3, \alpha, x)$ and $N_\varphi(3, \alpha, x)$ for appropriate values of α. This includes an unconditional proof of (1) for $k = 3$.

THEOREM 1. *For $\alpha > \alpha_0$ and $x > x_0(\alpha)$ we have*

$$N_\sigma(3, \alpha, x) > \frac{x}{\log^2 x}; \tag{2}$$

especially,

$$\liminf_{n\to\infty} \frac{\sigma_3(n)}{n} < \infty.$$

THEOREM 2. *For $\alpha < \alpha_1$ and $x > x_1(\alpha)$ we have*

$$N_\varphi(3, \alpha, x) > \frac{x}{\log^2 x}. \tag{3}$$

It is possible to improve (2) and (3) as follows:

$$N_\sigma(3, \alpha, x) > \frac{x}{\log^2 x}(\log\log x)^t \tag{4}$$

for $\alpha > \alpha_0$ and $x > x_0(\alpha, t)$, where t is arbitrarily large; and

$$N_\varphi(3, \alpha, x) > \frac{x}{\log^2 x}(\log\log x)^t \tag{5}$$

for $\alpha < \alpha_1$ and $x > x_1(\alpha, t)$, where t is arbitrarily large.

For the sake of simplicity we prove only (2), the proof of (3) being completely analogous. At the end we indicate the modifications which are necessary to prove (4) and (5).

2. Preliminary lemmas.

Notation. Throughout this paper the letters p, q always denote primes, and $c_1, c_2, \ldots$ denote absolute positive constants.

Put

$$\varrho(n) = \sum_{p|n} \frac{1}{p}, \qquad \lambda(p) = \frac{\sigma(p+1)}{p+1}.$$

LEMMA 1. *We have*

$$\frac{\sigma(n)}{n} \leqslant \exp(c_1 \varrho(n)). \tag{6}$$

Proof. Indeed, we get

$$\frac{\sigma(n)}{n} \leqslant \prod_{p|n} \left(1+\frac{1}{p-1}\right) = \exp\left\{\sum_{p|n} \log\left(1+\frac{1}{p-1}\right)\right\} \leqslant \exp(c_1 \varrho(n)).$$

Put

$$\pi(x, d, l) = \sum_{\substack{p \leqslant x \\ p \equiv l (\mathrm{mod}\, d)}} 1.$$

LEMMA 2 (Brun-Titchmarsh theorem). *We have*

$$\pi(x, d, l) < \frac{3x}{\varphi(d)\log(x/d)} \quad \textit{if } (d, l) = 1, \ 1 \leqslant d < x.$$

For the proof see [3], p. 110, Theorem 3.8.

Put

$$M(\beta, v) = \sum_{\substack{2 \leqslant p < v \\ \lambda(p) \geqslant \beta}} 1, \quad \text{where } \beta > 1, v > 1.$$

LEMMA 3. *We have*

$$M(\beta, v) \leqslant c_2 \frac{v}{\log v \log \beta}. \tag{7}$$

Proof. A simple computation together with the application of Lemma 2 gives

$$\sum_{2 \leqslant p \leqslant v} \varrho(p+1) \leqslant \sum_{2 \leqslant p \leqslant v} \sum_{\substack{q \leqslant v^{1/2} \\ q|p+1}} \frac{1}{q} + \sum_{2 \leqslant p \leqslant v} v^{-1/2}$$

$$\leqslant c_3 \sum_{q \leqslant v^{1/2}} \frac{1}{q} \sum_{\substack{p \leqslant v \\ p \equiv -1 (\mathrm{mod}\, q)}} 1 + v^{1/2} \leqslant c_4 \frac{v}{\log v},$$

which together with (6) implies (7).

LEMMA 4 (Bombieri [1]). *For every $D > 0$ there exists an $E > 0$ such that*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-E}} \max_{y \leqslant x} \max_{(d,l)=1} \left|\pi(y, d, l) - \frac{\mathrm{li}\, y}{\varphi(d)}\right| = O\left(x(\log x)^{-D}\right),$$

where

$$\mathrm{li}\, x = \int_2^x \frac{dt}{\log t}.$$

3. Notation and results from sieve theory. The proof of Theorem 1 is mainly based on sieve results. We adopt the notation, which is used by Halberstam and Richert [3] in their treatment of the Selberg sieve.

Let $\mathfrak{A}$ denote a finite sequence of (not necessarily distinct) integers, and $\mathfrak{P}$ a subset of the set P of primes, especially put $\mathfrak{P}_K = \{p\colon p \nmid K\}$ for every integer K. Let $z \geq 2$ and

$$S(\mathfrak{A}, \mathfrak{P}, z) = \left|\{a \in \mathfrak{A}\colon (a, P(z)) = 1\}\right|, \quad \text{where } P(z) = \prod_{\substack{p<z \\ p \in \mathfrak{P}}} p.$$

Let ω denote a multiplicative function defined for all square-free numbers so that $\omega(p) = 0$ if $p \notin \mathfrak{P}$. Choose an approximation $X > 1$ for $|\mathfrak{A}|$ and define for $\mu(d) \neq 0$

$$R_d = |\{a \in \mathfrak{A}\colon a \equiv 0 \pmod{d}\}| - \frac{\omega(d)}{d} X.$$

Put

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right).$$

We then have

LEMMA 5. *Let ω satisfy the conditions*

$$(\Omega_1) \qquad 0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1},$$

$$(\Omega_2(\varkappa)) \qquad \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} \leq \varkappa \log \frac{z}{w} + A_2 \quad \text{if } 2 \leq w \leq z,$$

where $\varkappa > 0$, $A_1 \geq 1$, $A_2 \geq 1$ are suitable constants, and let $\xi \geq z$. Then

$$S(\mathfrak{A}, \mathfrak{P}, z) = XW(z)\{1 + O_{\varkappa, A_1, A_2}(\exp\{-\tau(\log \tau + 1)\})\} + \theta \sum_{\substack{d < \xi^2 \\ d | P(z)}} 3^{\nu(d)} |R_d|,$$

where

$$\tau = \frac{\log \xi}{\log z}, \quad |\theta| \leq 1, \quad \nu(d) = \sum_{p|d} 1.$$

For the proof see [3], Theorem 7.1, p. 206.

4. Proof of Theorem 1. Let x be a sufficiently large real number. Apply Lemma 5 with

$$\mathfrak{A} = \{p+1\colon 2 < p+1 \leq x\}, \quad \mathfrak{P} = \mathfrak{P}_2, \quad X = \operatorname{li} x,$$

$$\omega(p) = \frac{p}{p-1} \quad (p > 2).$$

We have

$$R_d = \left|\pi(x-1, d, -1) - \frac{\operatorname{li} x}{\varphi(d)}\right|.$$

Using the trivial estimate $|R_d| \leqslant x/d$ we obtain

$$\sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)} |R_d| \leqslant x^{1/2} \Big(\sum_{\substack{d<\xi^2 \\ d|P(z)}} \mu^2(d) \frac{9^{\nu(d)}}{d} \Big)^{1/2} \Big(\sum_{\substack{d<\xi^2 \\ d|P(z)}} |R_d| \Big)^{1/2}.$$

A simple computation gives (see [3], p. 115, Lemma 3.4)

$$\sum_{d<\xi^2} \mu^2(d) \frac{9^{\nu(d)}}{d} \leqslant \sum_{d_1 \ldots d_9 < \xi^2} \frac{\mu(d_1 \ldots d_9)}{d_1 \ldots d_9} \leqslant \Big(\sum_{n<\xi^2} \frac{1}{n} \Big)^9 \leqslant (\log \xi^2 + 1)^9.$$

Lemma 4 implies, if we choose $\xi^2 = x(\log x)^{-E}$, where E is sufficiently large, the inequality

$$\sum_{d<\xi^2} |R_d| \leqslant x(\log x)^{-14}.$$

In total we have

$$\sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)} |R_d| = o\Big(\frac{x}{\log^2 x}\Big).$$

From a well-known theorem of Mertens which states that

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + a + o(1)$$

we get

$$W(z) = \frac{c_5}{\log z} (1+o(1)) \quad (z \to \infty).$$

All these estimates now imply that for arbitrarily large $C > 0$ there exists a $\gamma = \gamma(C) > 0$ such that

$$S(\mathfrak{A}, \mathfrak{P}_2, x^\gamma) > C \frac{x}{\log^2 x}; \tag{8}$$

in the opposite direction Lemma 5 gives

$$S(\mathfrak{A}, \mathfrak{P}_2, x^{1/3}) \leqslant c_6 \frac{x}{\log^2 x}. \tag{9}$$

From (8) and (9) we conclude that there exist a $\gamma > 0$ independent of x and a set $\mathfrak{B} = \mathfrak{B}(x)$ of primes such that

$$|\mathfrak{B}| > \frac{2x}{\log^2 x}, \tag{10}$$

$$p \in \mathfrak{B} \Rightarrow 2 < p+1 \leqslant x, \tag{11a}$$

$$p \in \mathfrak{B} \Rightarrow (p+1,\, P(x^{\gamma})) = 1, \tag{11b}$$

$$p \in \mathfrak{B} \Rightarrow (q \neq 2,\, q \mid p+1 \Rightarrow x^{\gamma} \leqslant q \leqslant x^{1-\gamma}). \tag{11c}$$

We now remove from $\mathfrak{B}$ the subset $\mathfrak{C} = \mathfrak{C}(x)$ consisting of those p for which $p+1$ is divisible by the square of a prime $q \geqslant x^{\gamma}$ or by any prime $q \geqslant x^{\gamma}$ with $\lambda(q) \geqslant \beta$, β to be determined later. We set

$$\mathfrak{B}(q) = \{nq(nq-1)\colon n \leqslant x/q\}.$$

For $\mathfrak{C}$ we have the estimate

$$|\mathfrak{C}| \leqslant \sum_{\substack{x^{\gamma} \leqslant q \leqslant x^{1-\gamma} \\ \lambda(q) \geqslant \beta}} S(\mathfrak{B}(q),\, \mathfrak{P}_2,\, x^{\delta}) + O(x^{1-\gamma}), \quad \text{where } 0 < \delta < \gamma. \tag{12}$$

Once more we apply Lemma 5 with $\mathfrak{A} = \mathfrak{B}(q)$, $\mathfrak{P} = \mathfrak{P}_2$, $\xi^2 = x^{\gamma-\varepsilon}$, $z = x^{\gamma/2}$, and we obtain

$$S(\mathfrak{B}(q),\, \mathfrak{P}_2,\, x^{\delta}) \leqslant c_7 \frac{x}{q \log^2 x}.$$

Using (12) we now get

$$|\mathfrak{C}| \leqslant c_8 \frac{x}{\log^2 x} \sum_{\substack{x^{\gamma} \leqslant q \leqslant x^{1-\gamma} \\ \lambda(q) \geqslant \beta}} \frac{1}{q} + O(x^{1-\gamma}).$$

Applying Lemma 2 in estimating the last sum we have

$$\sum_{\substack{x^{\gamma} \leqslant q \leqslant x^{1-\gamma} \\ \lambda(q) \geqslant \beta}} \frac{1}{q} \leqslant \frac{c_9}{\log \beta}$$

and, consequently,

$$|\mathfrak{C}| \leqslant c_{10} \frac{x}{\log \beta \log^2 x}.$$

For sufficiently large β we obtain

$$|\mathfrak{C}| \leqslant \frac{x}{\log^2 x}. \tag{13}$$

Put $\mathfrak{D} = \mathfrak{B} - \mathfrak{C}$. Then (10) and (13) give

$$|\mathfrak{D}| > \frac{x}{\log^2 x}. \tag{14}$$

The primes $p \in \mathfrak{D}$ satisfy (11a), (11b), and

$$q \neq 2,\ q \mid p+1 \Rightarrow \frac{\sigma(q+1)}{q+1} \leqslant \beta.$$

We now estimate $\sigma_3(p)/p$ for $p \in \mathfrak{D}$. Set $[1/\gamma] = L$. We then have

$$p+1 = 2^r q_1, \ldots, q_s, \quad \text{where } s \leqslant L,$$

and

$$\sigma_3(p) = \sigma_2(p+1) \leqslant 2^{r+1} \prod_{\substack{q_i \mid p+1 \\ q_i \neq 2}} \sigma(q_i+1)$$

$$\leqslant 2^{r+1} \beta^L \prod_{\substack{q_i \mid p+1 \\ q_i \neq 2}} (q_i+1) \leqslant 2(2\beta)^L (p+1).$$

Therefore

$$\frac{\sigma_3(p)}{p} \leqslant 2(2\beta)^L, \tag{15}$$

which together with (14) proves Theorem 1.

5. Improvement of Theorem 1. For proving the sharper estimate (4) we use the following theorem of Erdös ([2], Theorem 2):

For every t the number of integers $m \leqslant y$ for which $\sigma_2(m) < 2m$ is larger than

$$c_{11} \frac{y}{\log y} (\log\log y)^t.$$

Set $\mathfrak{E} = \{m < x^{1/4}\colon \sigma_2(m) < 2m\}$. Instead of sieving the single set $\mathfrak{A} = \{p+1 \leqslant x\}$ we sieve the family of sets

$$\mathfrak{A}'(m) = \left\{\frac{p+1}{m}\colon p \leqslant x,\ p+1 \equiv 0 \ (\mathrm{mod}\, m)\right\}, \qquad m \in \mathfrak{E}.$$

The theorem of Bombieri allows us to control the remainder terms in average ("for almost all" m).

From the sifted sets we again remove the elements which are divisible by any q for which $\lambda(q)$ is large. The resulting estimates yield (4); the estimate (5) is obtained analogously.

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Reçu par la Rédaction le 11. 6. 1980