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Changing measurable into accessible cardinals

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CONTENTS

INTRODUCTION

1. A large part of the content of this thesis is of the following type. We start with a countable transitive model $\mathfrak{M}$ of ZF + AC. Let $\varkappa$ be a cardinal of $\mathfrak{M}$ satisfying certain properties. We construct a suitable extension $\mathfrak{N}$ of $\mathfrak{M}$ by methods of P. Cohen and study how the properties of $\varkappa$ are altered in $\mathfrak{N}$. An example of a property we can ask to be satisfied by $\varkappa$ in $\mathfrak{M}$ is measurability. In many of the constructions certain combinatorial properties play a principal role alongside the methods of P. Cohen, both as the chief tool in the construction and as a subject of the investigation. These properties are close to those studied in [7] from the published literature and still closer to [13]. Many theorems are statements of relative consistency with respect to ZF + AC + "an additional hypothesis" of various conditions concerning uncountable cardinals.

More specifically, our main results are as follows. Let combinatorial property $(\varkappa, \overline{R(\omega_1)})$ (Definition 1.24) hold in $\mathfrak{M}$. Then there is a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{M}$, $\mathfrak{N}$ have the same cardinals and $cf(\varkappa) = \omega_0$ in $\mathfrak{N}$. In particular, the construction can be carried out, if $\varkappa$ is measurable, or if there is an ω_1-saturated $\varkappa$-complete ideal in $\mathscr{P}(\varkappa)$. This answers a question raised by J. Silver and R. Solovay, whether the cofinality of a cardinal can be changed by a Cohen extension preserving cardinalities. The singular cardinal $\varkappa$ obtained in this way has many interesting properties in $\mathfrak{N}$. We mention here the following. There is a tree of cardinality $\varkappa^+$ in $\mathfrak{N}$ such that every branch and every level have cardinality $\leqslant \varkappa$ in $\mathfrak{N}$. This is the first example of a singular cardinal with this property. Actually, the method gives the relative consistency of the existence of such a tree for the pair $\aleph_\omega$, $\aleph_{\omega+1}$. The problem, whether such a tree exists for every singular cardinal $\varkappa$ was stated in [25] and again in [8]. Another property of $\varkappa$ is that there is no Jónsson algebra of cardinality $\varkappa$ in $\mathfrak{N}$. We give still another example of such a singular cardinal in Theorem 1.29. These are the only known examples of singular cardinals for which there is no Jónsson algebra of that cardinality. (For successor cardinals the problem of the existence of Jónsson algebras is solved in [6], assuming GCH. Assuming $V = L$ the problem is solved completely in [9].) Indeed, we prove a relative consistency of the existence of some $\varkappa$ such that

$cf(\varkappa) = \omega_0$ and, for every $\lambda < \varkappa$ and every $f\colon [\varkappa]^{<\omega_0} \to \lambda$, there is a set $U \subseteq \varkappa$ for which $|U| = \varkappa$ and $|f^*([U]^{<\omega_0})| \leqslant \aleph_0$. This does not exhaust the list of interesting properties which we prove that $\varkappa$ satisfies. Another application of this construction of a Cohen extension is to the theory of complete Boolean algebras. If $\varkappa$ is a measurable cardinal or if $\varkappa = \lim_{n\in\omega_0} \varkappa_n$, $\varkappa_n$ measurable, then there is a complete Boolean algebra $\mathscr{B}$ such that the distributive law $(\lambda, 2)$ holds in $\mathscr{B}$ for every $\lambda < \varkappa$ and the $(\aleph_0, \varkappa)$ distributive law fails in $\mathscr{B}$. This is proved at the end of Chapter 2. This type of problem was investigated in [22] and [14]. Our result represents a strengthening of the results of [14] for special cardinals.

In Theorems 1.39, 1.44, we deduce the existence of nonregular ultrafilters in $\mathscr{P}(\varkappa)$ from even weaker properties than those used for changing cofinalities of cardinals. The problem whether every ultrafilter is regular was raised in [10a].

In Chapter 3 we answer some questions of J. Keisler and A. Tarski [10]. In particular, we prove that if it is consistent that there is a measurable cardinal, then it is consistent that there is an ω_1-saturated $2^{\aleph_0}$ -complete ideal in $\mathscr{P}(2^{\aleph_0})$. It is also proved that the existence of such an ideal does not imply the existence of a measure on $\mathscr{P}(2^{\aleph_0})$, answering a question of A. Tarski. W. Sierpiński [16] studies the problem whether every set of real numbers of second category can be decomposed into uncountably many disjoint such sets. Some results concerning this are obtained in Chapter 3. This problem is closely tied up with the problem of the existence of saturated ideals.

2. We define some concepts used frequently in the text. Let A be a set. $|A|$ denotes the cardinality of A. Cardinals are initial ordinals and an ordinal is the set of smaller ordinals. Letters $\varkappa$, λ, ν will always denote cardinals. $\varkappa^+$ denotes the least cardinal greater than $\varkappa$. If $\mathscr{I} \subseteq \mathscr{P}(A)$, $\mathscr{I} \neq 0$ and (1) $A \notin \mathscr{I}$, (2) $(\forall X, Y)(X, Y \in \mathscr{I} \to X \cup Y \in \mathscr{I})$, (3) $(\forall X, Y)$ $(X \in \mathscr{I} \;\&\; Y \subset X \to Y \in \mathscr{I})$, then $\mathscr{I}$ is said to be an *ideal* in $\mathscr{P}(A)$. Let $\mathscr{I}$ be an ideal in $\mathscr{P}(A)$. $\mathscr{I}$ is *non-principal* if $\bigcup \mathscr{I} \notin \mathscr{I}$. $\mathscr{I}$ is *maximal* if, for every $X \in \mathscr{P}(A)$, $X \in \mathscr{I}$ or $A - X \in \mathscr{I}$. $\mathscr{I}$ is *$\varkappa$-complete* (abbrev. $\varkappa$-comp.) if, for every $S \subset \mathscr{I}$ such that $|S| < \varkappa$, $\bigcup S \in \mathscr{I}$. $\mathscr{I}$ is *$\varkappa$-saturated* ($\varkappa$-sat.) if, for every disjointed $S \subseteq \mathscr{P}(\varkappa) - \mathscr{I}$, $|S| < \varkappa$; S is said to be *disjointed*, if for all $X, Y \in S$, $X \neq Y$ implies $X \cap Y = 0$. Φ is said to be a *filter* in $\mathscr{P}(A)$, if $\Phi \subseteq \mathscr{P}(A)$ and $\{X \mid A - X \in \Phi\} = \mathscr{I}$ is an ideal in $\mathscr{P}(A)$. If Φ and $\mathscr{I}$ are as above, then Φ or $\mathscr{I}$ is said to be dual to $\mathscr{I}$ or Φ, respectively. Let Φ be a filter in $\mathscr{P}(A)$. Φ is an *ultrafilter* if its dual ideal is maximal, Φ is *uniform* if, for every $X \in \Phi$, $|X| = |A|$. Φ is $\varkappa$-complete if its dual ideal is $\varkappa$-complete. From now on, by "an ideal" we shall always mean a non-principal ideal containing all finite sets, unless specified otherwise.

Let μ be a real-valued function defined on $\mathscr{P}(A)$ such that for every $X \in \mathscr{P}(A)$, $0 \leqslant \mu(X) \leqslant 1$, $\mu(A) = 1$, $\mu(\{x\}) = 0$ for every $x \in A$, and, for every sequence $\{X_i\}_{i \in \omega_0}$ of disjoint elements of $\mathscr{P}(A)$, $\mu(\bigcup_{i \in \omega_0} X_i) = \sum_{i=0}^{\infty} \mu(X_i)$. Then μ is called a *measure* on $\mathscr{P}(A)$. Let μ be a measure on $\mathscr{P}(A)$. μ is *2-valued* if, for every $X \in \mathscr{P}(A)$, $\mu(X) = 0$ or $\mu(X) = 1$. Let $\mathscr{I}_\mu = \{X \mid \mu(X) = 0\}$. μ is said to be *$\varkappa$-additive* ($\varkappa$-addit.) if $\mathscr{I}_\mu$ is $\varkappa$-comp. The following statements are well known or obvious from the definitions. $\mathscr{I}_\mu$ is ω_1-comp. and ω_1-sat. μ is 2-valued iff $\mathscr{I}_\mu$ is maximal.

If s, r are sequences of elements of A and $x \in A$, then $\widehat{s\,x}$, $\widehat{s\,r}$ will denote the sequences obtained by letting x follow all terms of s, or by letting the terms of r follow the terms of s, respectively. The order of s and r is left unchanged. If $n \in \omega_0$, then $s|n$ denotes the initial segment of s of length n. $[A]^n$ is the family of subsets of A of cardinality n.

$$[A]^{<\omega_0} \overset{\text{def.}}{=} \bigcup_{n \in \omega_0} [A]^n.$$

There is an obvious one-to-one correspondence between elements of $[\varkappa]^{<\omega_0}$ and finite increasing sequences of elements of $\varkappa$ for every cardinal $\varkappa$. Because of this we shall allow ourselves to treat finite increasing sequences of elements of $\varkappa$ as finite subsets of $\varkappa$ and vice versa.

If f is a function, then $f^*(A)$ is the image of A under f. Other commonly used notation will be adopted without any explanation.

Remark. Professor Tarski pointed out that if the axiom of choice holds, then our definition of *a* $\varkappa$-addit. measure is equivalent to the following: measure μ on $\mathscr{P}(A)$ is $\varkappa$-addit. if for every disjointed family $S \subseteq \mathscr{P}(A)$ such that $|S| < \varkappa$, $\mu(\bigcup S) = \sum_{X \in S} \mu(X)$. Here

$$\sum_{X \in S} \mu(X) = \sup\{\sum_{X \in H} \mu(X) \mid H \in [S]^{<\omega_0}\}.$$

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CHAPTER 1

In this chapter we introduce various combinatorial concepts. Almost all facts appearing here serve as lemmas needed for the development in Chapters 2 and 3. However, because of their common combinatorial character, they are collected in Chapter 1, rather than being proved in connection with a particular result to which they pertain most. Many of these facts were previously known and are attributed to their original author. If they are only simple modifications of known results, the relationship is pointed out. There are however a few theorems that are new and of independent interest. We mention particularly Theorems 1.29, 1.39 and 1.44. We also introduce a property $(\varkappa, \underline{R(\omega_1)})$ (or more generally $(\varkappa, \underline{R(\nu)})$—see Definition 1.24—which has not been previously formulated explicitly and has many interesting applications. Both measurable cardinals and cardinals $\varkappa$ such that there is an ω_1-saturated $\varkappa$-complete ideal in $\mathscr{P}(\varkappa)$ satisfy $(\varkappa, \underline{R(\omega_1)})$. However, it follows from our results that the above-mentioned cardinals are not the only examples of cardinals $\varkappa$ such that $(\varkappa, \underline{R(\omega_1)})$ holds. (Compare Theorem 1.29 and also Theorems 2.28 and 2.29.) If $(\varkappa, \underline{R(\omega_1)})$ holds, then $\varkappa$ has to be a regular limit cardinal or $cf(\varkappa) = \omega_0$. (Compare Proposition 1.23.) These are the only known restrictions. A few other useful properties are introduced and no uncountable cardinal is known for which they fail. They of course apply to a great variety of so called large cardinals, even to some singular ones. See for example Definition 1.42 and Theorem 2.26. These new properties are applied at the end of Chapter 1 to study the problem of the existence of non-regular ultrafilters raised by J. Keisler [10a]. See Theorems 1.39 and 1.44.

An acquaintance with Section 2 of the introduction is necessary for reading Chapter 1. Some conventions were made there, which are not in common usage. Thus, for example, by a measure we mean a rather non-trivial case of what is commonly accepted to be a measure.

1. DEFINITION 1.1. A cardinal $\varkappa$ is called *measurable* if there is a $\varkappa$-addit., 2-valued measure on $\mathscr{P}(\varkappa)$.

DEFINITION 1.2. Let μ be a measure on $\mathscr{P}(\varkappa)$. μ is *normal* if μ is $\varkappa$-addit., 2-valued, and for every function $f \in \varkappa^{\varkappa}$ with the property $f(\xi) < \xi$ for all $\xi \in \varkappa$, $\xi > 0$, there is a set $U \in \mathscr{P}(\varkappa)$, $\mu(U) = 1$, such that $|f^*(U)| = 1$.

THEOREM 1.3. (D. Scott, see [10], Theorem 1.8, p. 241). *Let $\varkappa$ be a measurable cardinal. Then there is a normal measure on $\mathscr{P}(\varkappa)$.*

LEMMA 1.4. *Let $0 \neq S \subseteq [\varkappa]^{<\omega_0}$, μ be a normal measure on $\mathscr{P}(\varkappa)$, and $\mu(U_s) = 1$ for every $s \in S$. Let $U = \{\xi | \xi \in \bigcap_{\max s < \xi} U_s\}$. Then $\mu(U) = 1$.*

Proof. $\varkappa - U = \{\xi | (\exists s \in S)(\max s < \xi \ \&\ \xi \notin U_s)\}$. Put

$$g(\xi) = \min\{\eta | (\exists s \in S)(\max s = \eta \ \&\ \xi \notin U_s)\} \qquad \text{for} \qquad \xi \in \varkappa - U,$$

and $g(\xi) = 0$ for $\xi \in U$. It is obvious that $g(\xi) < \xi$ for all ξ such that $0 < \xi < \varkappa$. By the normality of μ, there is a set $V \subseteq \varkappa$, $\mu(V) = 1$ and an ordinal η_0 such that $g(\xi) = \eta_0$ for all $\xi \in V$. If $\mu(\varkappa - U) = 1$, then $\mu\big((\varkappa - U) \cap V\big) = 1$ and for every $\xi \in (\varkappa - U) \cap V$ there is an $s \in S$, $\max s = \eta_0$ and $\xi \notin U_s$. Since $|\{s | s \in S \ \&\ \max s = \eta_0\}| < \varkappa$, there is a set $W \subseteq (\varkappa - U) \cap$ $\cap V$, $\mu(W) = 1$, and an $s_0 \in S$, $\max s_0 = \eta_0$, such that $\xi \notin U_{s_0}$ for all $\xi \in W$. This contradicts $\mu(U_{s_0}) = 1$. Hence $\mu(\varkappa - U) = 0$ and $\mu(U) = 1$.

Remark 1.5. Lemma 1.4 is well known. It has the following special case. Let μ be as in Lemma 1.4 and $\mu(U_\xi) = 1$ for all $\xi \in \varkappa$. Then

$$\mu(\{\eta | \eta \in \bigcap_{\xi < \eta} U_\xi\}) = 1.$$

Remark 1.6. It is an easy consequence of Definition 1.2, that if μ is a normal measure on $\mathscr{P}(\varkappa)$, $\mu(U) = 1$ and $f(\xi) < \xi$ for all $\xi \in U$, $\xi \neq 0$, then there is a set $V \subseteq U$, $\mu(V) = 1$ and $|f^*(V)| = 1$. (Put $\bar{f}(\xi) = f(\xi)$ for $\xi \in U$ and $\bar{f}(\xi) = 0$ for $\xi \in \varkappa - U$ and apply normality to $\bar{f}$). Since μ is always assumed to be a normal measure, we shall not repeat this in Lemma 1.7 and Theorems 1.8, 1.9.

LEMMA 1.7. *Suppose $U \in \mathscr{P}(\varkappa)$, $\mu(U) = 1$, $n \in \omega_0$, $n > 0$ and f: $[U]^n \to \varkappa$ is such that, for every $s \in [U]^n$, $f(s) < \min s$ or $\min s = 0$. Then there is a set $W \subseteq U$ for which $\mu(W) = 1$ and $|f^*([W]^n)| = 1$.*

Proof. We proceed by induction on n. For $n = 1$ this is Remark 1.6. Assume that Lemma 1.7 is true for $n-1$ for every U and every f as above. For every $\xi \in U$ let $U_\xi = \{\eta | \eta > \xi \ \&\ \eta \in U\}$. Obviously $\mu(U_\xi) = 1$ for all $\xi \in U$. We define f_ξ: $[U_\xi]^{n-1} \to \varkappa$ by $f_\xi(s) = f(\{\xi\} \cup s)$ for all $\xi \in U$ and $s \in [U_\xi]^{n-1}$. By hypothesis about f and definition of U_ξ, $f_\xi(s) < \xi$ for all $s \in [U_\xi]^{n-1}$ and all $\xi \in U$, $\xi \neq 0$. Thus $f_\xi(s) < \min s$ for all $s \in [U_\xi]^{n-1}$ and all $\xi \in U$, $\xi \neq 0$. By the induction hypothesis, there is a set $V_\xi \subseteq U_\xi$, $\mu(V_\xi) = 1$ and an ordinal $\alpha_\xi < \xi$ such that $f_\xi(s) = \alpha_\xi$ for all $s \in [V_\xi]^{n-1}$ and all $\xi \in U$, $\xi \neq 0$. By $\mu(U) = 1$ and Remark 1.6 there is a set $V \subseteq U$, $\mu(V) = 1$, and an ordinal α such that $\alpha_\xi = \alpha$ for all $\xi \in V$. Let

$$W = \{\eta | \eta \in V \ \&\ \eta \in \bigcap_{\xi < \eta} V_\xi\}.$$

By Lemma 1.4, $\mu(W) = 1$ and obviously $W \subseteq V$. Let $s \in [W]^n$. Let $\xi = \min s$. By the definition of W, $s - \{\xi\} \subseteq V_\xi$. Thus $f_\xi(s - \{\xi\}) = a_\xi$ and $a_\xi = a$ by $\xi \in V$. But $f_\xi(s - \{\xi\}) = f(\{\xi\} \cup (s - \{\xi\})) = f(s)$. Hence $f(s) = a$ for all $s \in [W]^n$. This proves the lemma.

Lemma 1.7 gives easily

THEOREM 1.8. *Let $U \in \mathscr{P}(\varkappa)$, $\mu(U) = 1$ and f: $[U]^{<\omega_0} \to \varkappa$ such that for every $s \in [U]^{<\omega_0}$, $f(s) < \min s$ or $\min s = 0$. Then there is a set $V \subseteq U$ such that $\mu(V) = 1$ and for every $n \in \omega_0$, $n > 0$, $|f^*([V]^n)| = 1$.*

Proof. For every $n \in \omega_0$, $n > 0$, there is a set $V_n \subseteq U$ for which $\mu(V_n) = 1$ and $|f^*([V_n]^n)| = 1$. This is by Lemma 1.7. Take $V = \bigcap_{n \in \omega_0} V_n$. By $\varkappa$-addit. of μ, $\mu(V) = 1$, and the rest is obvious. Theorem 1.8 generalizes the following result due to F. Rowbottom [13]. The proof of Theorem 1.8 is essentially the same as his proof of Theorem 1.9.

THEOREM 1.9 ([13]). *Let $\mu(U) = 1, \lambda < \varkappa$ and f: $[U]^{<\omega_0} \to \lambda$. Then there is a set $V \subseteq U$, $\mu(V) = 1$ and for every $n \in \omega_0$, $n > 0$, $|f^*([V]^n)| = 1$.*

Proof. Let $U' = \{\eta | \eta \in U \;\&\; \eta > \lambda\}$. Obviously $\mu(U') = 1$. Let $f' = f | [U']^{<\omega_0}$. f', U' satisfy the assumptions of Theorem 1.8. Let $V \subset U'$, $\mu(V) = 1$ and for every $n \in \omega_0$, $n > 0$, $|f'^*([V]^n)| = 1$. Then also $|f^*([V]^n)| = 1$ for all $n \in \omega_0$, $n > 0$.

The next lemma appears in J. Silver's work on indiscernables.

LEMMA 1.10. *Let μ be a normal measure on $\mathscr{P}(\varkappa)$, $U \in \mathscr{P}(\varkappa)$ and $\mu(U) = 1$. Let $0 < m, n \in \omega_0$ and f: $[U]^{m+n} \to \varkappa$ such that $\max(r|m) < f(r) < \min(r - r|m)$ for all $r \in [U]^{m+n}$. Then there is a set $V \subseteq U$, $\mu(V) = 1$, such that for every $s \in [V]^m$ and every $s', s'' \in [\{\xi | \xi > \max s \;\&\; \xi \in V\}]^n$, $f(s \cup s') = f(s \cup s'')$.*

Proof. For every $s \in [U]^m$ let $U_s = \{\xi |\ \xi > \max s \;\&\; \xi \in U\}$. Thus $\mu(U_s) = 1$. We define f_s: $[U_s]^n \to \varkappa$ by $f_s(s') = f(s \cup s')$ for all $s \in [U]^m$ and all $s' \in [U_s]^n$. By our hypothesis about f we get $\max s < f(s \cup s') < \min s'$, i.e. $f_s(s') < \min s'$, for all $s \in [U]^m$ and all $s' \in [U_s]^n$. By Lemma 1.7, there is a set $V_s \subseteq U_s$, $\mu(V_s) = 1$, and f_s is constant on $[V_s]^n$ for all $s \in [U]^m$. Let

$$V = \{\xi | \xi \in \bigcap_{\max s < \xi} V_s\}.$$

By Lemma 1.4, $\mu(V) = 1$. It is claimed that V has the required properties. Let $s \in [V]^m$. By the definition of V,

$$\{\xi | \xi > \max s \;\&\; \xi \in V\} = V_s' \subseteq V_s.$$

Hence, if $s', s'' \in [V_s']^n$, then $s', s'' \in [V_s]^n$. By the choice of V_s and definition of f_s,

$$f(s \cup s') = f_s(s') = a_s = f_s(s'') = f(s \cup s'').$$

This proves the lemma.

DEFINITION 1.11. Let $\nu < \varkappa$, ν regular. Let $\mathscr{I}$ be a $\varkappa$-comp. ν-sat. ideal in $\mathscr{P}(\varkappa)$. $\mathscr{I}$ is *normal* if, for every $U \in \mathscr{P}(\varkappa)$, $U \notin \mathscr{I}$ and every $f\colon U \to \varkappa$ such that $f(\xi) < \xi$ for all $\xi \in U$, $\xi \neq 0$, there is a set $V \subseteq U$, $V \notin \mathscr{I}$ and $|f^*(V)| = 1$.

THEOREM 1.12. (R. Solovay, [23], [24]). *Let ν, $\varkappa$ be as in Definition 1.11. If there is a $\varkappa$-comp. ν-sat. ideal $\mathscr{I}$ in $\mathscr{P}(\varkappa)$, then there is a normal $\varkappa$-comp. ν-sat. ideal $\mathscr{I}$ in $\mathscr{P}(\varkappa)$.*

Theorem 1.12 is an analog of Theorem 1.3.

THEOREM 1.13. (R. Solovay, unpublished). *Let ν, $\varkappa$ be as in Definition 1.11. Let $\mathscr{I}$ be a normal $\varkappa$-comp. ν-sat. ideal in $\mathscr{P}(\varkappa)$. Let $U \in \mathscr{P}(\varkappa)$, $U \notin \mathscr{I}$. Then for every $\lambda < \varkappa$ and every $f\colon [U]^{<\omega_0} \to \lambda$ there is a set $V \subseteq U$ such that $U - V \in \mathscr{I}$ and $|f^*([V]^{<\omega_0})| < \nu$.*

We sketch a proof of this result.

LEMMA 1.14. *Let ν, $\varkappa$, $\mathscr{I}$, λ and U be as in Theorem 1.13. Let $g\colon U \to \lambda$. Then there is a set $V \subseteq U$, $U - V \in \mathscr{I}$ and $|g^*(V)| < \nu$.*

Proof. Let $S = \{\alpha \mid \alpha \in \lambda \mathbin{\&} g^{-1}(\alpha) \notin \mathscr{I}\}$. By the ν-saturatedness of $\mathscr{I}$, $|S| < \nu$. By $\lambda < \varkappa$ and the $\varkappa$-completeness of $\mathscr{I}$, $\bigcup_{\alpha \in \lambda - S} g^{-1}(\alpha) \in \mathscr{I}$. Hence we can take $V = \bigcup_{\alpha \in S} g^{-1}(\alpha)$.

LEMMA 1.15. *Let ν, $\varkappa$, $\mathscr{I}$, U, f, λ be as in Theorem 1.13 and $n \in \omega_0$, $n > 0$. Then there is a set $V \subseteq U$, $U - V \in \mathscr{I}$ and $|f^*([V]^n)| < \nu$.*

Proof. For $n = 1$ Lemma 1.15 follows easily from Lemma 1.14. Assume that Lemma 1.15 holds for $n-1$ for all λ, f, U as above. We prove by induction on λ that Lemma 1.15 holds for n. If $\lambda < \nu$, there is nothing to prove. Let Lemma 1.15 hold for all $\lambda' < \lambda$. We shall consider two cases. Case I: $cf(\lambda) < \nu$. Hence $\lambda = \lim_{\varrho \in \theta} \lambda_\varrho$, $\lambda_\varrho < \lambda$, $\theta < \nu$. Put $f_\varrho(s) = f(s)$, if $f(s) < \lambda_\varrho$, and $f_\varrho(s) = 0$ otherwise, for all $s \in [U]^n$ and $\varrho < \theta$. By the inductive hypothesis, there are sets $V_\varrho \subseteq U$, $U - V_\varrho \in \mathscr{I}$ and $|f_\varrho^*([V_\varrho]^n)| < \nu$. Put $V = \bigcap_{\varrho \in \theta} V_\varrho$. By $\varkappa$-completeness, $U - V \in \mathscr{I}$. Obviously $f^*([V]^n) \subseteq \bigcup_{\varrho \in \theta} f_\varrho^*([V_\varrho]^n)$. Simple cardinality computations show $|\bigcup_{\varrho \in \theta} f_\varrho^*([V_\varrho]^n)| < \nu$. Case II. $cf(\lambda) \geqslant \nu$. We put $U_\xi = \{\eta \mid \eta \in U \mathbin{\&} \eta > \xi\}$ and $f_\xi(s) = f(s \cup \{\xi\})$ for all $s \in [U_\xi]^{n-1}$ and all $\xi \in U$. Obviously $U_\xi \notin \mathscr{I}$ for all $\xi \in U$. Let $V_\xi \subseteq U_\xi$, $U_\xi - V_\xi \in \mathscr{I}$ and $|f_\xi^*([V_\xi]^{n-1})| < \nu$ for all $\xi \in U$. Such sets V_ξ exist by the induction hypothesis. Let $\alpha_\xi = \sup f_\xi^*([V_\xi]^{n-1})$ for all $\xi \in U$. By $cf(\lambda) \geqslant \nu$, $\alpha_\xi < \lambda$ for all $\xi \in U$. Apply Lemma 1.14 to get $U' \subseteq U$, $U - U' \in \mathscr{I}$ such that $|\{\alpha_\xi \mid \xi \in U'\}| < \nu$. Again by $cf(\lambda) \geqslant \nu$, $\sup\{\alpha_\xi \mid \xi \in U'\} = \alpha < \lambda$. Let $V' = \{\eta \mid \eta \in \bigcap_{\xi < \eta} U^1 \mathbin{\&} \eta \in U'\}$. Similarly as in Lemma 1.4 (or Remark 1.5) one can prove that $U' - V' \in \mathscr{I}$. By reasoning analogous to that of Lemma 1.7, we get $f^*([V']^n) \subseteq \alpha$. Since $\alpha < \lambda$, we can apply the inductive hypothesis to get a set $V \subseteq V'$, $V' - V \in \mathscr{I}$ and $|f^*([V]^n)| < \nu$. The rest is obvious.

The passage from Lemma 1.15 to Theorem 1.13 is similar to the one from Lemma 1.7 to Theorem 1.8. A theorem parallel to 1.8 holds here as well. We state it without proof.

THEOREM 1.16. *Let* ν, $\varkappa$, $\mathscr{I}$, U *be as in Theorem* 1.13 *and let* $f\colon [U]^{<\omega_0} \to \varkappa$ *be such that for every* $s \in [U]^{<\omega_0}$, $f(s) < \min s$ *or* $\min s = 0$. *Then there is a set* $V \subseteq U$ *such that* $U - V \in \mathscr{I}$ *and* $|f^*([V]^{<\omega_0})| < \nu$.

2. DEFINITION 1.17. $\varkappa$ *has the property* $R(\nu)$ if, for every $\lambda < \varkappa$ and every $f\colon [\varkappa]^{<\omega_0} \to \lambda$, there is a set $V \in \mathscr{P}(\varkappa)$ such that $|V| = \varkappa$ and $|f^*([V]^{<\omega_0})| < \nu$. Instead of saying $\varkappa$ has the property $R(\nu)$, we shall often write $(\varkappa, R(\nu))$.

THEOREM 1.18. (F. Rowbottom [13]). *Let* $(\varkappa, R(\omega_1))$ *hold. Then for every* $\mathfrak{A} = \langle \varkappa, \lambda, R_n \rangle_{n \in \omega_0}$, $\lambda < \varkappa$, *there is an* $\mathfrak{A}' \leqslant \mathfrak{A}$, $\mathfrak{A}' = \langle A, A \cap \lambda, R_n | A \rangle_{n \in \omega_0}$ *such that* $|A| = \varkappa$ *and* $|A \cap \lambda| \leqslant \aleph_0$.

An analogous statement holds if $(\varkappa, R(\nu))$ for some $\nu < \varkappa$.

DEFINITION 1.19. $\langle A, f_n \rangle_{n \in \omega_0}$ is a *Jónsson algebra* if it has no proper subalgebra of the same cardinality.

The next theorem was known to F. Rowbottom.

THEOREM 1.20. *If* $(\varkappa, R(\omega_1))$, *then there is no Jónsson algebra of cardinality* $\varkappa$.

Theorem 1.20 continues to hold if $(\varkappa, R(\omega_1))$ is replaced by $(\varkappa, R(\nu))$ for some $\nu < \varkappa$.

It has not been known whether, for any singular cardinal $\varkappa$, $(\varkappa, R(\nu))$ holds with some $\nu < \varkappa$, or weaker, whether there is no Jónsson algebra of cardinality $\varkappa$. However, J. Keisler and F. Rowbottom [9] proved that the axiom of constructibility implies the existence of Jónsson algebras of every cardinality. Compare Theorems 1.29, 2.28, 2.29.

In order to discuss the applicability of certain constructions to appear later, we indicate how to obtain the following known

PROPOSITION 1.21. *If* $\varkappa = \lambda^+$, *then* $\varkappa$ *does not satisfy the property* $R(\omega_1)$.

Proof. For every $\alpha \in \lambda^+ - \lambda$ let f_α be a one-to-one function mapping α onto λ. We shall consider $\mathfrak{A} = \langle \lambda^+, \lambda, R(\alpha, \beta, \gamma) \rangle$, where $R(\alpha, \beta, \gamma)$ iff $f_\alpha(\beta) = \gamma$. Let $\langle A, A \cap \lambda, R' = R | A \rangle = \mathfrak{A}' \leqslant \mathfrak{A}$ and $|A| = \lambda^+$. By $\mathfrak{A}' \leqslant \mathfrak{A}$ we obtain $R'(\alpha, \beta, \gamma)$ iff $f_\alpha(\beta) = \gamma$ and $\alpha, \beta, \gamma \in A$. Also f_α gives a one to one correspondence between $\alpha \cap A$ and $\lambda \cap A$ for every $\alpha \in A$. By $|A| = \lambda^+$ we get an $\alpha_0 \in A \cap (\lambda^+ - \lambda)$ such that $|\alpha_0 \cap A| = \lambda$. Thus by $f^*_{\alpha_0}(\alpha_0 \cap A) = \lambda \cap A$, $|\lambda \cap A| = \lambda$. It follows from Theorem 1.18 that $(\varkappa, R(\omega_1))$ fails.

PROPOSITION 1.22. *Let λ be regular, $\omega_0 < \lambda < \varkappa$ and $\varkappa = \lim_{\xi\in\lambda} \varkappa_\xi$, where $\{\varkappa_\xi\}_{\xi\in\lambda}$ is a continuous increasing sequence of cardinals. Then $(\varkappa, R(\omega_1))$ fails.*

Proof. Put $f(\{\alpha\}) = \xi$ iff $\alpha\in\varkappa_{\xi+1}-\varkappa_\xi$ and $f(s) = 0$ for $s\in[\varkappa]^{<\omega_0}$, $|s| > 1$. Clearly, if $U \subseteq \varkappa$ and $|U| = \varkappa$, then $|f^*([U]^{<\omega_0})| = \lambda$. E. Klineberg too found Proposition 1.22.

Combining Propositions 1.21, 1.22 we obtain

PROPOSITION 1.23. *If $(\varkappa, R(\omega_1))$ holds, then $\varkappa$ is a regular limit cardinal or $cf(\varkappa) = \omega_0$. An analogous proposition holds, if $(\varkappa, R(\nu))$ for some $\nu < \varkappa$.*

DEFINITION 1.24. Let Φ be a uniform filter in $\mathscr{P}(\varkappa)$. *$\varkappa$ has the property $R(\omega_1)$ with respect to Φ*, abbrev. $(\varkappa, R(\omega_1), \Phi)$, if for every $\lambda < \varkappa$, and every $f\colon [\varkappa]^{<\omega_0} \to \lambda$ there is a set $U\in\Phi$ such that $|f^*([U]^{<\omega_0})| \leqslant \aleph_0$. We put $(\varkappa, \underline{R}(\omega_1))$ iff $(\exists\Phi)\,(\varkappa, R(\omega_1), \Phi)$.

Property $(\varkappa, \underline{R}(\nu))$ can be defined analogously. However, the most important case seems to be $(\varkappa, \underline{R}(\omega_1))$. Obviously, if $(\varkappa, \underline{R}(\omega_1))$, then $(\varkappa, R(\omega_1))$. The following is easy to see. Suppose $(\varkappa, R(\omega_1), \Phi)$ and $U\in\Phi$. Let $\lambda < \varkappa$ and $f\colon [U]^{<\omega_0} \to \lambda$. Then there is a set $V\in\Phi$, $V \subseteq U$ such that $|f^*([V]^{<\omega_0})| \leqslant \aleph_0$. It is clear from Theorems 1.9, 1.13, that, if $\varkappa$ is measurable, or if there is an ω_1-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$, then $(\varkappa, \underline{R}(\omega_1))$ holds. In the first case,

$$(\varkappa, R(\omega_1), \Phi) \quad \text{holds for} \quad \Phi = \{X|X \subseteq \varkappa \;\&\; \mu(X) = 1\},$$

where μ is a normal measure on $\mathscr{P}(\varkappa)$. In the second case, we take

$$\Phi = \{X|X \subseteq \varkappa \;\&\; \varkappa - X\in\mathscr{I}\},$$

where $\mathscr{I}$ is a normal ω_1-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$.

3. We shall now give examples of singular cardinals $\varkappa$ such that $(\varkappa, \underline{R}(\nu))$ holds for some $\nu < \varkappa$. By specializing ν, it will be possible to get cardinals $\varkappa$ such that, e.g., $(\varkappa, \underline{R}(\omega_1))$ holds and a fortiori, $(\varkappa, R(\omega_1))$. Compare remarks following Theorem 1.20.

Throughout this section let $\{\varkappa_\xi\}_{\xi\in\nu}$ be an increasing continuous sequence of cardinals such that $\varkappa_\xi$ is measurable for every successor $\xi = \eta+1$, $\xi\in\nu$, ν is a regular cardinal, $\varkappa_0 = 0$ and $\nu < \varkappa_1$. Let μ_ξ be a normal measure on $\mathscr{P}(\varkappa_\xi)$ for every $\xi = \eta+1$. Let $\varkappa = \lim_{\xi\in\nu} \varkappa_\xi$. Let $r\in[\varkappa]^{<\omega_0}$. We define the *type of* r (relative to $\{\varkappa_\xi\}_{\xi\in\nu}$) as follows:

$$\text{(0)} \quad \mathrm{tp}^0(r) = \langle\xi_0\xi_1\ldots\xi_k\rangle \leftrightarrow [\xi_0 < \xi_1 < \ldots < \xi_k \;\& \\ (\forall i \leqslant k)\,(\varkappa_{\xi_i+1}-\varkappa_{\xi_i}) \cap r \neq 0) \;\&\; r = \bigcup_{i\leqslant k} (\varkappa_{\xi_i+1}-\varkappa_{\xi_i}) \cap r],$$

$$\text{(1)} \quad \mathrm{tp}^1(r) = \langle m_0\ldots m_k\rangle \leftrightarrow \\ (\exists\xi_0\ldots\xi_k)\,[\mathrm{tp}^0(r) = \langle\xi_0\ldots\xi_k\rangle \;\&\; (\forall i \leqslant k)\,(|r \cap (\varkappa_{\xi_i+1}-\varkappa_{\xi_i})| = m_i)].$$

Obviously $\mathrm{tp}^0(r), \mathrm{tp}^1(r)$ exist for every $r \in [\varkappa]^{<\omega_0}$ and are uniquely determined. We put $\mathrm{tp}(r) = \langle \mathrm{tp}^0(r), \mathrm{tp}^1(r) \rangle$.

DEFINITION 1.25. Let $\lambda < \varkappa$ and $f\colon [\varkappa]^{<\omega_0} \to \lambda$. We say that $V \subseteq \varkappa$ is *homogeneous of order n, $n \in \omega_0$ with respect to f and λ*, abbrev. $\mathrm{hom}^n(V, f, \lambda)$, if

(1) $$(\forall \xi \in \nu)\big(\varkappa_{\xi+1} > \lambda \to \mu_{\xi+1}(V \cap \varkappa_{\xi+1}) = 1\big),$$

(2) $$(\forall r, s \in [V]^{<\omega_0})\big(\mathrm{tp}(r) = \mathrm{tp}(s) \ \&\ |\mathrm{tp}^0(r)| = n \to f(r) = f(s)\big).$$

In the remaining part of this section we are using notation just described and assuming $\lambda < \varkappa_1$ unless specified otherwise. For such a λ, and f as above, we shall write $\mathrm{hom}^n(V, f)$ instead of $\mathrm{hom}^n(V, f, \lambda)$.

LEMMA 1.26. *For every $n \in \omega_0$ and every f as above there is a set $V \subseteq \varkappa$ such that* $\mathrm{hom}^n(V, f)$.

Proof is by induction on n. For $n = 1$ it follows easily from $\lambda < \varkappa_1$ and $\varkappa_1$-addit. of $\mu_{\xi+1}$ for all $\xi \in \nu$. Assume the lemma to be true for all $m \leqslant n$ for all f as above. We prove that it holds for $n+1$ as well. Let $r \in [\varkappa]^{<\omega_0}$ and $\mathrm{tp}^0(r) = \langle \xi_0 \ldots \xi_n \rangle$. Obviously $r = p \cup q$, where $\mathrm{tp}^0(p) = \langle \xi_0 \ldots \xi_{n-1} \rangle$ and $\mathrm{tp}^0(q) = \langle \xi_n \rangle$. Such a decomposition is unique.

If $p \in [\varkappa]^{<\omega_0}$, $\mathrm{tp}^0(p) = \langle \xi_0 \ldots \xi_{n-1} \rangle$ and $\mathrm{tp}^0(q) = \langle \xi \rangle$, where $\xi > \xi_{n-1}$, we put $f_p^\xi(q) = f(p \cup q)$. By Theorem 1.9 applied to f_p^ξ we get a set X_p^ξ such that $X_p^\xi \subseteq \varkappa_{\xi+1} - \varkappa_\xi$, $\mu_{\xi+1}(X_p^\xi) = 1$ and, for any $q, q' \subseteq X_p^\xi$, if $|q| = |q'|$, then $f_p^\xi(q) = f_p^\xi(q')$. We put

$$X^\xi = \bigcap \{X_p^\xi \mid (\exists \xi_0, \ldots \xi_{n-1} < \xi)\big(\mathrm{tp}^0(p) = \langle \xi_0 \ldots \xi_{n-1} \rangle\big)\}.$$

Obviously $\big|\{p \mid (\exists \xi_0 \ldots \xi_{n-1} < \xi)\big(\mathrm{tp}^0(p) = \langle \xi_0 \ldots \xi_{n-1} \rangle\big)\}\big| = \varkappa_\xi < \varkappa_{\xi+1}$. Thus $\mu_{\xi+1}(X^\xi) = 1$. We put $g_\xi^l(p) = f(p \cup q_\xi^l)$ for a fixed $q_\xi^l \subseteq X^\xi$, $|q_\xi^l| = l \in \omega_0$, and any $p \in [\varkappa]^{<\omega_0}$. If $\mathrm{tp}^0(p) = \langle \xi_0 \ldots \xi_{n-1} \rangle$ for some $\xi_0, \ldots \xi_{n-1} < \xi$, then:

(*) $$g_\xi^l(p) = f(p \cup q) \quad \text{for any} \quad q \subseteq X^\xi,\ |q| = l,$$

since for such a p we have $X^\xi \subseteq X_p^\xi$. By applying the inductive hypothesis we get a set $Y^{l,\xi}$ such that $\mathrm{hom}^n(Y^{l,\xi}, g_\xi^l)$. Thus

(**) for any $p, p' \subseteq Y^{l,\xi}$, if $\mathrm{tp}(p) = \mathrm{tp}(p')$ and $|\mathrm{tp}^0(p)| = n$, then $g_\xi^l(p) = g_\xi^l(p')$.

Let

($\overset{*}{**}$) $$\mathrm{tp}^0(p) = \mathrm{tp}^0(p') = \langle \xi_0 \ldots \xi_{n-1} \rangle \quad \text{for some} \quad \xi_0, \ldots \xi_{n-1} < \xi.$$

Hence, if $q, q' \subseteq X^\xi$, $|q| = l = |q'|$ and p, p' are as in (**) and ($\overset{*}{**}$), then by (*), $g_\xi^l(p) = f(p \cup q)$, $g_\xi^l(p') = f(p' \cup q')$. By (**), we get $f(p \cup q) = f(p' \cup q')$. Let $X = \bigcup_{\xi \in \nu} X^\xi$, $Y = \bigcap_{\xi \in \nu, l \in \omega_0} Y^{l,\xi}$.

Let $V = X \cap Y$. That $\mu_{\xi+1}\big(Y \cap (\varkappa_{\xi+1} - \varkappa_\xi)\big) = 1$, for all $\xi \in \nu$, follows from $\mathrm{hom}^n(Y^{l,\xi})$ and $\nu < \varkappa_1$. It is now obvious that $\mu_{\xi+1}\big(V \cap (\varkappa_{\xi+1} - \varkappa_\xi)\big)$

$= 1$ for all $\xi \in \nu$. It is claimed that $\mathrm{hom}^{n+1}(V, f)$. Let $r, r' \in [V]^{<\omega_0}$, $\mathrm{tp}(r) = \mathrm{tp}(r')$, $\mathrm{tp}^0(r) = \langle \xi_0 \dots \xi_{n-1} \xi_n \rangle$. We write $r = p \cup q$, $r' = p' \cup q'$, where $\mathrm{tp}^0(p) = \mathrm{tp}^0(p') = \langle \xi_0 \dots \xi_{n-1} \rangle$ and $\mathrm{tp}^0(q) = \mathrm{tp}^0(q') = \langle \xi_n \rangle$. It is now easy to see that p, p', q, q' satisfy conditions of $(**)$, $(^*_{**})$ for $\xi = \xi_n$. Hence $f(p \cup q) = f(p' \cup q')$, $f(r) = f(r')$.

LEMMA 1.27. *For every f as above there is a $V \subseteq \varkappa$ such that for all $n \in \omega_0$, $\mathrm{hom}^n(V, f)$.*

Proof. By Lemma 1.26, for every $n \in \omega_0$ there is a $V_n \subseteq \varkappa$ such that $\mathrm{hom}^n(V_n, f) \cdot V = \bigcap_{n \in \omega_0} V_n$ works.

We shall now eliminate the assumption $\lambda < \varkappa_1$.

THEOREM 1.28. *Let $\lambda < \varkappa$ and $f\colon [\varkappa]^{<\omega_0} \to \lambda$. Then there is a set $V \subseteq \varkappa$ such that, for all $n \in \omega_0$, $\mathrm{hom}^n(V, f, \lambda)$.*

Proof. Let ξ_0 be the least ordinal such that $\varkappa_{\xi_0} > \lambda$. Thus ξ_0 is a successor ordinal. We define a sequence $\{\varkappa'_\zeta\}_{\zeta \in \nu}$ as follows: $\varkappa'_0 = \varkappa_0$, $\varkappa'_1 = \varkappa_{\xi_0}$. If $\varkappa'_\zeta$ have been defined for $\zeta < \eta$, then $\varkappa'_\eta$ is the least ordinal $\varkappa_\xi$, $\xi \in \nu$, such that $\varkappa_\xi > \varkappa'_\zeta$ for all $\zeta < \eta$. We have $\varkappa'_1 > \lambda$ and can apply Lemma 1.27 to get a set $V \subseteq \varkappa$ such that for all $n \in \omega_0$, $\mathrm{hom}^n(V, f)$ holds with respect to the sequence $\{\varkappa'_\zeta\}_{\zeta \in \nu}$. But it is easy to see that $\mathrm{hom}^n(V, f, \lambda)$ holds now for all $n \in \omega_0$ with respect to the sequence $\{\varkappa_\xi\}_{\xi \in \nu}$ as well.

Let $\Phi = \big\{V \,|\, (\exists \xi \in \nu)\, (\forall \eta \in \nu)\, \big(\eta \geqslant \xi \to \mu_{\eta+1}(V \cap \varkappa_{\eta+1}) = 1\big)\big\}$. Obviously, Φ is a filter. If $V \in \Phi$, let $\xi(V) = \min\big\{\xi \,|\, \xi \in \nu \;\&\; (\forall \eta \in \nu)\, (\eta \geqslant \xi \to \mu_{\eta+1}(V \cap \varkappa_{\eta+1}) = 1)\big\}$. Let $\xi_0 \in \nu$, $\lambda < \varkappa_{\xi_0+1}$ and $\{V_\alpha\}_{\alpha \in \lambda}$ be such that $V_\alpha \in \Phi$ and $\xi(V_\alpha) \leqslant \xi_0$ for all $\alpha \in \lambda$. Then $\bigcap_{\alpha \in \lambda} V_\alpha = V \in \Phi$ and $\xi(V) \leqslant \xi_0$. This is an obvious consequence of the fact that $\mu_{\eta+1}$ is $\varkappa_{\xi_0+1}$-addit. for all $\eta \geqslant \xi_0$. Whenever there exists a function $f\colon [\varkappa]^{<\omega_0} \to \lambda$, where $\lambda < \varkappa$ such that $\mathrm{hom}^n(V, f, \lambda)$ for all $n \in \omega_0$, it follows that $V \in \Phi$.

THEOREM 1.29. *$\big(\varkappa, R(\nu^+), \Phi\big)$ holds, where Φ as above.*

Proof. Observe that $|\{\mathrm{tp}(r) | r \in [\varkappa]^{<\omega_0}\}| = \nu$. The theorem now follows from Theorem 1.28 and the above stated fact that, if $\mathrm{hom}^n(V, f, \lambda)$ holds for all $n \in \omega_0$, then $V \in \Phi$.

In particular, for $\nu = \omega_0$, we get $\big(\varkappa, R(\omega_1), \Phi\big)$.

4. In this section we state some lemmas of technical nature.

LEMMA 1.30. (F. Rowbottom [13]). *Let $\{f_n\}_{n \in \omega_0}$ be arbitrary functions such that $\mathscr{D}(f_n) \subseteq [\varkappa]^{<\omega_0}$ for all $n \in \omega_0$. Then there is a function f such that $\mathscr{D}(f) \subseteq [\varkappa]^{<\omega_0}$ and, for every set $S \subseteq \varkappa$ without last element, $\bigcup_{n \in \omega_0} f_n^*([S]^{<\omega_0}) = f^*([S]^{<\omega_0})$. We say that f replaces $\{f_n\}_{n \in \omega_0}$.*

Proof. Let g be an arbitrary function such that $\mathscr{D}(g) \subseteq [\varkappa]^{<\omega_0}$. Put $g_n(s) = g(s)$ for every $s \in \mathscr{D}(g) \cap [\varkappa]^n$ and g_n undefined otherwise. It is clear that for every $S \subseteq \varkappa$, $g^*([S]^{<\omega_0}) = \bigcup_{n \in \omega_0} g_n^*([S]^{<\omega_0})$. We can assume that for every $n \in \omega_0$, $\mathscr{D}(f_n) \subseteq [\varkappa]^{m_n}$ for some $m_n \in \omega_0$. Since if this is not the case, we can replace each f_n by countably many functions as above without altering the image of any set, and arrange the new functions into a sequence of type ω_0. We define a new sequence of functions $\{h_n\}_{n\in\omega_0}$ by induction. We put $h_0 = f_0$ and $j_0 = m_0$. Assume that h_n has been defined and $\mathscr{D}(h_n) \subseteq [\varkappa]^{j_n}$. Let $j_{n+1} = \max(j_n, m_{n+1})+1$. We put $\mathscr{D}(h_{n+1}) = \{s | s \in [\varkappa]^{j_{n+1}} \& \, s|m_{n+1} \in \mathscr{D}(f_{n+1})\}$ and $h_{n+1}(s) = f_{n+1}(s|m_{n+1})$ for all $s \in \mathscr{D}(h_{n+1})$. If $n \neq n'$, then $\mathscr{D}(h_n) \cap \mathscr{D}(h_{n'}) = 0$. Hence we can define f on $\bigcup_{n\in\omega_0} \mathscr{D}(h_n)$ by $f(s) = h_n(s)$ iff $s \in \mathscr{D}(h_n)$. The construction has the following properties: Let S be as in the hypothesis of the lemma. Let $s \in [S]^{<\omega_0}$ and $s \in \mathscr{D}(f_n)$ for some n. Let $r \in [S]^{j_n}$ and $r|m_n = s$. Such an r exists, because S has no last element and $j_n \geqslant m_n$. By the definition of h_n, $h_n(r) = f_n(s)$. It is now easy to see that

$$\bigcup_{n\in\omega_0} f_n^*([S]^{<\omega_0}) = \bigcup_{n\in\omega_0} h_n^*([S]^{<\omega_0}).$$

It is immediate that

$$\bigcup_{n\in\omega_0} h_n^*([S]^{<\omega_0}) = f^*([S]^{<\omega_0}).$$

Thus f has the required properties. We mention that Lemma 1.30 plays an important role in the proof of Theorem 1.18.

LEMMA 1.31. *Let f be a function such that $\mathscr{D}(f) \subseteq [\varkappa]^{<\omega_0} \times [\varkappa]^{<\omega_0}$. Then there is a function g such that $\mathscr{D}(g) \subseteq [\varkappa]^{<\omega_0}$ and, for every set $S \subseteq \varkappa$ without last element, $f^*([S]^{<\omega_0} \times [S]^{<\omega_0}) = g^*([S]^{<\omega_0})$. Again, we say that g replaces f.*

Proof. Let $\langle ab \rangle \in [\omega_0]^{<\omega_0} \times [\omega_0]^{<\omega_0}$. We put

$$E^{\langle ab \rangle} = \{\langle rs \rangle | \langle rs \rangle \in [\varkappa]^{<\omega_0} \times [\varkappa]^{<\omega_0} \ \& \ \langle a \cup b, a, b, \varepsilon \rangle \cong \langle r \cup s, r, s, \varepsilon \rangle\}$$

and $F^{\langle ab \rangle} = \{r \cup s | \langle rs \rangle \in E^{\langle ab \rangle}\}$. It is immediate that for every $t \in F^{\langle ab \rangle}$ there is exactly one pair $\langle rs \rangle \in E^{\langle ab \rangle}$ such that $t = r \cup s$. For this unique $\langle rs \rangle$ we put $\langle rs \rangle = t^{\langle ab \rangle}$. We define a function $f^{\langle ab \rangle}$ by $f^{\langle ab \rangle}(t) = f(t^{\langle ab \rangle})$ for $t \in F^{\langle ab \rangle}$ such that $t^{\langle ab \rangle} \in \mathscr{D}(f)$ and $f^{\langle ab \rangle}$ undefined otherwise. Let $\{g_n\}_{n\in\omega_0}$ be an arrangement of functions $f^{\langle ab \rangle}$ into a sequence of type ω_0 and let g replace $\{g_n\}_{n\in\omega_0}$ (see Lemma 1.30). Let S be as in the hypothesis of the lemma. For every $\langle rs \rangle \in [S]^{<\omega_0} \times [S]^{<\omega_0}$ there is an $\langle ab \rangle \in [\omega_0]^{<\omega_0} \times [\omega_0]^{<\omega_0}$ such that $\langle rs \rangle \in E^{\langle ab \rangle}$ and thus $r \cup s \in F^{\langle ab \rangle} \cap [S]^{<\omega_0}$. Vice versa, if $t \in F^{\langle ab \rangle} \cap [S]^{<\omega_0}$, then $t^{\langle ab \rangle} \in [S]^{<\omega_0} \times [S]^{<\omega_0}$. These two remarks give

$$f^*([S]^{<\omega_0} \times [S]^{<\omega_0}) = \bigcup_{n\in\omega_0} g_n^*([S]^{<\omega_0}).$$

The rest follows from the fact that g replaces $\{g_n\}_{n\in\omega_0}$.

LEMMA 1.32. *Let $(\varkappa, R(\omega_1), \Phi)$ hold, $\lambda < \varkappa$, $U \in \Phi$ and $f\colon [U]^{<\omega_0} \times [U]^{<\omega_0} \to \lambda$. Then there is a set $V \in \Phi$ such that $V \subseteq U$ and $|f^*([V]^{<\omega_0} \times [V]^{<\omega_0})| \leqslant \aleph_0$.*

Proof. By Lemma 1.31 there is a function g such that g replaces f. By the remark following Definition 1.24 there is a set $V \subseteq U$, $V \in \Phi$ such that $|g^*[V]^{<\omega_0}| \leqslant \aleph_0$. But since g replaces f and since V has no last element, $|f^*([V]^{<\omega_0} \times [V]^{<\omega_0})| \leqslant \aleph_0$.

5. The following combinatorial facts and principles are weaker than those discussed so far. The main difference is that while the above stated properties apply to families of all finite subsets of a set, those described next deal with more general collections of finite sets. The reason for mentioning these weaker properties is that in the subsequent applications they can be often used instead of those discussed above, sometimes with corresponding modifications of the conclusions. Also, there are a few facts which we do not know how to obtain otherwise. Another merit of these more general combinatorial principles is that they might apply to a wider class of cardinals.

DEFINITION 1.33. Let Φ be a filter in $\mathscr{P}(\varkappa)$. A family $U \subseteq [\varkappa]^{<\omega_0}$ is called *Φ-extendable* if

(1) $\{x \mid \langle x\rangle \in U\} \in \Phi$,

(2) $(\forall s \in U)\,(\{x \mid s\widehat{\ }x \in U\} \in \Phi\}$,

(3) $(\forall s \in U)\,(\forall k \in \omega_0)\,(s|k \in U)$.

It is obvious that if U, V are Φ-extendable, then $U \cap V$ is Φ-extendable.

DEFINITION 1.34. Let Φ be a uniform filter in $\mathscr{P}(\varkappa)$. $\varkappa$ has *weak $R(\omega_1)$ property with respect to Φ*, abbrev. $(\varkappa, w-R(\omega_1), \Phi)$, if for every $\lambda < \varkappa$, and every $f\colon [\varkappa]^{<\omega_0} \to \lambda$, there is a Φ-extendable family U such that $|f^*(U)| \leqslant \aleph_0$. We put

$$\big(\varkappa, w-R(\omega_1)\big) \leftrightarrow (\exists\Phi)\,\big(\varkappa, w-R(\omega_1), \Phi\big).$$

If $\big(\varkappa, w-R(\omega_1), \Phi\big)$, U is a Φ-extendable family, $\lambda < \varkappa$, and $f\colon U \to \lambda$, then there is a Φ-extendable family $V \subseteq U$ such that $|f^*(V)| \leqslant \aleph_0$. This is analogous to the remark following Definition 1.24. It is obvious, that if $\big(\varkappa, R(\omega_1), \Phi\big)$ holds, then $\big(\varkappa, w-R(\omega_1), \Phi\big)$ holds as well. The assumption $\big(\varkappa, w-R(\omega_1)\big)$ is the weakest condition on $\varkappa$ under which we know how to change the cofinality of $\varkappa$ by a Cohen extension preserving cardinalities. Chapter 2 deals in detail with the problem of changing cofinalities.

DEFINITION 1.35. Let μ_i be a $\varkappa_i$-addit. 2-valued measure on $\mathscr{P}(X_i)$ for $i = 1, 2, \ldots n, \ldots, n \in \omega_0$. Let $A \subseteq \bigcup_{1 \leqslant i \in \omega_0} X_1 \times \ldots \times X_i = [\prod X_i]^{<\omega_0}$ and

(1) $\mu_1(\{x \mid \langle x\rangle \in A\}) = 1$,

(2) $(\forall s \in A)\,\big(\mu_{|s|+1}(\{x \mid s\widehat{\ }x \in A\}) = 1\big)$,

(3) $(\forall s \in A)\,(\forall k \in \omega_0)\,(s|k \in A)$.

Then A is called *$\{\mu_i\}_{i\in\omega_0}$-extendable.*

THEOREM 1.36. (We shall use the notation of Definition 1.35.) *Let* $\lambda < \min\{\varkappa_i | i \in \omega_0\}$. *Let* $f\colon [\prod X_i]^{<\omega_0} \to \lambda$. *Then there is a set* $A \subseteq [\prod X_i]^{<\omega_0}$ *such that* A *is* $\{\mu_i\}_{i\in\omega_0}$*-extendable and* $|f^*(A \cap (X_1 \times \ldots \times X_i))| = 1$ for all $i \in \omega_0$.

Proof. Let $s \in X_1 \times \ldots \times X_i$. We put $f'(s) = \alpha$ iff $\mu_{i+1}(\{x | f(\widehat{s\,x}) = a\}) = 1$. It follows from the $\varkappa_{i+1}$-addit. of μ_{i+1} and $\lambda < \varkappa_{i+1}$ that $f'(s)$ is well defined. We put also $f = f^{(0)}$ and $f^{(n+1)} = f^{(n)\prime}$. Suppose $Y_n \subset X_1$, $\mu_1(Y_n) = 1$ and $f^{(n)}$ is constant on $\{\langle y\rangle | y \in Y_n\}$, say $f^{[n]}(\langle y\rangle) = a_n$ for $y \in Y_n$. The existence of sets Y_n satisfying the above conditions is a consequence of $\varkappa_1$-addit. of μ_1, $\lambda < \varkappa_1$ and $f^{(n)}\colon \{\langle x\rangle | x \in X_1\} \to \lambda$. Let $Y = \bigcap_{n\in\omega_0} Y_n$; $\mu_1(Y) = 1$. We now define elements of A by induction on the length of sequences. If $|s| = 1$, we put $s \in A$ iff $s = \langle y\rangle$ for some $y \in Y$. Assume that all sequences of A of length $\leqslant m$ were defined in such a way that

$$(*)\qquad (\forall s \in A)\Big[\big(|s| = m \to f(s) = a_{m-1} = a_{|s|-1}\big) \;\&\; (\forall n \in \omega_0)\big(f^{(n+1)}(s) = a_{n+m} = a_{n+|s|}\big)\Big].$$

Let $s \in A$ and $|s| = m$. Put $Z_n^s = \{y | f^{(n)}(\widehat{s\,y}) = a_{n+m}\}$. By $f^{(n+1)}(s) = a_{n+m}$ and the definition of $f^{(n+1)}$, we get $\mu_{m+1}(Z_n^s) = 1$. Let $Z^s = \bigcap_{n\in\omega_0} Z_n^s$; $\mu_{m+1}(Z^s) = 1$. We put $\widehat{s\,y} \in A$ iff $y \in Z^s$. If we let $n = 0$ in $(*)$, we get $f'(s) = a_m = f(\widehat{s\,y}) = a_{|\widehat{s\,y}|-1}$ for all $y \in Z^s$ since $Z^s \subseteq Z_0^s$. And $f^{(n)}(\widehat{s\,y}) = a_{n+m} = a_{n-1+m+1} = a_{n-1+|\widehat{s\,y}|}$ for $n = 1, 2, \ldots$ and all $y \in Z^s$, because $Z^s \subseteq Z_n^s$. Since s was an arbitrary element of A such that $|s| = m$, we see that the inductive hypothesis for sequences of length m implies the corresponding assertion for sequences of A of length $m+1$. So the construction can be continued for all $m \in \omega_0$. The set A defined as above is obviously $\{\mu_i\}_{i\in\omega_0}$-extendable and if $s \in A$, $|s| = m$, then $f(s) = a_{m-1}$ for $m = 1, 2, \ldots$

The usual remark about functions defined on $\{\mu_i\}_{i\in\omega_0}$-extendable families, instead of the whole $[\prod X_i]^{<\omega_0}$, applies.

6. In this paragraph we shall discuss the problem of the existence of non-regular ultrafilters raised in [10a].

DEFINITION 1.37. $(\varkappa, \nu, \lambda, \Phi)$ holds if Φ is a uniform filter in $\mathscr{P}(\varkappa)$ such that for every $f \in \lambda^\varkappa$ there is a set $U \in \Phi$ such that $|f^*(U)| < \nu$. We put

$(\varkappa, \nu, \Phi) \leftrightarrow (\forall \lambda < \varkappa)\,(\varkappa, \nu, \lambda, \Phi)$,

$(\varkappa, \nu, \lambda) \leftrightarrow (\exists \Phi)\,(\varkappa, \nu, \lambda, \Phi)$,

$(\varkappa, \nu) \leftrightarrow (\exists \Phi)\,(\varkappa, \nu, \Phi)$.

Remark 1.37.1. $\big(\varkappa, R(\nu), \Phi\big) \to \big(\varkappa, w-R(\nu), \Phi\big) \to (\varkappa, \nu, \Phi)$ etc. See Definitions 1.24, 1.34. The implications in the opposite direction are not known.

Remark 1.37.2. Let $(\varkappa, \nu, \lambda, \Phi)$ hold and $\nu \leqslant cf(\lambda)$. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be such that $U_\alpha \in \Phi$ and $U_\beta \subseteq U_\alpha$ for all $\alpha, \beta \in \varkappa$, $\alpha < \beta$. Then $\bigcap_{\alpha\in\lambda} U_\alpha \in \Phi$.

Remark 1.37.3. Suppose $(\varkappa, \nu, \lambda, \Phi)$ holds and $\nu \leqslant \lambda$. Then the ideal $\mathscr{I}$ dual to Φ is ν-saturated.

Remark 1.37.4. Suppose $(\varkappa, \nu, \lambda, \Phi)$ holds and $\nu \leqslant \lambda < \varkappa$. Then $|\Phi| \geqslant \varkappa^+$. This is seen as follows. Let $\mathscr{I}$ be the ideal dual to Φ and assume $|\Phi| \leqslant \varkappa$. We can construct pairwise disjoint sets U_α, $\alpha\in\lambda$ such that $U_\alpha \notin \mathscr{I}$ for all $\alpha\in\lambda$. To assure that $U_\alpha \notin \mathscr{I}$ it is enough to have $U_\alpha \cap U \neq 0$ for every $U\in\Phi$. This is however done easily by transfinite induction using $|\Phi| \leqslant \varkappa$ and the uniformity of Φ. This contradicts Remark 1.37.3.

Definition 1.38. (H. J. Keisler [10a]). A uniform ultrafilter $\mathscr{U}$ in $\mathscr{P}(\varkappa)$ is *regular*, if there is a family $\{U_\alpha\colon \alpha\in\varkappa\} \subseteq \mathscr{U}$ such that for every $S \subseteq \varkappa$, if $|S| \geqslant \aleph_0$, then $\bigcap_{\alpha\in S} U_\alpha = 0$. H. J. Keisler raised the question whether every ultrafilter is regular.

Theorem 1.39. *Let $(\varkappa, \nu, \nu, \Phi)$ hold for some $\nu < \varkappa$. Let $\mathscr{U}$ be an arbitrary uniform ultrafilter in $\mathscr{P}(\varkappa)$, $\mathscr{U} \supseteq \Phi$. Then $\mathscr{U}$ is not regular.*

Theorem 1.39 will follow easily from

Lemma 1.40. *Let $(\varkappa, \nu, \nu, \Phi)$ hold for some $\nu < \varkappa$. Let $\{U_\alpha\}_{\alpha\in\nu}$ be such that $U_\alpha \cap U \neq 0$ for every $U\in\Phi$ and every $\alpha\in\nu$. Then there is an infinite set $S \subseteq \nu$ such that $\bigcap_{\alpha\in S} U_\alpha \neq 0$.*

Proof. Let $U_\alpha^1 = U_\alpha$, $U_\alpha^0 = \varkappa - U_\alpha$ for all $\alpha\in\nu$. Let $U^\varphi = \bigcap_{\alpha\in\nu} U_\alpha^{\varphi(\alpha)}$ for every $\varphi\in 2^\nu$. Let $D = \{\varphi | U^\varphi \neq 0\}$. Thus $\bigcup_{\varphi\in D} U^\varphi = \varkappa$ and $U^\varphi \cap U^\psi = 0$ for all $\varphi \neq \psi$. Let $S_\varphi = \{\alpha | \varphi(\alpha) = 1\}$ for every $\varphi\in 2^\nu$. If there is a $\varphi\in D$ such that S_φ is infinite, we are done. Otherwise S_φ is finite for every $\varphi\in D$. Hence $|D| \leqslant \nu$. Define $f\colon \varkappa \to D$ by $f(\alpha) = \varphi \leftrightarrow \alpha\in U^\varphi$. By $(\varkappa, \nu, \nu, \Phi)$, there is a set $U\in\Phi$ such that $|f^*(U)| < \nu$. Let $f^*(U) = \{\varphi_\xi | \xi\in\lambda\}$ for some $\lambda < \nu$. This means nothing else but $\{\varphi | \varphi\in D \ \&\ U \cap U^\varphi \neq 0\} = \{\varphi_\xi | \xi\in\lambda\}$. So if $U_\alpha \cap U \neq 0$, then $U_\alpha \cap U^{\varphi_\xi} \neq 0$ for some $\xi\in\lambda$. But $\{\alpha | U_\alpha \cap U^{\varphi_\xi} \neq 0\} = S_{\varphi_\xi}$. Hence $\{\alpha | U_\alpha \cap U \neq 0\} \subseteq \bigcup_{\xi\in\lambda} S_{\varphi_\xi} = \Sigma$. $|\Sigma| \leqslant \lambda$ since every S_{φ_ξ} is finite. Let $\alpha_0 \in \nu - \Sigma$. Then $U_{\alpha_0} \cap U = 0$. This contradicts the hypothesis of the lemma and $U\in\Phi$. Thus there is a $\varphi\in D$ such that S_φ is infinite. We take $S = S_\varphi$ for such a φ.

Proof of Theorem 1.39. Let $\mathscr{U}$ be an arbitrary uniform ultrafilter in $\mathscr{P}(\varkappa)$, $\mathscr{U} \supseteq \Phi$. Thus for every $V\in\mathscr{U}$ and $U\in\Phi$, $V \cap U\in\mathscr{U}$. This implies $V \cap U \neq 0$. The rest now follows easily from Definition 1.38 and Lemma 1.40.

Proceeding similarly as above, we can prove e.g.

THEOREM 1.41. *Let* $(\varkappa, \omega_1, \Phi)$ *and* $\mathscr{U}$ *be a uniform ultrafilter in* $\mathscr{P}(\varkappa)$, $\Phi \subseteq \mathscr{U}$. *Let* $\{U_a | a \in \nu\} \subseteq \mathscr{U}$ *where* $\omega_0 < \nu < \varkappa$, ν regular and $\sum_{\delta<\nu} 2^\delta < \varkappa$. Then there is a set $S \subseteq \nu$, $|S| = \nu$ and $\bigcap_{a \in S} U_a \neq 0$.

Professor C. C. Chang informed me, that if GCH holds, then (ω_n, ω_1) fails for all $n \geqslant 2$, $n \in \omega_0$. This is proved in [2]. It is open, whether $(\omega_\omega, \omega_1)$ holds. We give examples of cardinals $\varkappa$ such that $cf(\varkappa) = \omega_0$ and $(\varkappa, \omega_1)$ holds. See Theorem 1.29 and Remark 1.37.1. Theorem 2.21 gives the consistency of the existence of such a cardinal, provided that it is consistent relative to ZF that there is a measurable cardinal. Compare also Theorem 2.26. No non-trivial information concerning the existence of non-regular uniform ultrafilters in $\mathscr{P}(\omega_1)$ can be deduced from the above results. We make an attempt to fill up this gap.

DEFINITION 1.42. Let Φ be a uniform filter in $\mathscr{P}(\varkappa)$. Φ is said to be *weakly normal*, if for every $f \in \varkappa^\varkappa$ such that $f(\xi) < \xi$ for all $\xi \in \varkappa$, $\xi \neq 0$, there is a set $U \in \Phi$ such that $|f^*(U)| < \varkappa$.

For a few instances of cardinals $\varkappa$ such that there is a weakly normal filter in $\mathscr{P}(\varkappa)$ see Theorems 1.3, 1.8, 1.16, 2.28. Also the $\varkappa$ discussed in Section 3 of this chapter has that property. We did not prove it, but it is easy to see on the basis of results given. The author does not know whether it is consistent relative to ZF + AC + MC that there is a weakly normal filter Φ in $\mathscr{P}(\omega_1)$.

In the next lemma we shall assume that for every $X \subseteq \varkappa$, if $|X| < \varkappa$, then $\varkappa - X \in \Phi$. This assumption could easily be avoided, but nothing substantial is lost, if we make it.

LEMMA 1.43. *Let* $\varkappa$ *be a regular cardinal,* Φ *a weakly normal filter in* $\mathscr{P}(\varkappa)$, *and* $\mathscr{I}$ *the ideal dual to* Φ. *Let* $U_a \subseteq \varkappa$, $U_a \notin \mathscr{I}$ *for all* $a \in \varkappa$. *Then there is an infinite set* $S \subseteq \varkappa$ *such that* $\bigcap_{a \in S} U_a \neq 0$.

Proof. Let $\xi_\beta = \sup(\{a | a < \beta \;\&\; \beta \in U_a\})$ for all $\beta \in \varkappa$. Hence $\xi_\beta \leqslant \beta$ for all $\beta \in \varkappa$. Let $U = \{\beta | \xi_\beta < \beta\}$. We prove that $U \notin \Phi$. Assume $U \in \Phi$. Define $f \in \varkappa^\varkappa$ by $f(\beta) = \xi_\beta$ for $\beta \in U$ and $f(\beta) = 0$ otherwise. For all $\beta \in \varkappa$, if $\beta \neq 0$, then $f(\beta) < \beta$. By weak normality, there is a set $V \in \Phi$ such that $|f^*(V)| < \varkappa$. By regularity there is an $a \in \varkappa$ such that $f^*(V) \subseteq a_0$. It is easy to see that $\{\beta | \beta \in V \cap U \;\&\; \beta > a_0\} = W \in \Phi$. Thus if $\beta \in W$, then $f(\beta) = \xi_\beta < a_0 < \beta$. By definitions of ξ_β, U and W, we conclude that if $\beta \in W$, then $\beta \notin U_{a_0}$. Hence $U_{a_0} \in \mathscr{I}$, contradiction. By $U \notin \Phi$, we get $\varkappa - U = \{\beta | \xi_\beta = \beta\} \notin \mathscr{I}$. Thus $|\varkappa - U| > 1$ and it follows from the definition of ξ_β that every element of $\varkappa - U$ is a limit ordinal. Let $\beta \in \varkappa - U$, $\beta \neq 0$. Since β is a limit ordinal and $\xi_\beta = \beta$, the set $\{a | a < \beta \;\&\; \beta \in U_a\} = S$ is infinite and $\beta \in \bigcap_{a \in S} U_a$. This proves the lemma.

THEOREM 1.44. *Let $\varkappa$, Φ be as in Lemma* 1.43. *Let $\mathscr{U}$ be a uniform ultrafilter such that $\Phi \subseteq \mathscr{U}$ and $U_\alpha \in \mathscr{U}$ for all $\alpha \in \varkappa$. Then there is an infinite set $S \subseteq \varkappa$ such that $\bigcap_{\alpha\in S} U_\alpha \neq 0$. Hence $\mathscr{U}$ is not regular.*

Proof. Proceed as in the proof of Theorem 1.39. Apply Lemma 1.43 instead of Lemma 1.40.

We state without proof the following observation. Let $\varkappa$, Φ, $\mathscr{I}$ be as in Lemma 1.43. Let $U_\alpha \subseteq \varkappa$ and $\bigcup_{\beta\in\alpha} U_\beta \in \mathscr{I}$ for all $\alpha \in \varkappa$. Then $\{\alpha | \alpha \in \bigcup_{\beta<\alpha} U_\beta\} \in \mathscr{I}$.

The following unpublished result is due to J. Silver.

THEOREM 1.45. *Let $\nu < \varkappa$, ν regular. Let $\mathscr{I}$ be a ν-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$. Let $\{X_\alpha\colon \alpha \in \nu\} \subseteq \mathscr{P}(\varkappa) - \mathscr{I}$. Then there is a set $A \subseteq \nu$, $|A| = \nu$, such that $\bigcap_{\alpha\in A} X_\alpha \neq 0$.*

A sketch of the proof. Let $\alpha_0 = 0$, $\alpha_{\xi+1} < \nu$ be the least ordinal $> \alpha_\xi$ such that $\bigcup_{\alpha_\xi \leqslant \alpha < \nu} X_\alpha - \bigcup_{\alpha_\xi \leqslant \alpha < \alpha_{\xi+1}} X_\alpha \in \mathscr{I}$, and $\alpha_\eta = \lim_{\xi<\eta} \alpha_\xi$ for η limit. The existence of $\alpha_{\xi+1}$ follows from the ν-saturatedness of $\mathscr{I}$ and regularity of ν. Let $Y_\xi = \bigcup_{\alpha_\xi \leqslant \alpha < \alpha_{\xi+1}} X_\alpha$. It is easy to see, that if $\eta < \xi$, then $Y_\xi - Y_\eta \in \mathscr{I}$ and $Y_\xi \notin \mathscr{I}$. Hence by another application of ν-saturatedness of $\mathscr{I}$ and regularity of ν, we get $\xi_0 \in \nu$ such that $Y_{\xi_0} - Y_\xi \in \mathscr{I}$ for $\xi \geqslant \xi_0$. By $\nu < \varkappa$ and $\varkappa$-completeness of $\mathscr{I}$, $Y_{\xi_0} - \bigcap_{\xi_0 \leqslant \xi} Y_\xi \in \mathscr{I}$. Thus $\bigcap_{\xi_0 \leqslant \xi} Y_\xi \notin \mathscr{I}$, since $Y_{\xi_0} \notin \mathscr{I}$. A fortiori $\bigcap_{\xi_0 \leqslant \xi} Y_\xi \neq 0$. So by a distributivity law and definition of Y_ξ there must be a family $\{X_{\beta_\xi} | \xi_0 \leqslant \xi < \nu\}$ such that $\alpha_\xi \leqslant \beta_\xi < \alpha_{\xi+1}$ and $\bigcap_{\xi\in\nu} X_{\beta_\xi} \neq 0$.

The following observation is due to J. Silver and the author.

THEOREM 1.46. *Let $\varkappa$, ν, $\mathscr{I}$ be as in Theorem* 1.45. *Let $T \subseteq \mathscr{P}(\varkappa) - \mathscr{I}$ such that for every $X, Y \in T$, if $X \neq Y$, then either $X \cap Y = 0$, or $X \subset Y$ or $Y \subset X$. Should $X \subset Y$, or $Y \subset X$ happen, then $Y - X \notin \mathscr{I}$, or $X - Y \notin \mathscr{I}$ respectively. Moreover, assume that T is well-founded with respect to reverse inclusion. Then $|T| < \nu$.*

Proof. By contradiction. Let T contain a family $S = \{X_\alpha | \alpha \in \nu\}$ of cardinality ν. By Theorem 1.45, there is a set $A \subseteq \nu$, $|A| = \nu$, and $\bigcap_{\alpha\in A} X_\alpha \neq 0$. The family $\{X_\alpha | \alpha \in A\}$ is well-ordered by reverse inclusion, since for any $\alpha, \beta \in A$, $X_\alpha \cap X_\beta \neq 0$. This leads to a disjointed family $F \subseteq \mathscr{P}(\varkappa) - \mathscr{I}$, $|F| = \nu$. A contradiction.

The author does not know, whether Theorem 1.46 holds, if the assumption of well-foundedness of T is removed. The relationship of theorem 1.45 to the problem concerning regular ultrafilters is easily seen. Compare Theorem 1.39, Lemma 1.40, Lemma 1.43 and Theorem 1.44. Theorem 1.45 will be applied in Chapter 3. The next theorem is due to A. Tarski [26].

THEOREM 1.47. *Let* $\sum_{\xi\in\nu} 2^{\xi} < \varkappa$. *Then any* $\varkappa$*-complete* ν*-saturated ideal in* $\mathscr{P}(\varkappa)$ *can be extended to a* $\varkappa$*-complete maximal ideal in* $\mathscr{P}(\varkappa)$, *i.e.* $\varkappa$ *is a measurable cardinal.*

We take the opportunity to recall once more the convention made in the introduction that by an ideal in $\mathscr{P}(\varkappa)$ we mean a non-principal ideal in $\mathscr{P}(\varkappa)$ containing all finite subsets of $\varkappa$.

CHAPTER 2

In this chapter, $\mathfrak{M}$ will be a countable transitive model of ZF, where ZF includes the axiom of choice. $\mathfrak{N}$ will always be a Cohen extension of $\mathfrak{M}$. The reasonings will be carried out in $\mathfrak{M}$ or in $\mathfrak{N}$. The words "in $\mathfrak{M}$", "in $\mathfrak{N}$", will be omitted, if no confusion can arise.

J. Silver and R. Solovay raised independently the question, whether there is a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ such that both $\mathfrak{M}$, $\mathfrak{N}$ have the same cardinals and for some cardinal $\varkappa$, $cf^{\mathfrak{M}}(\varkappa) \neq cf^{\mathfrak{N}}(\varkappa)$. In this chapter, it is proved that the answer is affirmative, if sufficiently strong conditions are imposed on $\varkappa$. The assumption $\big(\varkappa, w-R(\omega_1)\big)$ is sufficient (see Definition 1.34). For simplicity we shall use the property $\big(\varkappa, \underline{R(\omega_1)}\big)$ (Definition 1.24). Thus in particular our construction can be used to change the cofinality of a measurable cardinal, or of a cardinal $\varkappa$ such that there is an ω_1-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$. It is well known, that if $\varkappa$ satisfies one of the last two conditions, then $\varkappa$ is a regular limit cardinal ([10], [23], [24], [28]). In $\mathfrak{N}$, $\varkappa$ becomes cofinal with ω_0. It turns out that some properties which $\varkappa$ has in $\mathfrak{M}$ are transferred to $\mathfrak{N}$. For example, $\big(\varkappa, \underline{R(\omega_1)}\big)$, or still stronger statements, hold in $\mathfrak{N}$. This gives the relative consistency of such properties for singular cardinals.

We shall study the notion of forcing for a partial ordering $(\mathscr{P}, \leqslant)$ defined below. The elements of $\mathscr{P}$ will be called conditions. If p, q are conditions, we shall sometimes say p extends q instead of $p \leqslant q$. Conditions p, q are compatible, if there is a condition r such that $r \leqslant p$, $r \leqslant q$. Otherwise p, q are incompatible. A set $\mathscr{D} \subseteq \mathscr{P}$ is called a dense subset of $\mathscr{P}$, if for every $p \in \mathscr{P}$ there is a $q \leqslant p$, $q \in \mathscr{D}$. If $p \in \mathscr{P}$ and σ is a statement of the Cohen language associated with $(\mathscr{P}, \leqslant)$, then "$p \Vdash \sigma$" means "p forces σ", and "$p \parallel \sigma$" means "p decides σ", i.e. "$p \Vdash \sigma \vee p \Vdash \neg \sigma$". The forcing used will be so called "weak forcing" having the property $\neg(p \Vdash \varphi) \to (\exists q)\,(q \leqslant p \;\&\; q \Vdash \neg \varphi)$. This modification of P. Cohen's original notion is due to R. Solovay and others. It is well known that a statement φ holds in every Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$, associated with $(\mathscr{P}, \leqslant)$, iff $\{p|p \Vdash \varphi\}$ is a dense subset of $\mathscr{P}$. The set of sentences φ of the Cohen language associated with $(\mathscr{P}, \leqslant)$, for which $\{p|p \Vdash \varphi\}$ is a dense set, is a consistent set of sentences, provided the axioms of set theory are consistent. As always, the use of countable models can be avoided. However, for reasons of efficiency we shall not do so. For more details concerning our type of approach to forcing see e.g. J. Silver [21].

1. Let $\varkappa$ has the property $R(\omega_1)$ with respect to Φ as in Definition 1.24. We define a partial ordering $(\mathscr{P}, \leqslant)$ as follows:

DEFINITION 2.1. $\mathscr{P}$ is the set of ordered pairs $\langle s, A\rangle$, where $s\in[\varkappa]^{<\omega_0}$ and $A\in\Phi$. If $\langle s, A\rangle\in\mathscr{P}$, and $\langle r, B\rangle\in\mathscr{P}$, then $\langle r, B\rangle \leqslant \langle s, A\rangle$, iff s is an initial segment of r, $B\subseteq A$, and $r-s\subseteq A$.

Throughout this thesis, $\mathfrak{N}$ shall be any Cohen extension of $\mathfrak{M}$ that is associated with the $(\mathscr{P}, \leqslant)$ under discussion. In Sections 1, 2, 3 of Chapter 2, the $(\mathscr{P}, \leqslant)$ under discussion will always be the one just defined for the $\varkappa$ considered.

LEMMA 2.2. *Let $\langle s, A\rangle\in\mathscr{P}$, $\langle s, B\rangle\in\mathscr{P}$. Then $\langle s, A\rangle$, $\langle s, B\rangle$ are compatible.*

Proof. Since Φ is a filter, and $A, B\in\Phi$, we have $A\cap B\in\Phi$ as well. Thus $\langle s, A\cap B\rangle\in\mathscr{P}$. Obviously $\langle s, A\cap B\rangle\leqslant\langle s, A\rangle$, $\langle s, A\cap B\rangle\leqslant\langle s, B\rangle$.

LEMMA 2.3. *Suppose $\langle s, A\rangle\in\mathscr{P}$, $\langle r, B\rangle\in\mathscr{P}$, and $\langle s, A\rangle$, $\langle r, B\rangle$ are compatible. Then either r is an initial segment of s, or s is an initial segment of r. If r is an initial segment of s, then $s-r\subseteq B$.*

Proof. Let $\langle t, C\rangle\leqslant\langle s, A\rangle$, $\langle t, C\rangle\leqslant\langle r, B\rangle$, by compatibility. Then by Definition 2.1, r, s are initial segments of t. Hence either r is an initial segment of s, or vice versa. Let r be an initial segment of s. Then $s-r\subseteq t-r$, and $t-r\subseteq B$ by $\langle t, C\rangle\leqslant\langle r, B\rangle$. Thus $s-r\subseteq B$.

LEMMA 2.4. *Let σ be a statement of the Cohen language associated with $(\mathscr{P}, \leqslant)$. Let $p, q\in\mathscr{P}$ be compatible, $p\Vdash\sigma$ and $q\|\sigma$. Then $q\Vdash\sigma$.*

Proof. If $q\Vdash\neg\sigma$ and $q'\leqslant p$, $q'\leqslant q$, we get $q'\Vdash\sigma$, $q'\Vdash\neg\sigma$. Contradiction. Hence $q\Vdash\sigma$.

Instead of writting $\langle 0, \varkappa\rangle\Vdash\sigma$, we write $\Vdash\sigma$. This is the same as to say, for every $p\in\mathscr{P}$, $p\Vdash\sigma$.

LEMMA 2.5. *Let $a<\varkappa$ and a be a cardinal in $\mathfrak{M}$. Then a is a cardinal in any Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$, associated with $(\mathscr{P}, \leqslant)$.*

Proof. It is sufficient to prove the claim under the assumption that a is regular, because the limit of a sequence of cardinals is a cardinal. Let β be a cardinal, $\beta<a$. Let τ be a term of the Cohen language such that $\Vdash\tau\colon\beta\to a$. Let $\langle s, A\rangle\in\mathscr{P}$, $\langle s, A'\rangle\in\mathscr{P}$, $\xi\in\beta$, $\eta\in a$, $\eta'\in a$, $\langle s, A\rangle\Vdash\tau(\xi)=\eta$, and $\langle s, A'\rangle\Vdash\tau(\xi)=\eta'$. Since $\Vdash$ (τ is a function), we get by Lemmas 2.2, 2.4, $\eta=\eta'$. Thus for every $s\in[\varkappa]^{<\omega_0}$, the set

$$E_s=\{\eta\,|\,(\exists\xi)\,(\exists A)\,(\xi\in\beta\ \&\ A\in\Phi\ \&\ \langle s, A\rangle\Vdash\tau(\xi)=\eta)\}$$

has cardinality $\leqslant\beta$, because every $\xi\in\beta$ contributes at most one η. Also $E_s\subseteq a$ (by $\Vdash\tau\colon\beta\to a$). Therefore by the regularity of a, $\sup E_s<a$.

We put $\tilde{\tau}(s) = \sup E_s$ for every $s \in [\varkappa]^{<\omega_0}$. Since $a < \varkappa$ and $\tilde{\tau}\colon [\varkappa]^{<\omega_0} \to a$, by $(\varkappa, R(\omega_1), \Phi)$ there is a set $U \in \Phi$ such that $|\tilde{\tau}^*([U]^{<\omega_0})| \leqslant \aleph_0$. Let $\eta_0 = \sup \tilde{\tau}^*([U]^{<\omega_0})+1$. Again since a is regular, $\eta_0 < a$. It is claimed that $\langle 0, U\rangle \Vdash \tau^*(\beta) \subseteq \eta_0$. Let $\langle s, A\rangle \leqslant \langle 0, U\rangle$, $\xi \in \beta$ and $\langle s, A\rangle \Vdash \tau(\xi) = \eta$. Thus $\eta \in E_s$ and therefore $\eta \leqslant \sup E_s = \tilde{\tau}(s)$. By $\langle s, A\rangle \leqslant \langle 0, U\rangle$ we get $s \in [U]^{<\omega_0}$. Thus $\tilde{\tau}(s) < \eta_0$ by the definition of η_0 which gives $\eta < \eta_0$. Thus no extension of $\langle 0, U\rangle$ can force $(\exists \eta \geqslant \eta_0)\big(\eta \in \tau^*(\beta)\big)$. This proves the claim.

However, a completely analogous argument shows that for every $p \in \mathscr{P}$, we can find a $q \in \mathscr{P}$, $q \leqslant p$, and an ordinal $\eta_q \in a$ such that $q \Vdash \tau^*(\beta) \subseteq \eta_q$. For an arbitrary $p \in \mathscr{P}$, we have $p = \langle r, B\rangle$ for some $r \in [\varkappa]^{<\omega_0}$, $B \in \Phi$. Let $B' = \{\varphi | \varphi \in B \;\&\; \varphi > \max r\}$. Thus $\langle r, B'\rangle \leqslant p$. We would consider conditions of the form $\langle \widehat{r\,s}, A\rangle$ where $s \in [B']^{<\omega_0}$ and $A \subseteq B'$. $\tilde{\tau}$ would be defined on $[B']^{<\omega_0}$. Using the remark following Definition 1.24, we can get a set $U \subseteq B'$, $U \in \Phi$ such that $|\tau^*([U]^{<\omega_0})| \leqslant \aleph_0$ and so on. Thus we have shown that the set of conditions q which force an ordinal $\eta_q \in a$ to be an upper bound of $\tau^*(\beta)$ is a dense set of conditions. Since τ was an arbitrary term such that $\Vdash \tau\colon \beta \to a$, there can be no map of β onto a in the extension. This is true for every cardinal $\beta < a$. Thus a is a cardinal in the extension.

LEMMA 2.6. *$(\mathscr{P}, \leqslant)$ satisfies the $\varkappa$-chain condition, i.e. every set S of mutually incompatible elements of $\mathscr{P}$ has cardinality at most $\varkappa$.*

Proof. By Lemma 2.2, for every $s \in [\varkappa]^{<\omega_0}$ and every $A, B \in \Phi$, $\langle s, A\rangle$, $\langle s, B\rangle$ are compatible. Hence S can contain at most one condition $\langle s, X\rangle$ for every $s \in [\varkappa]^{<\omega_0}$. But $|[\varkappa]^{<\omega_0}| = \varkappa$.

LEMMA 2.7. *If $(\mathscr{P}, \leqslant)$ satisfies the $\varkappa$-chain condition, then all cardinals $\lambda > \varkappa$ remain cardinals in $\mathfrak{N}$.*

Proof. This is well known. The proof is analogous to the one in P. Cohen ([4], Theorem 2, p. 132). Compare also [15].

DEFINITION 2.8. If $p \in \mathscr{P}$, $p = \langle s, A\rangle$, we put $s_p = s$.

LEMMA 2.9. *For every $a \in \varkappa$,*

$$\{q | (\exists \beta)\, (\beta \in \varkappa \;\&\; \beta \geqslant a \;\&\; \beta \in s_q)\}$$

is a dense subset of $\mathscr{P}$.

Proof. Let $p \in \mathscr{P}$. Thus $p = \langle s, A\rangle$ for some $s \in [\varkappa]^{<\omega_0}$ and $A \in \Phi$. Since Φ is a uniform filter, $|A| = \varkappa$. Let $\beta \in A$, $\beta > \max(a, \max s)$. Put $q = \langle \widehat{s\,\beta}, A\rangle$. Then $q \leqslant \langle s, A\rangle$, $\beta \in s_q$ and $\beta > a$. This proves the lemma.

LEMMA 2.10. $|\mathscr{P}| \leqslant 2^{\varkappa}$.

Proof. This follows easily from Definition 2.1. From Lemmas 2.6, 2.10 and from some results in [15] follows, that if $\lambda \geqslant \varkappa$, then $2^{\lambda} = \aleph_a$ in $\mathfrak{N}$ iff $2^{\lambda} = \aleph_a$ in $\mathfrak{M}$. Therefore in particular, if $2^{\lambda} = \lambda^+$ holds in $\mathfrak{M}$ for all $\lambda \geqslant \varkappa$, then the same remains true in $\mathfrak{N}$.

LEMMA 2.11. *There is an increasing sequence of ordinals* $\varrho = \{\varrho_n\}_{n\in\omega_0} \in \mathfrak{N}$ *such that* $\lim\limits_{n\in\omega_0} \varrho_n = \varkappa$, *and* $\varrho \notin \mathfrak{M}$.

Proof. $\mathfrak{N}$ is obtained by adjoining a set $\mathcal{T}$ of true conditions to $\mathfrak{M}$. $\mathcal{T}$ has the following properties:

(1) $(\forall p, q \in \mathcal{T})\,(p, q$ are compatible$)$,

(2) $(\forall \mathcal{D})\,(\mathcal{D}$ is a dense subset of $\mathcal{P} \to (\exists p \in \mathcal{D})\,(p \in \mathcal{T}))$,

(3) $(\forall p, q)\,(p \in \mathcal{T} \;\&\; q \geqslant p \to q \in \mathcal{T})$.

Let $t = \bigcup\{s_p | p \in \mathcal{T}\}$. By Lemma 2.9 and (2),

$$(*) \qquad (\forall \alpha \in \varkappa)\,(\exists \beta \geqslant \alpha)\,(\beta \in \varkappa \cap t).$$

It is claimed that $t_\alpha = \{\beta | \beta \leqslant \alpha \;\&\; \beta \in t\}$ is finite for every $\alpha \in t$. For some $p_0 \in \mathcal{T}$, $\alpha \in s_{p_0}$. Let $q \in \mathcal{T}$ arbitrary. By (1), p_0, q are compatible. By Lemma 2.3, either s_{p_0} is an initial segment of s_q, or s_q is an initial segment of s_{p_0}. Hence in any case, $s_q \cap t_\alpha \subseteq s_{p_0} \cap t_\alpha$. Thus $t_\alpha = \{\beta | \beta \leqslant \alpha \;\&\; \beta \in s_{p_0}\}$. Since s_{p_0} is finite, t_α must be finite. This proves the claim. So there is a sequence $\varrho = \{\varrho_n\}_{n\in\omega_0}$ such that for all $n \in \omega_0$, $\varrho_n < \varrho_{n+1}$, and $t = \{\varrho_n | n \in \omega_0\}$. By $(*)$, $\lim\limits_{n\in\omega_0} \varrho_n = \varkappa$. It remains to show that $\varrho \notin \mathfrak{M}$. If $\varrho' \in (\varkappa^{\omega_0})^{\mathfrak{M}}$, we put $\mathcal{D}_{\varrho'} = \{q | s_q$ is not an initial segment of $\varrho'\}$. It is claimed that $\mathcal{D}_{\varrho'}$, is a dense set of conditions. Let $p = \langle s, U\rangle$. Let $\alpha \in U$, $\alpha > \max s$, and α be different from all elements of ϱ'. Such an α exists, since s is finite, ϱ' is a countable sequence, and $|U| = \varkappa$ by $U \in \Phi$. Then obviously $\langle \widehat{s\,\alpha}, U\rangle \in \mathcal{D}_{\varrho'}$. This proves the claim. By (2), there is a $p \in \mathcal{D}_{\varrho'} \cap \mathcal{T}$ for every $\varrho' \in (\varkappa^{\omega_0})^{\mathfrak{M}}$. But it follows easily from the previous part of the proof that s_p is an initial segment of ϱ for every $p \in \mathcal{T}$. Thus $\varrho \neq \varrho'$ for every $\varrho' \in (\varkappa^{\omega_0})^{\mathfrak{M}}$. This proves the lemma.

THEOREM 2.12. *Cardinals of* $\mathfrak{N}$ *are the same as cardinals of* $\mathfrak{M}$. *Moreover* $cf^{\mathfrak{N}}(\varkappa) = \omega_0$.

Proof. For cardinals $< \varkappa$ or $> \varkappa$ this follows from Lemmas 2.5, 2.6, 2.7. $cf^{\mathfrak{N}}(\varkappa) = \omega_0$ follows from Lemma 2.11. That $\varkappa$ itself remains a cardinal in $\mathfrak{N}$ would be clear, if $\varkappa$ is a limit cardinal in $\mathfrak{M}$. This however follows from Proposition 1.23 and $(\varkappa, R(\omega_1))$ in $\mathfrak{M}$.

Let $\Phi' = \{V | V \in \mathcal{P}^{\mathfrak{N}}(\varkappa) \;\&\; (\exists U)\,(U \in \Phi \;\&\; U \subseteq V)\}$.

THEOREM 2.13. $(\varkappa, R(\omega_1), \Phi')$ holds in $\mathfrak{N}$.

Proof. Let $\lambda < \varkappa$ and $\Vdash \tau\colon [\varkappa]^{<\omega_0} \to \lambda$. Similarly as in the proof of Lemma 2.5, we observe that if for some $s \in [\varkappa]^{<\omega_0}$, $h \in [\varkappa]^{<\omega_0}$, A, A', η, η', $\langle s, A\rangle \Vdash \tau(h) = \eta$ and $\langle s, A'\rangle \Vdash \tau(h) = \eta$, then $\eta = \eta'$. Hence we can define a function $\tilde{\tau}\colon ([\varkappa]^{<\omega_0} \times [\varkappa]^{<\omega_0}) \to \lambda$ as follows:

$$\tilde{\tau}(\langle s, h\rangle) = 0, \text{ if for no } A \in \Phi \text{ and no } \eta \in \lambda,\ \langle s, A\rangle \Vdash \tau(h) = \eta;$$

$$\tilde{\tau}(\langle s, h\rangle) = \eta, \quad \text{if for some} \quad A \in \Phi,\ \langle s, A\rangle \Vdash \tau(h) = \eta.$$

By $(\varkappa, R(\omega_1), \Phi)$ and Lemma 1.32, there is a set $U \in \Phi$ such that $|\tilde{\tau}^*[U]^{<\omega_0} \times \times [U]^{<\omega_0})| \leqslant \aleph_0$. It is claimed that $\langle 0, U\rangle \Vdash |\tilde{\tau}^*([U]^{<\omega_0})| \leqslant \aleph_0$. In fact, let $\langle s, V\rangle \leqslant \langle 0, U\rangle$, $h \in [U]^{<\omega_0}$, and $\langle s, V\rangle \Vdash \tau(h) = \eta$. Thus $\tilde{\tau}(\langle s, h\rangle) = \eta$. By $\langle s, V\rangle \leqslant \langle 0, U\rangle$, we get $s \in [U]^{<\omega_0}$. Therefore $\eta \in \tilde{\tau}^*([U]^{<\omega_0} \times [U]^{<\omega_0}) = S$. So $\langle 0, U\rangle \Vdash \tau^*([U]^{<\omega_0}) \subseteq S$. But $|S| \leqslant \aleph_0$ in $\mathfrak{M}$. This proves the claim. A remark like that in Lemma 2.5, concerning arbitrary conditions, completes the proof.

2. To prove our next result, we shall assume that Φ is $\varkappa$-complete. It follows (see Remark 1.37.3 and Theorem 1.47) that $2^{\aleph_0} \geqslant \varkappa$, or $\varkappa$ is a measurable cardinal.

THEOREM 2.14. *Let Φ be $\varkappa$-complete. Let $\nu < \varkappa$ and $\{a_\xi | \xi \in \nu\} = E$ be a set of ordinals in $\mathfrak{N}$. Then there are sets E_n, $n \in \omega_0$ such that $E_n \in \mathfrak{M}$ for all $n \in \omega_0$ and $\bigcup_{n \in \omega_0} E_n = E$.*

Theorem 2.14. has the following corollaries:

COROLLARY 2.15. *Let E be a set of ordinals in $\mathfrak{N}$, $|E| = \varkappa$. Then there are sets E_n, $n \in \omega_0$, such that $\bigcup_{n \in \omega_0} E_n = E$ and $E_n \in \mathfrak{M}$ for all $n \in \omega_0$.*

This follows easily from Theorem 2.14 and $cf^{\mathfrak{N}}(\varkappa) = \omega_0$.

COROLLARY 2.16. *Let $E \subseteq \mathfrak{M}$, $E \in \mathfrak{N}$, $|E| \leqslant \varkappa$. Then there are sets $E_n \in \mathfrak{M}$, $n \in \omega_0$ such that $E = \bigcup_{n \in \omega_0} E_n$.*

Proof. By the $\varkappa$-chain condition for $\mathscr{P}$ (see Lemma 2.6), there is a set $F \supseteq E$, $F \in \mathfrak{M}$, $|F| \leqslant \varkappa$. F can be well-ordered in $\mathfrak{M}$ in type $\leqslant \varkappa$. We apply Corollary 2.15 or Theorem 2.14, according as $|E| = \varkappa$ or $|E| < \varkappa$, to the set S of indices of elements of E in that well-ordering. In this way S is decomposed into countably many sets $S_n \in \mathfrak{M}$. Hence the corresponding sets $E_n \subseteq E$ are in $\mathfrak{M}$, since the whole well-ordering and S_n are both in $\mathfrak{M}$.

DEFINITION 2.17. Let ν, λ be cardinals and τ a term such that $\Vdash \tau: \nu \to \lambda$. Let $\langle s, V\rangle \in \mathscr{P}$ be such that for all $W \in \Phi$, all $\xi \in \nu$ and all $a \in \lambda$,

$$\big(\langle s, W\rangle \Vdash \tau(\xi) = a\big) \to \big(\langle s, V\rangle \Vdash \tau(\xi) = a\big).$$

Then we say that $\langle s, V\rangle$ is *full with respect to* τ.

LEMMA 2.18. *Let ν, λ, τ be as in Definition 2.17 and $\nu < \varkappa$. Let $\langle s, U\rangle \in \mathscr{P}$. Then there is a condition $\langle s, V\rangle \leqslant \langle s, U\rangle$ such that $\langle s, V\rangle$ is full with respect to τ.*

Proof. Let $S = \{\xi \in \nu | (\exists W \in \Phi, a \in \lambda)\, (\langle s, W\rangle \Vdash \tau(\xi) = a)\}$. For every $\xi \in S$, let V_ξ, a_ξ be such that $\langle s, V_\xi\rangle \Vdash \tau(\xi) = a_\xi$. Let $V = U \cap \bigcap_{\xi \in S} V_\xi$. By $\varkappa$-completeness of Φ, $V \in \Phi$. $\langle s, V\rangle \leqslant \langle s, U\rangle$ and $\langle s, V\rangle \leqslant \langle s, V_\xi\rangle$ for $\xi \in S$. Hence, for all $\xi \in S$, $\langle s, V\rangle \Vdash \tau_\xi = a_\xi$. By the definition of S, if $\xi \notin S$, then for all $W \in \Phi$ and all $a \in \lambda$, $\neg\big(\langle s, W\rangle \Vdash \tau(\xi) = a\big)$. If $\xi \in S$ and

$\langle s, W\rangle \Vdash \tau(\xi) = a$, then $\langle s, W \cap V\rangle \Vdash \big(\tau(\xi) = a_\xi \mathbin{\&} \tau(\xi) = a\big)$. Thus $a_\xi = a$. This shows that $\langle s, V\rangle$ is full with respect to τ.

LEMMA 2.19. *For every $n \in \omega_0$, the set*

$$\mathscr{D}_n = \{\langle s, V\rangle \mid |s| \geqslant n \mathbin{\&} \langle s, V\rangle \text{ is full with respect to } \tau\}$$

is a dense subset of $\mathscr{P}$.

Proof. This is an obvious consequence of Lemma 2.18 and of the definition of a dense set of conditions.

Proof of Theorem 2.14. The set E is the range of a one-to-one function $f\colon \nu \to \lambda$ for some cardinal λ. Let τ be a term denoting f in $\mathfrak{N}$. Let $\mathscr{T}$ be as in Lemma 2.11. By property (2) of $\mathscr{T}$ and by Lemma 2.19, there is an $\langle s_n, V_n\rangle \in \mathscr{D}_n \cap \mathscr{T}$ for every $n \in \omega_0$. Let

$$E_n = \big\{a \mid (\exists \xi \in \nu)\big(\langle s_n, V_n\rangle \Vdash \tau(\xi) = a\big)\big\}.$$

Since the notion of forcing is expressible in $\mathfrak{M}$, we get $E_n \in \mathfrak{M}$ for all $n \in \omega_0$. It remains to prove that for every $a \in E$ there is an $n \in \omega_0$ such that $a \in E_n$. Let $a \in E = f^*(\nu)$. Hence for some $\xi \in \nu$, $a = f(\xi)$ holds in $\mathfrak{N}$. Therefore for some $\langle s, W\rangle \in \mathscr{T}$, $\langle s, W\rangle \Vdash \tau(\xi) = a$. Let $n \in \omega_0$, $n \geqslant |s|$. Since $\langle s_n, V_n\rangle$, $\langle s, W\rangle$ are both in $\mathscr{T}$, they must be compatible by (1), Lemma 2.11. Thus by Lemma 2.3, we get $s_n - s \subseteq W$ which gives $\langle s_n, W\rangle \leqslant \langle s, W\rangle$ (see Definition 2.1). Hence $\langle s_n, W\rangle \Vdash \tau(\xi) = a$. Now $\langle s_n, V_n\rangle$ is full with respect to τ. We conclude $\langle s_n, V_n\rangle \Vdash \tau(\xi) = a$, i.e. $a \in E_n$. This completes the proof of Theorem 2.14.

COROLLARY 2.20. *Let $U_\xi \in \Phi$ for all $\xi \in \nu \in \varkappa$, $cf(\nu) > \omega_0$ and $\{U_\xi\}_{\xi\in\nu} \in \mathfrak{N}$. Then there is a set $S \subseteq \nu$ such that $|S| = \nu$ and $\bigcap_{\xi\in S} U_\xi \in \Phi$.*

Proof. By Corollary 2.16, $\{U_\xi \mid \xi \in \nu\} = \bigcup_{n\in\omega_0} E_n$ and $E_n \in \mathfrak{M}$ for all $n \in \omega_0$. Hence, for some $n_0 \in \omega_0$, $|E_{n_0}| = \nu$. Since $E_{n_0} \in \mathfrak{M}$ and Φ is closed under intersections of cardinality $< \varkappa$ in $\mathfrak{M}$, we get $\bigcap E_{n_0} \in \Phi$. Take $S = \{\xi \mid U_\xi \in E_{n_0}\}$.

Another result of this type is

THEOREM 2.21. *Let $\mathscr{U}$ be an arbitrary ultrafilter in $\mathscr{P}^{\mathfrak{N}}(\varkappa)$ such that $\Phi \subseteq \mathscr{U} \in \mathfrak{N}$. Then $\mathscr{U}$ is non-regular.*

Proof. This follows from Theorems 2.13 and 1.39.

3. In this section, unless specified otherwise, we shall assume that $\varkappa$ is measurable and $\Phi = \{U \mid \mu(U) = 1\}$, where μ is a fixed normal measure on $\mathscr{P}^{\mathfrak{M}}(\varkappa)$. Hence $\big(\varkappa, R(\omega_1), \Phi\big)$ holds by Theorem 1.9 and all results proved thus far about $(\mathscr{P}, \leqslant)$ continue to hold.

LEMMA 2.22. *Let σ be a statement of the Cohen language associated with $(\mathscr{P}, \leqslant)$. Then there is a condition $\langle 0, A\rangle$ such that $\langle 0, A\rangle \| \sigma$.*

Proof. Put

$$S_0 = \{s | s \in [\varkappa]^{<\omega_0} \,\&\, (\exists U)(\langle s, U\rangle \Vdash \neg\sigma)\},$$
$$S_1 = \{s | s \in [\varkappa]^{<\omega_0} \,\&\, (\exists U)(\langle s, U\rangle \Vdash \sigma)\},$$
$$S_2 = [\varkappa]^{<\omega_0} - (S_0 \cup S_1).$$

It is an immediate consequence of Lemmas 2.2, 2.4 that $S_0 \cap S_1 = 0$. Thus $S_i \cap S_j = 0$ for $i \neq j$, $i, j = 0, 1, 2$. By Theorem 1.9, there is a set $A \in \Phi$ such that for every $n \in \omega_0$ there is an $i = 0, 1$, or 2 such that $[A]^n \subseteq S_i$. It is claimed that $\langle 0, A\rangle \| \sigma$, i.e. $\langle 0, A\rangle \Vdash \sigma$, or $\langle 0, A\rangle \Vdash \neg\sigma$. Assume not. Then there are conditions $\langle r, U\rangle \leqslant \langle 0, A\rangle$, $\langle s, V\rangle \leqslant \langle 0, A\rangle$ such that $\langle r, U\rangle \Vdash \neg\sigma$ and $\langle s, V\rangle \Vdash \sigma$. We can assume that $|r| = |s|$. Since if for example $|r| < |s|$, we could replace $\langle r, U\rangle$ by a condition $\langle r', U'\rangle$ such that $\langle r', U'\rangle \leqslant \langle r, U\rangle$ and $|r'| = |s|$. By $\langle r, U\rangle \leqslant \langle 0, A\rangle$, we get $r \in [A]^{<\omega_0}$. Similarly $s \in [A]^{<\omega_0}$. Hence $r, s \in [A]^n$ for some $n \in \omega_0$. Therefore $[A]^n \cap S_0 \neq 0$, $[A]^n \cap S_1 \neq 0$. Thus by the choice of A, $[A]^n \subseteq S_0$, $[A]^n \subseteq S_1$. But $[A]^n \neq 0$, and $S_0 \cap S_1 = 0$, which is a contradiction. This proves the lemma.

Lemma 2.23. *Let $\lambda < \varkappa$ and $\Vdash \tau \subseteq \lambda$. Then there is a condition $\langle 0, A\rangle$ and a set $a \subseteq \lambda$ such that $\langle 0, A\rangle \Vdash \tau = a$.*

Proof. Let $\alpha \in \lambda$ arbitrary. By Lemma 2.22 there is a condition $\langle 0, A_\alpha\rangle$ such that $\langle 0, A_\alpha\rangle \Vdash \alpha \in \tau$ or $\langle 0, A_\alpha\rangle \Vdash \alpha \notin \tau$. Let $A = \bigcap_{\alpha \in \lambda} A_\alpha$. We have $\mu(A) = 1$, since $\mu(A_\alpha) = 1$ for every $\alpha \in \lambda$ and μ is $\varkappa$-addit. Thus $\langle 0, A\rangle \in \mathscr{P}$ and $\langle 0 A\rangle \leqslant \langle 0, A_\alpha\rangle$ for every $\alpha \in \lambda$. Therefore $\langle 0, A\rangle \Vdash \alpha \in \tau$ iff $\langle 0, A_\alpha\rangle \Vdash \alpha \in \tau$, and $\langle 0, A\rangle \Vdash \alpha \notin \tau$ iff $\langle 0, A_\alpha\rangle \Vdash \alpha \notin \tau$, for every $\alpha \in \lambda$. It follows that $\langle 0, A\rangle \Vdash \tau = a$, where $a = \{\alpha | \langle 0, A\rangle \Vdash \alpha \in \tau\}$.

Proceeding by analogy with Lemmas 2.22, 2.23 (compare also Lemma 2.5), we can establish

Lemma 2.24. *Let λ, τ be as in Lemma 2.23 and $\langle s, A\rangle \in \mathscr{P}$. Then there is a condition $\langle s, B\rangle$, $\langle s, B\rangle \leqslant \langle s, A\rangle$ and a set $a \subseteq \lambda$ such that $\langle s, B\rangle \Vdash \tau = a$.*

Theorem 2.25. *Let $\lambda < \varkappa$ and $a \in \mathscr{P}^{\mathfrak{N}}(\lambda)$. Then $a \in \mathscr{P}^{\mathfrak{M}}(\lambda)$.*

Proof. This is an immediate consequence of Lemma 2.24.

The next theorem gives a normality property of Φ in $\mathfrak{N}$.

Theorem 2.26. *The following statement holds in $\mathfrak{N}$: For every $f \in \varkappa^\varkappa$ such that $f(\eta) < \eta$ for all $\eta \in \varkappa$, $\eta \neq 0$, there is a set $U \in \Phi$ such that $|f^*(U)| \leqslant \aleph_0$.*

Remark. Let $\Phi' = \{V | V \in \mathscr{P}^{\mathfrak{N}}(\varkappa) \,\&\, (\exists U \in \Phi)(U \subseteq V)\}$. It follows that Φ' is a weakly normal filter in $\mathscr{P}^{\mathfrak{N}}(\varkappa)$ in $\mathfrak{N}$. For definition of a w. norm. filter see Definition 1.42.

Proof of Theorem 2.26. We shall need some auxiliary notions. Let A be the family of all finite structures $\mathfrak{A} = \langle a, \varepsilon_a, c, d\rangle$, where ε_a

is the membership relation ε on ω_0 restricted to $a, c \epsilon a, d \epsilon a$ and $d \epsilon c$. Obviously $|A| = \aleph_0$. We could identify isomorphic elements of A, but this is not absolutely necessary for our purpose. Let τ be a term such that

$$\Vdash \big(\tau\colon \varkappa \to \varkappa \; \& \; (\forall \eta \epsilon \varkappa)\big(\tau(\eta) < \eta \vee \eta = 0\big)\big).$$

We define sets $E^{\mathfrak{A}} \subseteq [\varkappa]^{<\omega_0}$ for $\mathfrak{A} \epsilon A$ as follows:

$s \epsilon E^{\mathfrak{A}}$

$\leftrightarrow (\exists \eta \epsilon s)\,(\exists a \notin s)\,(\exists U)\,\big[\big(\langle s - \{\eta\}, U\rangle \Vdash \tau(\eta) = a\big) \;\&\; \langle s \cup \{a\}, \varepsilon, \eta, a\rangle \cong \mathfrak{A}\big]$.

If $\mathfrak{A}$ is given and $s \epsilon E^{\mathfrak{A}}$, then η, a as above are uniquely determined. Indeed, there is at most one ε-preserving isomorphism between $a - \{d\}$ and s. η is the image of c under this isomorphism. The existence of a unique a now follows by $\Vdash$ (τ is a function) and Lemma 2.2, much as in Theorem 2.13. This unique η will be denoted by $\eta_s^{\mathfrak{A}}$, and the unique a, $a_s^{\mathfrak{A}}$. We put also $(s)_{\mathfrak{A}} = \{\xi | \xi \epsilon s \;\&\; \xi < a_s^{\mathfrak{A}}\}$, $(s)^{\mathfrak{A}} = s - (s)_{\mathfrak{A}}$. Clearly, if $s, s' \epsilon E^{\mathfrak{A}}$, then $|s| = |s'|$, $|(s)_{\mathfrak{A}}| = |(s')_{\mathfrak{A}}|$ and $|(s)^{\mathfrak{A}}| = |(s')^{\mathfrak{A}}|$. In fact, $|s| = |a| - 1$ for all $s \epsilon E^{\mathfrak{A}}$. By Theorem 1.9 there is a set $U_{\mathfrak{A}} \epsilon \mathscr{P}(\varkappa)$ such that $\mu(U) = 1$, and either $[U_{\mathfrak{A}}]^{|a|-1} \subseteq E^{\mathfrak{A}}$, or $[U_{\mathfrak{A}}]^{|a|-1} \cap E^{\mathfrak{A}} = 0$. Assume first that

(*) $$[U_{\mathfrak{A}}]^{|a|-1} \subseteq E^{\mathfrak{A}}.$$

There are two cases:

Case I. $|(s)_{\mathfrak{A}}| = 0$ for some (and hence all) $s \epsilon E^{\mathfrak{A}}$. Then $(s)^{\mathfrak{A}} = s$ and $a_s^{\mathfrak{A}} < \min s$ for all $s \epsilon E^{\mathfrak{A}}$. By Lemma 1.7 there is a set $V_{\mathfrak{A}} \subseteq U_{\mathfrak{A}}$ such that $\mu(V_{\mathfrak{A}}) = 1$ and $a_s^{\mathfrak{A}} = a_{s'}^{\mathfrak{A}} = a^{\mathfrak{A}}$ for all $s, s' \epsilon [V_{\mathfrak{A}}]^{|a|-1}$ and some fixed $a^{\mathfrak{A}}$.

Case II. $|(s)_{\mathfrak{A}}| = n_{\mathfrak{A}} > 0$ for some (and hence all) $s \epsilon E^{\mathfrak{A}}$. We know that $|(s)^{\mathfrak{A}}| = n^{\mathfrak{A}} > 0$ by $\eta_s^{\mathfrak{A}} > a_s^{\mathfrak{A}}$. Thus for every $s \epsilon [U_{\mathfrak{A}}]^{|a|-1}$, $\max s | n_{\mathfrak{A}} < a_s < \min(s - s | n_{\mathfrak{A}})$.

By Lemma 1.10, there is a set $V_{\mathfrak{A}} \subseteq U_{\mathfrak{A}}$ such that $\mu(V_{\mathfrak{A}}) = 1$ and, for every $r \epsilon [V_{\mathfrak{A}}]^{n_{\mathfrak{A}}}$ and every $t', t'' \epsilon [\{\xi | \xi > \max r \;\&\; \xi \epsilon V_{\mathfrak{A}}\}]^{n^{\mathfrak{A}}}$, $a_{r \cup t'}^{\mathfrak{A}} = a_{r \cup t''}^{\mathfrak{A}} = a_r^{\mathfrak{A}}$.

If $[U_{\mathfrak{A}}]^{|a|-1} \cap E^{\mathfrak{A}} = 0$, put $V_{\mathfrak{A}} = U_{\mathfrak{A}}$. Let $V = \bigcap_{\mathfrak{A} \epsilon A} V_{\mathfrak{A}}$; $\mu(V) = 1$. It is claimed that $\langle 0, V\rangle \Vdash |\tau^*(V)| \leqslant \aleph_0$. It is enough to prove that if $\langle 0, V\rangle \epsilon \mathscr{T}$ (see Lemma 2.11) and τ denotes g in $\mathfrak{N}$, then $|g^*(V)| \leqslant \aleph_0$. Let t be as in Lemma 2.11. Hence

$$\{t|n\colon n \epsilon \omega_0\} = \{t_n | n \epsilon \omega_0 \;\&\; t_n = t|n\} = \{s | (\exists U)\,(\langle s, U\rangle \epsilon \mathscr{T}\}.$$

If $\eta \epsilon V$ and $g(\eta) = a$, then there is a condition $\langle t_n, U\rangle \epsilon \mathscr{T}$ such that $\langle t_n, U\rangle \Vdash \tau(\eta) = a$. Since $\langle t_n, U\rangle \epsilon \mathscr{T}$ and $\langle 0, V\rangle \epsilon \mathscr{T}$, $\langle t_n, U\rangle$, $\langle 0, V\rangle$ must be compatible by Lemma 2.11, (1). Thus by Lemma 2.3, $t_n \epsilon [V]^n$. Assume that $\eta \notin t_n$, $a \notin t_n$. Let $s = t_n \cup \{\eta\}$. For some $\mathfrak{A} \epsilon A$, $\langle s \cup \{a\}, \varepsilon, \eta, a\rangle \cong \mathfrak{A}$

$= \langle a, \varepsilon_a, c, d\rangle$. We have $|s| = n+1 = |a|-1$. So $s \in [V]^{n+1} \subseteq [V_{\mathfrak{A}}]^{|a|-1}$. Obviously $s \in E^{\mathfrak{A}}$. It follows that $(*)$ happened for $U_{\mathfrak{A}}$. By the choice of $V_{\mathfrak{A}}$ and cases I, II, either $\alpha = a^{\mathfrak{A}}$, or $\alpha = a^{\mathfrak{A}}_{s|n_{\mathfrak{A}}}$. Consider case II, i.e. $\eta = a^{\mathfrak{A}}_{s|n_{\mathfrak{A}}}$. We have $\eta > \alpha > \max s|\ n_{\mathfrak{A}}$. This, together with $s = t_n \cup \{\eta\}$, gives $t_n|n_{\mathfrak{A}} = s|n_{\mathfrak{A}}$. Hence $\alpha = a^{\mathfrak{A}}_{t_n|n_{\mathfrak{A}}}$. It follows that

$$g^*(V) \subseteq \{a^{\mathfrak{A}}|\mathfrak{A} \in A\} \cup \{a^{\mathfrak{A}}_{t_n|n_{\mathfrak{A}}}|\mathfrak{A} \in A \ \&\ n \in \omega_0\} \cup \\ \cup \{g(\eta)|(\exists n \in \omega_0)\,(\eta \in t_n)\} \cup \{a|(\exists n \in \omega_0)\,(a \in t_n)\}.$$

Thus $g^*(V) \leqslant \aleph_0$. A remark concerning arbitrary conditions, as in Lemma 2.5, has to be made.

An analogous, but somewhat more complicated proof shows

THEOREM 2.27. *Let* $f\colon [\varkappa]^{<\omega_0} \to \varkappa$ *be such that* $f(s) < \min s$, *or* $\min s = 0$, $f \in \mathfrak{N}$. *Then there is a set* U *such that* $\mu(U) = 1$ *and* $|f^*([U]^{<\omega_0})| \leqslant \aleph_0$.

Theorem 2.27 remains true, if measurability of $\varkappa$ is replaced by the existence of a $\varkappa$-comp. ω_1-sat. ideal $\mathscr{I}$ in $\mathscr{P}(\varkappa)$ and $\mu(U) = 1$ by $U \in \Phi$, where Φ is dual to $\mathscr{I}$.

Theorem 2.25 implies that 2^{λ} remains the same aleph in $\mathfrak{N}$ as it was in $\mathfrak{M}$ for all $\lambda < \varkappa$. Combining this with the remark following Lemma 2.10, we see that if GCH holds in $\mathfrak{M}$, then it holds in $\mathfrak{N}$. This gives, together with Theorems 2.12, 2.13: If ZF + GCH + "$\varkappa$ is meas." holds in $\mathfrak{M}$, then ZF + GCH + $(\exists\varkappa)\ \big((\varkappa, \underline{R(\omega_1)}) \ \&\ cf(\varkappa) = \omega_0\big)$ holds in $\mathfrak{N}$. Thus by well known methods, we obtain

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{MC} + \mathrm{GCH}) \to \mathrm{Con}\Big(\mathrm{ZF} + \mathrm{GCH} + (\mathrm{E}\varkappa)\ \big((\varkappa, R(\omega_1)) \ \& \\ cf(\varkappa) = \omega_0\big)\Big).$$

It is a theorem of J. Silver [20] that

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{MC}) \to \mathrm{Con}(\mathrm{ZF} + \mathrm{MC} + \mathrm{GCH}).$$

We can therefore state

THEOREM 2.28.

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{MC}) \to \mathrm{Con}\Big(\mathrm{ZF} + \mathrm{GCH} + (\exists\varkappa)\ \big((\varkappa, \underline{R(\omega_1)}) \ \&\ cf(\varkappa) = \omega_0\big)\Big).$$

Let $\mathscr{I}$ be a normal $\varkappa$-comp. ω_1-sat. ideal in $\mathscr{P}(\varkappa)$ and $2^{\aleph_0} \geqslant \varkappa$. Let $\Phi = \{U|\varkappa - U \in \mathscr{I}\}$. It follows from Theorem 1.13 that $(\varkappa, R(\omega_1), \Phi)$ holds.

Proceeding analogously we get:

$$\mathrm{Con}\Big(\mathrm{ZF} + (\exists\varkappa)\ \big(2^{\aleph_0} \geqslant \varkappa \ \&\ (\exists\mathscr{I})\ (\mathscr{I} \text{ is a } \varkappa\text{-comp. } \omega_1\text{-sat. ideal in } \mathscr{P}(\varkappa))\big)\Big) \\ \to \mathrm{Con}\Big(\mathrm{ZF} + (\exists\varkappa)\ \big(2^{\aleph_0} \geqslant \varkappa \ \&\ cf(\varkappa) = \omega_0 \ \&\ (\varkappa, R(\omega_1))\big)\Big).$$

We shall however prove (see Theorem 3.21) that

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{MC}) \to \\ \to \mathrm{Con}\Big(\mathrm{ZF} + (\exists\varkappa)\ \big(2^{\aleph_0} \geqslant \varkappa \ \&\ (\exists\mathscr{I})\ (\mathscr{I} \text{ is a } \varkappa\text{-comp. } \omega_1\text{-sat. ideal in } \mathscr{P}(\varkappa))\big)\Big).$$

This gives

THEOREM 2.29.

$$\mathrm{Con}(\mathrm{ZF}+\mathrm{MC}) \to \mathrm{Con}\Big(\mathrm{ZF}+(\exists \varkappa)\big(2^{\aleph_0} \geqslant \varkappa \,\&\, cf(\varkappa)=\omega_0 \,\&\, (\varkappa, R(\omega_1))\big)\Big).$$

We remark again, that if $(\varkappa, R(\omega_1))$ holds, then there is no Jónsson algebra of cardinality $\varkappa$. Compare Theorem 1.20 and the remark following it.

4. Let $(T, <^T)$ be a partial ordering such that for every $t \in T$, $\{x | x <^T t\}$ is well ordered by $<^T$. Then $(T, <^T)$ is called a *tree*. If $t \in T$, then the rank of t is the order type of the set $\{x | x <^T t\}$ in the ordering $<^T$. The set of all elements of T of a fixed rank is called a *level*. A subset of T totally ordered by $<^T$ is called a *branch*.

E. Specker [25] proved the following theorem. Let GCH hold and $\varkappa$ be a regular cardinal. Then there is a tree of cardinality $\varkappa^+$ such that each level and each branch have cardinality $\leqslant \varkappa$. It is not known,whether such a tree exists for singular $\varkappa$. The problem is open even for $\varkappa = \aleph_\omega$ and assuming GCH. Compare [8]. We shall prove that if ZF + MC is consistent, then ZF + "there is a tree of cardinality $\aleph_{\omega+1}$ such that each level and each branch have cardinality $\leqslant \aleph_\omega$", is consistent. Subsequently J. Silver and the author found a different method for obtaining this result. The assumption of the existence of a measurable cardinal is not needed. This result will appear elsewhere. However, J. Silver observed that the method presented here gives something stronger. Namely, for every $\mathfrak{A} = \langle A, U, R_n\rangle_{n\in\omega_0}$, $|A| = \aleph_1$, $|U| = \aleph_0$, there is an $\mathfrak{A}' = \langle A', U', R'_n\rangle_{n\in\omega_0}$ such that $\mathfrak{A}' \equiv \mathfrak{A}$ and $|A'| = \aleph_{\omega+1}$, $|U'| = \aleph_\omega$, holds in the model constructed below. This proves the consistency of Vaught's two-cardinal conjecture, ([29], p. 82), for $\aleph_{\omega+1}, \aleph_\omega$. No alternative method is known to obtain this result.

The following is proved in [25]. Suppose $\varkappa$ is a regular cardinal and the GCH holds. Then there is a total ordering $(\Omega, <^\Omega)$ of cardinality $\varkappa$ and a tree $(T, <^T)$ such that

(a) elements of T are sequences $\mathfrak{A} = \{a_\xi\}_{\xi\in\alpha}$ where $\alpha \in \varkappa^+$, and if $\xi < \eta$, $\eta \in \alpha$, then $a_\xi <^\Omega a_\eta$.

(b) $\mathfrak{A} <^T \mathfrak{B}$ iff $\mathfrak{A}, \mathfrak{B} \in T$ and $\mathfrak{A}$ is an initial segment of $\mathfrak{B}$.

(c) $|T| = \varkappa^+$ and every level and every branch of T has cardinality $\leqslant \varkappa$.

We retain the same meanings for $\mathfrak{M}$, $\mathfrak{N}$ as throughout most of Section 3 of this chapter, i.e. $\varkappa$ measurable and so on.

THEOREM 2.30. *"There is a tree $(T, <^T) \in \mathfrak{N}$ such that $|T| = \varkappa^+$, and every branch and every level of T have cardinality $\leqslant \varkappa$", holds in $\mathfrak{N}$.*

Proof. Since $\varkappa$ is measurable in $\mathfrak{M}$, $\varkappa$ is regular in $\mathfrak{M}$. We can also assume that GCH holds in $\mathfrak{M}$ (see [20]). Thus by just described results of [25], there is an ordering $(\Omega, <^{\Omega})$ of cardinality $\varkappa$ in $\mathfrak{M}$, and a tree $(T, <^{T}) \in \mathfrak{M}$ such that (a), (b), (c) hold in $\mathfrak{M}$. By Theorem 2.12 the cardinals of $\mathfrak{N}$ and $\mathfrak{M}$ are the same. Hence the statements "$|\Omega| = \varkappa$", "$|T| = \varkappa^+$", (a), (b), and "every level of $(T, <^{T})$ has cardinality $\leqslant \varkappa$" obviously hold in $\mathfrak{N}$. All these claims follow from the fact that Ω and T have the same elements in $\mathfrak{N}$ as in $\mathfrak{M}$ and thus the notion of level of T remains unchanged. The proof, that "every branch of $(T, <^{T})$ has cardinality $\leqslant \varkappa$", holds in $\mathfrak{N}$, is essentially the same as the proof that the corresponding statement holds for every $(T, <^{T})$ satisfying (a), (b), (c) in set theory. Let $\{\mathfrak{A}_\xi\}_{\xi \in \varkappa^+}$ be a branch of $(T, <^{T})$ of cardinality $\varkappa^+$ in $\mathfrak{N}$, i.e. $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$ for $\xi \neq \eta$, $\mathfrak{A}_\xi <^{T} \mathfrak{A}_\eta$ for $\xi \in \eta \in \varkappa^+$. By (b) and the assumptions just stated we can get $a_\xi \in \mathfrak{A}_{\xi+1} - \mathfrak{A}_\xi$ for all $\xi \in \varkappa^+$. It is easily seen that $a_\xi <^{\Omega} a_\eta$ for $\xi < \eta$ and $a_\xi \neq a_\eta$ for $\xi \neq \eta$. Therefore $|\Omega| \geqslant \varkappa^+$ holds in $\mathfrak{N}$. But also "$|\Omega| = \varkappa$" holds in $\mathfrak{N}$, which contradicts the fact that $\varkappa, \varkappa^+$ are distinct cardinals of $\mathfrak{N}$. Hence every branch of T has cardinality $\leqslant \varkappa$ in $\mathfrak{N}$. This finishes the proof of Theorem 2.30.

We sketch the proof of J. Silver that Vaught's two-cardinal conjecture holds in $\mathfrak{N}$ for $\varkappa^+, \varkappa$. Let $\mathfrak{A} = \langle A, U, R_n\rangle_{n \in \omega_0}$, $\mathfrak{A} \in \mathfrak{N}$, $|A|^{\mathfrak{N}} = \aleph_1$, $|U|^{\mathfrak{N}} = \aleph_0$.

It follows easily from Theorem 2.25 that we can assume $\mathfrak{A} \in \mathfrak{M}$. Since GCH holds in $\mathfrak{M}$, by applying a theorem of C.C. Chang (see [29], Theorem 1.2, p. 82), we get an $\mathfrak{A}' \equiv \mathfrak{A}$, $\mathfrak{A}' \in \mathfrak{M}$, $\mathfrak{A}' = \langle A', U', R_n'\rangle_{n \in \omega_0}$, $|A'|^{\mathfrak{M}} = \varkappa^+$, $|U'|^{\mathfrak{M}} = \varkappa$. It is however easy to see that $\mathfrak{A}' \equiv \mathfrak{A}$ holds in $\mathfrak{N}$ as well. The rest follows from the absoluteness of cardinals.

To extend these results to the case of $\aleph_\omega, \aleph_{\omega+1}$ we prove the following.

THEOREM 2.31. *There is a Cohen extension $\mathfrak{N}'$ of $\mathfrak{N}$ such that $\varkappa = \omega_\omega^{\mathfrak{N}'}$, $\varkappa^+ = \omega_{\omega+1}^{\mathfrak{N}'}$, $\mathscr{P}^{\mathfrak{N}'}(\omega_1^{\mathfrak{N}}) = \mathscr{P}^{\mathfrak{N}}(\omega_1^{\mathfrak{N}})$.*

It is clear that if $\mathfrak{N}'$ is as in Theorem 2.31, then there is a tree of cardinality $\omega_{\omega+1}^{\mathfrak{N}'}$ in $\mathfrak{N}'$ whose each level and each branch have cardinality $\leqslant \omega_\omega^{\mathfrak{N}'}$. In fact, we take the same tree T as in the proof of Theorem 2.30 and by essentially the same argument as there that it has the required properties. To prove that Vaught's two-cardinal conjecture for $\aleph_{\omega+1}$, $\aleph_\omega$ holds in $\mathfrak{N}'$, we proceed similarly as in the proof of that conjecture for $\varkappa^+, \varkappa$ in $\mathfrak{N}$. We use the fact that it holds in $\mathfrak{N}$ for $\varkappa^+, \varkappa$, $\varkappa^+ = \omega_{\omega+1}^{\mathfrak{N}'}$, $\varkappa = \omega_\omega^{\mathfrak{N}'}$, and $\mathscr{P}^{\mathfrak{N}'}(\omega_1^{\mathfrak{N}}) = \mathscr{P}^{\mathfrak{N}}(\omega_1^{\mathfrak{N}})$. We remark yet that $\mathscr{P}^{\mathfrak{N}'}(\omega_1^{\mathfrak{N}}) = \mathscr{P}^{\mathfrak{N}}(\omega_1^{\mathfrak{N}})$ has the following obvious consequences: $\omega_1^{\mathfrak{N}} = \omega_1^{\mathfrak{N}'}$, $\omega_2^{\mathfrak{N}} = \omega_2^{\mathfrak{N}'}$. Hence if we prove Theorem 2.31, we can consider to have proved

THEOREM 2.32. *If* ZF + MC *is consistent, then* ZF + "*there is a tree of cardinality $\aleph_{\omega+1}$ such that each level and each branch have cardinality $\leqslant \aleph_\omega$*" + "*Vaught's two-cardinal conjecture holds for $\aleph_{\omega+1}, \aleph_\omega$*" *is consistent.*

Actually, by a remark due to J. Silver, if two-cardinal conjecture holds for a pair α^+, α, then there is a tree as above for same pair. We can arrange for GCH to hold in $\mathfrak{N}$ and this will now be assumed (compare remark following Theorem 2.27). Let $\omega_3 \leqslant \varkappa_3 < \varkappa_4 < \ldots < \varkappa_n \ldots$, $3 \leqslant n \in \omega_0$, be an increasing sequence of regular cardinals of $\mathfrak{N}$ such that $\lim_{n \in \omega_0} \varkappa_n = \varkappa$. The intention is to make $\omega_1^{\mathfrak{N}} = \omega_1^{\mathfrak{N}'}$, $\omega_2^{\mathfrak{N}} = \omega_2^{\mathfrak{N}'}$, $\varkappa_3^{+\mathfrak{N}} = \omega_3^{\mathfrak{N}'}$, $\ldots \varkappa_n^{+\mathfrak{N}} = \omega_n^{\mathfrak{N}'}$ for $n \geqslant 3$. We shall write $\varkappa_n^+$ instead of $\varkappa_n^{+\mathfrak{N}}$. Let $A_2 = \omega_2 \times \varkappa_3$ and $A_n = (\varkappa_n^+ - \varkappa_n) \times \varkappa_{n+1}$ for $n \geqslant 3$. Let $A = \bigcup_{n \geqslant 2} A_n$. We define in $\mathfrak{N}$

$$\mathscr{P}' = \{P \mid P \subseteq A \;\&\; P \text{ is a function } \& \; |P \cap A_2| = \aleph_1^{\mathfrak{N}} \;\&\; (\forall n \geqslant 3) \\ (|P \cap A_n| \leqslant \varkappa_n)\}.$$

We shall construct an extension $\mathfrak{N}'$ of $\mathfrak{N}$ by taking elements of $\mathscr{P}'$ for conditions. By definition, if $P, Q \in \mathscr{P}'$ and $P \supseteq Q$, then P extends Q.

LEMMA 2.33. *Let $P_\xi \in \mathscr{P}'$ for all $\xi \in \omega_1^{\mathfrak{N}}$ and $P_\xi \subseteq P_\eta$ for all $\xi \in \eta \in \omega_1$. Then $\bigcup_{\xi \in \omega_1} P_\xi \in \mathscr{P}'$.*

Proof. This is obvious from the definition of $\mathscr{P}'$.

LEMMA 2.34. $\mathscr{P}^{\mathfrak{N}}(\omega_1^{\mathfrak{N}}) = \mathscr{P}^{\mathfrak{N}'}(\omega_1^{\mathfrak{N}})$.

Proof. This is a well known consequence of Lemma 2.33. See also Lemma 2.23 for an analogous proof.

LEMMA 2.35. *$|\omega_2^{\mathfrak{N}}| = |\varkappa_3|$ and $|\varkappa_n^+| = |\varkappa_{n+1}|$ for $n \geqslant 3$ hold in $\mathfrak{N}'$.*

Proof. $\mathfrak{N}' = \mathfrak{N}(\mathscr{G})$, where $\mathscr{G}$ is a generic subset of A. It is claimed that $\mathscr{G} \cap A_2$ is a function from $\omega_2^{\mathfrak{N}}$ onto $\varkappa_3$. This is well known and we therefore only outline the proof. Since $\mathscr{G}$ is generic, $\mathscr{G}$ has to contain a condition P from every dense subset of $\mathscr{P}'$. But for every $\beta \in \varkappa_3$, $\{P \mid (\exists \alpha \in \omega_2)(\langle \alpha, \beta \rangle \in P)\}$ is a dense subset of $\mathscr{P}'$. Thus for every $\beta \in \varkappa_3$, for some $\alpha \in \omega_2$, $\langle \alpha, \beta \rangle \in \mathscr{G}$, hence $\langle \alpha, \beta \rangle \in \mathscr{G} \cap A_2$. The remaining part of the lemma is proved similarly.

LEMMA 2.36. *The set of cardinals of $\mathfrak{N}'$ less than $\varkappa$ is included in the set $\{\omega_1^{\mathfrak{N}}, \omega_2^{\mathfrak{N}}, \varkappa_3^+, \varkappa_4^+, \ldots \varkappa_n^+, \ldots n \geqslant 3\}$.*

Proof. This is a consequence of Lemmas 2.34, 2.35.

DEFINITION 2.37. Let $P \in \mathscr{P}'$, $n \geqslant 2$. Then

$$(P)_n = P \cap (\bigcup_{k \leqslant n} A_k), \; (P)^n = P \cap (\bigcup_{k \geqslant n} A_n).$$

Obviously $(P)_n \cap (P)^n = 0$, $(P)_n \cup (P)^n = P$ for all $n \geqslant 2$, and all $P \in \mathscr{P}'$.

LEMMA 2.38. *Let $\{P_\xi\}_{\xi \in \varkappa_{n+1}}$ be such that $P_\xi \in \mathscr{P}'$, $(P_\xi)_n = P$ for all $\xi \in \varkappa_{n+1}$ and $P_\xi \subseteq P_\eta$ for all $\xi \in \eta \in \varkappa_{n+1}$, $n \geqslant 2$. Then $\bigcup_{\xi \in \varkappa_{n+1}} P_\xi \in \mathscr{P}'$.*

Proof. This is an easy consequence of the definition of $\mathscr{P}'$.

DEFINITION 2.39. $\mathscr{P}'_n = \{(P)_n \mid P \in \mathscr{P}'\}$ for $n \geqslant 2$.

LEMMA 2.40. $|\mathscr{P}'_n| = \varkappa_{n+1}$ *for* $n \geqslant 2$ *holds in* $\mathfrak{N}$.

Proof. This is an easy consequence of GCH.

LEMMA 2.41. *Let* $P \Vdash (\tau\colon \varkappa_{n+1} \to \varkappa^+)$ *for some* $n \in \omega_0$, $n \geqslant 2$. *Then there is a condition* $Q \supseteq P$, $(Q)_n = (P)_n$ *such that for every* $R \supseteq Q$, *every* $\alpha \in \varkappa_{n+1}$, *and every* $\beta \in \varkappa^+$, *if* $R \Vdash \tau(\alpha) = \beta$, *then* $Q \cup (R)_n \Vdash \tau(\alpha) = \beta$.

Proof. Well order $\mathscr{P}'_n$ so that $\mathscr{P}'_n = \{P_\xi \mid 1 \leqslant \xi \in \varkappa_{n+1}\}$ and every element of $\mathscr{P}'_n$ appears $\varkappa_{n+1}$-times in the sequence P_ξ. We shall define a sequence $\{Q_\xi\}_{\xi \in \varkappa_{n+1}}$ of conditions by induction. Let $Q_0 = P$. If Q_ξ have been defined for $\xi < \eta$, put $Q'_\eta = (Q_0)_n \cup \bigcup_{\xi < \eta} (Q_\xi)^n$.

P_η is the α-th occurrence of some condition $P' \in \mathscr{P}'_n$ in the sequence $\{P_\xi\}_{\xi \in \varkappa_{n+1}}$ for some $\alpha \in \varkappa_{n+1}$. Let $Q_\eta = Q'_\eta$ if there is no R and no $\beta \in \varkappa^+$, such that

$$(*) \qquad R \supseteq Q'_\eta,\ (R)_n = P_\eta \quad \text{and} \quad R \Vdash \tau(\alpha) = \beta.$$

If there exist R, $\beta \in \varkappa^+$ such that $(*)$ holds, let Q_η be such an R for some $\beta \in \varkappa^+$. Let $Q = (Q_0)_n \cup \bigcup_{\xi \in \varkappa_{n+1}} (Q_\xi)^n$. Using Lemma 2.38 it is easy to see that $Q \in \mathscr{P}'$. Obviously $Q \supseteq P$, $(Q)_n = (P)_n$. Let $R \supseteq Q$ and for some $\alpha \in \varkappa_{n+1}$, $\beta \in \varkappa^+$, $R \Vdash \tau(\alpha) = \beta$. Let P_{η_0} be the α-th occurrence of $(R)_n$ in the sequence $\{P_\xi\}_{\xi \in \varkappa_{n+1}}$. It follows easily from the definition of the sequence $\{Q_\xi\}_{\xi \in \varkappa_{n+1}}$ and from the properties of R just chosen that $Q_{\eta_0} \Vdash \tau(\alpha) = \beta'$ for some $\beta' \in \varkappa^+$, and $(Q_{\eta_0})_n = P_{\eta_0}$. Also $(Q_{\eta_0})^n \subseteq Q$. Thus $Q_{\eta_0} \subseteq R$ and therefore $R \Vdash \tau(\alpha) = \beta'$. Since τ is forced to be a function we get $\beta = \beta'$. Thus $Q_{\eta_0} \Vdash \tau(\alpha) = \beta$ and obviously $Q_{\eta_0} \subseteq Q \cup (R)_n$. Thus $Q \cup (R)_n \Vdash \tau(\alpha) = \beta$.

This proves the lemma.

LEMMA 2.42. *Let* $P \Vdash (\tau\colon \varkappa_{n+1} \to \varkappa^+)$. *Then there is a* $Q \supseteq P$ *and a set* $E \in \mathfrak{N}$, $|E| \leqslant \varkappa_{n+1}$ *such that* $Q \Vdash \tau^*(\varkappa_{n+1}) \subseteq E$.

Proof. Take Q as guaranteed by Lemma 2.41. Let

$$E = \big\{\beta \mid (\exists P' \in \mathscr{P}'_n)\,(\exists \alpha \in \varkappa_{n+1})\,\big(P' \cup Q \Vdash \tau(\alpha) = \beta\big)\big\}.$$

Let

$$E' = \big\{\beta \mid (\exists R \in \mathscr{P}')\,(\exists \alpha \in \varkappa_{n+1})\,\big(Q \cup R \Vdash \tau(\alpha) = \beta\big)\big\}.$$

It is obvious that $Q \Vdash \tau^*(\varkappa_{n+1}) \subseteq E'$. However by Lemma 2.41 we conclude easily that $E = E'$. Thus $Q \Vdash \tau^*(\varkappa_{n+1}) \subseteq E$. By Lemma 2.40 $|\mathscr{P}'_n| = \varkappa_{n+1}$. For every $P' \in \mathscr{P}'_n$ and every $\alpha \in \varkappa_{n+1}$, there is at most one β such that $P' \cup Q \Vdash \tau(\alpha) = \beta$. We conclude $|E| \leqslant \varkappa_{n+1}$.

LEMMA 2.43. *Let* $g \in \mathfrak{N}'$, $g\colon \varkappa_{n+1} \to \varkappa^+$, $2 \leqslant n \in \omega_0$. *Then there is a set* $E \in \mathfrak{N}$ *such that* $|E|^{\mathfrak{N}} \leqslant \varkappa_{n+1}$ *and* $g^*(\varkappa_{n+1}) \subseteq E$.

Proof. This is a consequence of Lemma 2.42 and of a density argument.

LEMMA 2.44. *$\varkappa_{n+1}^{+}$ is a cardinal in $\mathfrak{N}'$ for $n \geqslant 2$. $\varkappa^{+}$ is a cardinal in $\mathfrak{N}'$.*

Proof. Otherwise there is a map $g \in \mathfrak{N}'$, $g\colon \varkappa_{n+1} \xrightarrow{\text{onto}} \varkappa_{n+1}^{+}$. For every such a g, $g\colon \varkappa_{n+1} \to \varkappa^{+}$. By Lemma 2.43, there is a set $E \in \mathfrak{N}$, $|E|^{\mathfrak{N}} = \varkappa_{n+1}$ and $g^{*}(\varkappa_{n+1}) \subseteq E$. Since $\varkappa_{n+1}^{+}$ is a regular cardinal in $\mathfrak{N}$, there is a $\beta \in \varkappa_{n+1}^{+}$, $\beta > \sup E$. Thus for all $\alpha \in \varkappa_{n+1}$, $g(\alpha) \neq \beta$, contradiction. The remaining part is proved analogously.

LEMMA 2.45. $\varkappa = \lim\limits_{n\in\omega_0} \varkappa_n^{+}$.

Proof. This follows from $\varkappa = \lim\limits_{n\in\omega_0} \varkappa_n$.

LEMMA 2.46. $\varkappa = \omega_{\omega}^{\mathfrak{N}'}$, $\varkappa^{+} = \omega_{\omega+1}^{\mathfrak{N}'}$.

Proof. This follows from Lemmas 2.36, 2.44, 2.45.

Proof of theorem 2.31. See Lemmas 2.46, 2.34.

The type of construction used here appeared first in W. Easton [5].

5. We shall discuss one more construction closely related to those described in Sections 1–3 of this chapter. We adopt the notation described in Section 3, Chapter 1. We also take $\nu = \omega_0$. It will be proved that there is a system of conditions for adjoining a subset of $\varkappa$ in such a way that no new subset of λ is adjoined for all $\lambda < \varkappa$. Combinatorial results used this time will be Theorem 1.28 and the remark following it.

We repeat that μ_n is a normal measure on $\varkappa_n$ for $n \geqslant 1$, and

$$\Phi = \{V \mid (\exists m \in \omega_0)(\forall n \in \omega_0)\big(m \leqslant n \to \mu_n(V \cap \varkappa_n) = 1\big)\}.$$

For further information about Φ see the remark accompanying Theorem 1.28.

We define $(\mathscr{P}, \leqslant)$ as follows. Elements of $\mathscr{P}$ are ordered pairs $\langle s, U\rangle$ satisfying the following conditions: $s \in [\varkappa]^{<\omega_0}$, $s \cap (\varkappa_{m+1} - \varkappa_m)$ has at most one element for all $m \in \omega_0$, and if $s \cap (\varkappa_{m+1} - \varkappa_m) \neq 0$, $n < m$, then $s \cap (\varkappa_{n+1} - \varkappa_n) \neq 0$ as well. Furthermore, if m is the least natural number such that $s \cap (\varkappa_{m+1} - \varkappa_m) = 0$, then $\mu_{n+1}(\varkappa_{n+1} \cap U) = 1$ for all $n \geqslant m$. The relation $\leqslant$ is defined as in Definition 2.1. $\langle s, U\rangle \leqslant \langle r, V\rangle$ iff r is an initial segment of s, $s - r \subseteq V$ and $U \subseteq V$.

The definition of conditions differs somewhat from that in Definition 2.1. No restriction on the distribution of elements of s was imposed there. It is easy to see, that if we drop some of the requirements on s, a new set of natural numbers appears in the extension. For example, $\{n \mid$ the generic sequence contains no point of $\varkappa_{n+1} - \varkappa_n\}$ would be such a set.

Lemmas 2.2, 2.3, 2.4 are easily seen to hold for the present construction.

THEOREM 2.47. *Let $\lambda < \varkappa$. Then for every $a \in \mathscr{P}^{\mathfrak{N}}(\lambda)$, we have $a \in \mathscr{P}^{\mathfrak{M}}(\lambda)$.*

Proof. Let m_0 be such that $\varkappa_{m_0} > \lambda$. The set $\mathscr{D} = \{\langle s, U\rangle \mid \langle s, U\rangle \in \mathscr{P}$ & & $|s| \geqslant m_0\}$ is a dense subset of $\mathscr{P}$. Thus it is enough to prove the following: Let τ be a term of the Cohen language, $\langle s_0 U_0\rangle \in \mathscr{D}$ and $\langle s_0, U_0\rangle \Vdash \tau \subseteq \lambda$. Then there is a condition $\langle s_0, V\rangle \leqslant \langle s_0, U_0\rangle$ and a set $a \subseteq \lambda$ such that $\langle s_0, V\rangle \Vdash \tau = a$. To get an $\langle s_0, V\rangle$ as above, it is sufficient to get sets V_a such that $\langle s_0, V_a\rangle \| a \in \tau$ and $\xi(V_a) = |s_0|$ for all $a \in \lambda$. See remarks accompanying Theorem 1.28 for the definition of $\xi(X)$. $V' = \bigcap_{a \in \lambda} V_a \in \Phi$, and $\xi(V') = |s_0|$. Hence $\langle s_0, V'\rangle \leqslant \langle s_0, V_a\rangle$ for all $a \in \lambda$, and $\langle s_0, V'\rangle \Vdash \tau = \{a \mid \langle s_0, V_a\rangle \Vdash a \in \tau\} = a$. It is now enough to take $V = V' \cap U_0$ in order to get $\langle s_0, V\rangle \leqslant \langle s_0, U_0\rangle$, $\langle s_0, V\rangle \Vdash \tau = a$.

The next lemma completes the proof of Theorem 2.47.

LEMMA 2.48. *Let σ be a statement of the Cohen language and $\langle s_0, U_0\rangle$ as above. Then there is a set $V \in \Phi$ such that $\xi(V) = |s_0|$ and $\langle s_0, V\rangle \| \sigma$.*

Proof. We proceed in analogy with Lemma 2.22. Let

$$f(s) = 0 \leftrightarrow s \in [\varkappa]^{<\omega_0} \,\&\, (\exists U)\,(\langle \widehat{s_0 s}, U\rangle \Vdash \neg\sigma),$$

$$f(s) = 1 \leftrightarrow s \in [\varkappa]^{<\omega_0} \,\&\, (\exists U)\,(\langle \widehat{s_0 s}, U\rangle \Vdash \sigma),$$

$$f(s) = 2 \leftrightarrow s \in [\varkappa]^{<\omega_0} \,\&\, f(s) \neq 0 \,\&\, f(s) \neq 1.$$

As in Lemma 2.22, we see easily that for no s do we have $f(s) = 0$ and $f(s) = 1$ simultaneously. By applying Theorem 1.28 with $\lambda = 3$, or indeed Lemma 1.27, we get a set V' such that for all $n \in \omega_0$, $\mathrm{hom}^n(V', f)$. Let $V = V' \cap (\varkappa - \varkappa_{n_0})$ where $n_0 = |s_0|$. Thus $\xi(V) = n_0$. It is claimed that $\langle s_0 V\rangle \| \sigma$. If not, we can get, as in Lemma 2.22, $r, s \in [V]^{<\omega_0}$, $|r| = |s|$ and sets V_0, V_1 such that $\langle \widehat{s_0 r}, V_0\rangle \Vdash \neg\sigma$ and $\langle \widehat{s_0 s}, V_1\rangle \Vdash \sigma$. It is clear from the definition of $\mathscr{P}$ that $\mathrm{tp}(r) = \mathrm{tp}(s)$ (see Chapter 1, Section 3). But $f(r) = 0$, $f(s) = 1$ and $\mathrm{hom}^{|r|}(V', f)$ lead to a contradiction.

THEOREM 2.49. *There is an increasing sequence of ordinals $\varrho = \{\varrho_n\}_{n \in \omega_0} \in \mathfrak{N}$ such that $\lim_{n \in \omega_0} \varrho_n = \varkappa$ and $\varrho \notin \mathfrak{M}$.*

Proof. This is the same as in Lemma 2.11.

6. We will say a few words about further modifications of previous constructions. First of all, property $(\varkappa, \underline{R(\omega_1)})$ can be replaced by $(\varkappa, \underline{R(\nu)})$ where $\nu < \varkappa$. All cardinals $\lambda \geqslant \nu^+$ will remain absolute with respect to the Cohen extension in question. This case applies, for example, if there is a ν-sat. $\varkappa$-comp. ideal $\mathscr{I}$ in $\mathscr{P}(\varkappa)$. Since such a $\varkappa$ is a regular limit cardinal, its cofinality will be changed to ω_0. For another possible alteration compare remarks accompanying Definition 1.34 and introduction to this chapter.

A combinatorial principle, which can be used to generalize constructions using measurable cardinals, is given in Theorem 1.36. Still further results can be obtained in this way. Thus for example, assume that there is a uniform $\varkappa$-comp. ultrafilter Φ in $\mathscr{P}^{\mathfrak{M}}(\varkappa^+)$. Then there is a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ such that $|\varkappa^+|^{\mathfrak{N}} = \varkappa$, $cf^{\mathfrak{N}}(\varkappa^+) = \omega_0$, and, if $\lambda < \varkappa$ and $a \in \mathscr{P}^{\mathfrak{N}}(\lambda)$, then $a \in \mathscr{P}^{\mathfrak{M}}(\lambda)$. Let $\lim\limits_{n \in \omega_0} \varkappa_n = \varkappa$, $\varkappa_n < \varkappa_{n+1}$, and let Φ_n be $\varkappa_n$-complete uniform ultrafilter in $\mathscr{P}^{\mathfrak{M}}(\varkappa^+)$ for all $n \in \omega_0$. Then there is an extension $\mathfrak{N}$ of $\mathfrak{M}$ such that $|\varkappa^+|^{\mathfrak{N}} = \varkappa$, $cf^{\mathfrak{N}}(\varkappa^+) = \omega_0$, and, if $\lambda < \varkappa$ and $a \in \mathscr{P}^{\mathfrak{N}}(\lambda)$, then $a \in \mathscr{P}^{\mathfrak{M}}(\lambda)$. Hence if $2^\lambda = \lambda^+$ for all $\lambda < \varkappa$ and $2^{\varkappa^+} = \varkappa^{+++}$ holds in $\mathfrak{M}$, then $2^\lambda = \lambda^+$ and $2^\varkappa = \varkappa^{++}$ holds in $\mathfrak{N}$. Thus we would get an example of a singular cardinal $\varkappa$ such that the generalized continuum hypothesis holds below $\varkappa$ and fails at $\varkappa$.

We shall briefly discuss what assumptions are needed to change the cofinality of a cardinal $\varkappa$ to ω_1 by a construction of kind given above. As usual, $\prod\limits_{\iota \in I} A_\iota$ is the cartesian product of sets A_ι. $[\varkappa]^a$ denotes the set of all increasing sequences of elements of $\varkappa$ of order type a. We put $[\varkappa]^{<\omega_1} = \bigcup\limits_{a \in \omega_1} [\varkappa]^a$.

We would like to obtain the conjunction of Theorems 2.12, 2.25, where $cf(\varkappa) = \omega_0$ is replaced by $cf(\varkappa) = \omega_1$. For this it is sufficient that $\varkappa$ satisfies the following condition.

There exist $\varkappa$-addit. measures μ_a, $a \in \omega_1$, on $\mathscr{P}(\varkappa)$ such that for every $\lambda < \varkappa$, and every $f\colon [\varkappa]^{<\omega_1} \to \lambda$ there are sets A_a, $a \in \omega_1$ such that

(a) $\mu_a(A_a) = 1$ for all $a \in \omega_1$

(b) $|f^*([\varkappa]^a \cap \prod\limits_{\xi \in a} A_\xi)| = 1$ for all $a \in \omega_1$.

In above, ω_1 can be replaced by $\nu < \varkappa$, ν regular. Since all essential ideas needed to carry out the construction already appeared in previous examples, we say nothing more about this. A variation of the above assumption has been used by Y. Moschovakis and the author to extend results of D. A. Martin on the determinatness of Σ_1^1 subsets of $\omega_0^{\omega_0}$ to Σ_2^1 case. It has not been proved that any hypothesis of the above type about $\varkappa$ violates the axiom of choice.

7. In order to give some consequences of our results for the theory of Boolean algebras we need to state some results of [15]. Every Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ can be obtained by taking for $\mathscr{P}$ the set of non-zero elements of a complete Boolean algebra $\mathscr{B}$, and for all $a, b \in \mathscr{B}$, $b \leqslant a$ iff $b \subseteq a$ holds in $\mathscr{B}$. In what follows, $\mathscr{B}$ is a complete Boolean algebra in $\mathfrak{M}$; $\mathbf{1}, \mathbf{0}$ denote the unity and zero of $\mathscr{B}$ respectively. $\Vdash \varphi$ means $\mathbf{1} \Vdash \varphi$, or equivalently, for all $b \in \mathscr{B}$, $b \Vdash \varphi$.

THEOREM 2.50. *Let* λ *be a cardinal of* $\mathfrak{M}$. *The following statements are equivalent*:

(a) *There is a term* τ *such that* $\Vdash \tau \in (2^\lambda)^{\mathfrak{N}} - 2^\lambda)^{\mathfrak{M}}$.

(b) *There is a sequence* $\{b_\xi^i\}_{\xi\in\lambda}^{i=0,1} \in \mathfrak{M}$, $b_\xi^i \in \mathscr{B}$ *for all* $\xi \in \lambda$ *and* $i = 0,1$ *such that* $\bigcup_{i=0,1} b_\xi^i = \mathbf{1}$, $b_\xi^0 \frown b_\xi^1 = \mathbf{0}$ for all $\xi \in \lambda$ and

$$\bigcup_{f\in(2^\lambda)^{\mathfrak{M}}} \bigcap_{\xi\in\lambda} b_\xi^{f(\xi)} = \mathbf{0}.$$

Informally, there is a new subset of λ in every Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ associated with $\mathscr{B}$ iff the distributive law $(\lambda, 2)$ fails in $\mathfrak{M}$ relative to every element of $\mathscr{B}$.

THEOREM 2.51. *The following two statements are equivalent*:

(a) *For every term* τ, *if* $\Vdash \tau \in (2^\lambda)^{\mathfrak{N}}$, *then* $\Vdash \tau \in (2^\lambda)^{\mathfrak{M}}$.

(b) *For every sequence* $\{b_\xi^i\}_{\xi\in\lambda}^{i=0,1} \in \mathfrak{M}$, $b_\xi^i \in \mathscr{B}$ *for all* $\xi \in \lambda$ *and* $i = 0,1$ *such that* $b_\xi^0 \smile b_\xi^1 = \mathbf{1}$, $b_\xi^0 \frown b_\xi^1 = \mathbf{0}$ *for all* $\xi \in \lambda$, *we have*

$$\bigcup_{f\in(2^\lambda)^{\mathfrak{M}}} \bigcap_{\xi\in\lambda} b_\xi^{f(\xi)} = \mathbf{1}.$$

Less formally, the $(\lambda, 2)$ distributive law holds in $\mathscr{B}$ iff in every extension $\mathfrak{N}$ of $\mathfrak{M}$ corresponding to $\mathscr{B}$, $(2^\lambda)^{\mathfrak{N}} = (2^\lambda)^{\mathfrak{M}}$.

A result analogous to Theorem 2.50 is

THEOREM 2.52. *Let* $\varkappa$ *be a cardinal of* $\mathfrak{M}$. *Then the following statements are equivalent.*:

(a) *There is a term* τ *such that* $\Vdash \tau \in (\varkappa^{\omega_0})^{\mathfrak{N}} - (\varkappa^{\omega_0})^{\mathfrak{M}}$.

(b) *There is a sequence* $\{b_i^\xi\}_{i\in\omega_0}^{\xi\in\varkappa} \in \mathfrak{M}$, $b_i^\xi \in \mathscr{B}$ *for all* $\xi \in \varkappa$ *and* $i \in \omega_0$, $\bigcup_{\xi\in\varkappa} b_i^\xi = \mathbf{1}$, $b_i^\eta \frown b_i^\xi = \mathbf{0}$ *for every* $i \in \omega_0$ *and all* $\xi, \eta \in \varkappa$, $\xi \neq \eta$, *and*

$$(*) \qquad \bigcup_{f\in(\varkappa^{\omega_0})^{\mathfrak{M}}} \bigcap_{i\in\omega_0} b_i^{f(i)} = \mathbf{0}.$$

Theorem 2.52 says that there is a new countable sequence below $\varkappa$ in every Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ associated with $\mathscr{B}$ iff the $(\aleph_0, \varkappa)$ distributive law fails in $\mathfrak{M}$ relative to every element of $\mathscr{B}$. We outline a proof of Theorem 2.52. Theorems 2.50, 2.51 can be proved analogously.

(a) $\to$ (b). Let τ be a term such that $\Vdash \big(\tau \in (\varkappa^{\omega_0})^{\mathfrak{N}} - (\varkappa^{\omega_0})^{\mathfrak{M}}\big)$. Let $b_i^\xi = \bigcup\{b \mid b \Vdash \tau(i) = \xi\}$. Since $\Vdash$ (τ is a function), it is easy to see that $b_i^\xi \frown b_i^\eta = \mathbf{0}$ for $\xi \neq \eta$. Also $\bigcup_{\xi\in\varkappa} b_i^\xi = \mathbf{1}$. This follows from the fact that for every $b \in \mathscr{B}$ and every $i \in \omega_0$ there is a $b' \subseteq b$ and a $\xi \in \varkappa$ such that $b' \Vdash \tau(i) = \xi$. Assume that for some $f \in (\varkappa^{\omega_0})^{\mathfrak{M}}$, $b = \bigcap_{i\in\omega_0} b_i^{f(i)} \neq \mathbf{0}$. It is easy to see that $b \Vdash \tau = f$ contradicting the hypothesis $\Vdash \big(\tau \in (\varkappa^{\omega_0})^{\mathfrak{N}} - (\varkappa^{\omega_0})^{\mathfrak{M}}\big)$. (b) is now obvious.

(b) $\to$ (a). The terms of Cohen language are built in an inductive fashion from a basic term $\dot{\mathscr{T}}$ denoting the set $\mathscr{T}$ of true conditions in $\mathfrak{N}$ (see Lemma 2.11). Let $\{b_i^\xi\}_{i\in\omega_0}^{\xi\in\varkappa}$ be a sequence of elements of $\mathscr{B}$ such that (b) holds. We define a term τ by

$$(\forall i\in\omega_0,\ \xi\in\varkappa)\big(\tau(i) = \xi \leftrightarrow b_i^\xi\in\dot{\mathscr{T}}\big).$$

To prove that (b) $\to$ (a) it is enough to show that for every $f\in(\varkappa^{\omega_0})^{\mathfrak{M}}$, and every $b\in\mathscr{B}$, $b\neq \mathbf{0}$, there is a $b'\subseteq b$, $b'\neq\mathbf{0}$ such that $b'\Vdash\tau\neq f$. Let $f\in(\varkappa^{\omega_0})^{\mathfrak{M}}$ and $b\in\mathscr{B}$, $b\neq\mathbf{0}$. It follows easily from (*) that there is an $i\in\omega_0$ and $\xi,\eta\in\varkappa$, $\xi\neq\eta$ such that $b\cap b_i^\xi\neq\mathbf{0}$ and $b\cap b_i^\eta\neq\mathbf{0}$. For at least one of ξ,η we get $f(i)\neq\xi$, or $f(i)\neq\eta$. Say $f(i)\neq\xi$. Now $b\cap b_i^\xi\Vdash\tau(i)=\xi$, and $\xi\neq f(i)$. Hence $b\cap b_i^\xi\Vdash\tau\neq f$, and $b\cap b_i^\xi\subseteq b$. This proves (b) $\to$ (a).

Let $\varkappa$ be a measurable cardinal, or $\varkappa=\lim\limits_{n\in\omega_0}\varkappa_n$ where $\varkappa_n$ are measurable in $\mathfrak{M}$. Then there is a partial ordering $(\mathscr{P},\leqslant)\in\mathfrak{M}$ such that in every Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ associated with $(\mathscr{P},\leqslant)$, $\mathscr{P}^{\mathfrak{N}}(\lambda)=\mathscr{P}^{\mathfrak{M}}(\lambda)$ for every $\lambda<\varkappa$ and $(\varkappa^{\omega_0})^{\mathfrak{N}}\neq(\varkappa^{\omega_0})^{\mathfrak{M}}$. This follows from Theorems 2.25, 2.47, 2.49 and Lemma 2.11. As it was remarked previously, $(\mathscr{P},\leqslant)$ can be replaced by $(\mathscr{B},\subseteq)$ retaining all the properties of Cohen extensions. Taking into account Theorems 2.51, 2.52, we can state

THEOREM 2.53. *Let $\varkappa$ be measurable, or $\varkappa=\lim\limits_{n\in\omega_0}\varkappa_n$ where $\varkappa_n$ are measurable. Then there is a complete Boolean algebra $\mathscr{B}$ such that the $(\lambda,2)$ distributive law holds in $\mathscr{B}$ for all $\lambda<\varkappa$ and the $(\aleph_0,\varkappa)$ distributive law fails in $\mathscr{B}$.*

The methods connected with Theorem 1.36, as described in Section 6 of this chapter, make it possible to obtain more non-isomorphic examples of complete Boolean algebras satisfying conditions of Theorem 2.53. Actually, we sketched a proof that Theorem 2.53 holds in $\mathfrak{M}$. There is however a general method, see [15], which enables us to dispense with countable models and to obtain Theorem 2.53 as stated. But we will not go into this. Theorems analogous to Theorem 2.53 were proved by E. Smith and A. Tarski [22], and also D. Scott [14]. In fact, Theorem 2.53 represents a strengthening of the result of [14] for special cardinals $\varkappa$. For the reader interested in the connection of results stated here to those in [22], [14], we make the following remarks. The $(\lambda,2)$ d.l. is equivalent to the (λ,λ) d.l. If the $(\aleph_0,\varkappa)$ distributive law (d.l.) fails in $\mathscr{B}$, then the $(\varkappa,2)$ d.l. fails in $\mathscr{B}$. If $\varkappa$ is a singular cardinal and the $(\varkappa,2)$ d.l. fails in $\mathscr{B}$, then either the $(\lambda,2)$ d.l. fails in $\mathscr{B}$ for some $\lambda<\varkappa$, or the $\big(cf(\varkappa),\varkappa\big)$ d.l. fails in $\mathscr{B}$.

Remark. D. A. Martin and K. Kunen proved that the assumption in Section 6 of this chapter to change the cofinality of a cardinal to ω_1, is inconsistent. No use of this assumption except for the short remark in Section 6 of this chapter has been made in this thesis.

CHAPTER 3

The main results of this chapter concern saturated ideals. We shall need some auxiliary lemmas about models discovered by P. Cohen.

1. Thus let $\mathfrak{M}_0$ be a countable transitive model of ZF and λ an uncountable cardinal of $\mathfrak{M}_0$. Let $S \subseteq \lambda$. We put

$$\mathscr{P}(S) = \{p \mid |p| < \aleph_0 \;\&\; p \subset \lambda \times \omega_0 \times \omega_0 \;\&\; \& \;(\forall \xi \in \lambda, m \in \omega_0)(|\{n \mid \langle \xi, m, n\rangle \in p\}| \leqslant 1)\}.$$

Instead of $\mathscr{P}(\lambda)$ we shall write just $\mathscr{P}$. If $p, q \in \mathscr{P}$, then p extends q if $p \supseteq q$.

LEMMA 3.1. *The countable chain condition holds for $\mathscr{P}$, i.e. every family of mutually incompatible elements of $\mathscr{P}$ is at most countable.*

This is well known. It can be proved analogously to ([4], Lemma 2, p.131).

If $p \in \mathscr{P}$, we define the support of p, $\operatorname{supp} p = \{\xi \mid (\exists m, n \in \omega_0)(\langle \xi, m, n\rangle \in p)\}$. Obviously, if $\operatorname{supp} p \cap \operatorname{supp} q = 0$, then $p \cup q \in \mathscr{P}$ and hence p, q are compatible. If $p \in \mathscr{P}$ and $S \subseteq \lambda$, we put $p \mid S = \{\langle \xi, m, n\rangle \mid \langle \xi, m, n\rangle \in p \;\&\; \xi \in S\}$.

A generic subset of $\mathscr{P}$ over $\mathfrak{M}_0$ determines and is determined by a set $\mathscr{G} \subseteq \lambda \times \omega_0 \times \omega_0$ such that for every $\xi \in \lambda$, $m \in \omega_0$ there is exactly one $n \in \omega_0$ such that $\langle \xi, m, n\rangle \in \mathscr{G}$, and $\mathscr{G}$ includes a condition from every dense subset of $\mathscr{P}$ (compare Lemma 2.11). We put $\mathfrak{M}_1 = \mathfrak{M}_0(\mathscr{G})$. If $S \subseteq \lambda$, we put $\mathscr{G} \mid S = \mathscr{G} \cap (S \times \omega_0 \times \omega_0)$.

LEMMA 3.2. *Let $S \subseteq \lambda$ and $S \in \mathfrak{M}_0$. Then $\mathscr{G} \mid S$ is a generic subset of $\mathscr{P}(S)$.*

Proof. Let $\mathscr{D}$ be a dense subset of $\mathscr{P}(S)$. Let $\mathscr{D}' = \{p \cup q \mid p \in \mathscr{D} \;\&\; \& \; q \in \mathscr{P}(\lambda - S)\}$. We claim that $\mathscr{D}'$ is a dense subset of $\mathscr{P}$. To see this, let $p \in \mathscr{P}$. Hence $p = p_0 \cup p_1$, where $p_0 \in \mathscr{P}(S)$, $p_1 \in \mathscr{P}(\lambda - S)$. Since $\mathscr{D}$ is a dense subset of $\mathscr{P}(S)$, there is $q_0 \in \mathscr{D}$, $q_0 \supseteq p_0$. It follows that $q_0 \in \mathscr{P}(S)$ and therefore q_0 is compatible with p_1. Thus $q_0 \cup p_1 \in \mathscr{P}$, $q_0 \cup p_1 \supset p$. But clearly $q_0 \cup p_1 \in \mathscr{D}'$. We proved that for arbitrary $p \in \mathscr{P}$, there is a $q \in \mathscr{D}'$, $q \supseteq p$, i.e. $\mathscr{D}'$ is a dense subset of $\mathscr{P}$. Let $p \in \mathscr{D}'$, $p \subseteq \mathscr{G}$. Such a p exists by density of $\mathscr{D}'$ and genericity of $\mathscr{G}$. Then $p \mid S \in \mathscr{D}$, $p \mid S \subset \mathscr{G} \mid S$. Hence $\mathscr{G} \mid S$ contains a condition from every dense subset of $\mathscr{P}(S)$. This proves the lemma.

LEMMA 3.3. *Let $R \subseteq \lambda$, $S \subseteq \lambda$, $R \cap S = 0$, $R \in \mathfrak{M}_0$ and $S \in \mathfrak{M}_0$. Then $\mathscr{G} | R$ is generic over $\mathfrak{M}_0(\mathscr{G} | S)$.*

Proof. Let $\mathscr{D}$ be a dense subset of $\mathscr{P}(R)$, and $\mathscr{D} \in \mathfrak{M}_0(\mathscr{G} | S)$. By the previous lemma, $\mathfrak{M}_0(\mathscr{G} | S)$ is a Cohen extension of $\mathfrak{M}_0$ with respect to $\mathscr{P}(S)$. In the remaining part of this proof, the forcing which we have in mind is one with respect to $\mathscr{P}(S)$ over $\mathfrak{M}_0$. Let $\dot{\mathscr{D}}$ be a term denoting $\mathscr{D}$ in $\mathfrak{M}_0(\mathscr{G} | S)$. For some $p_0 \subseteq \mathscr{G} | S, p_0 \Vdash (\dot{\mathscr{D}}$ is dense in $\mathscr{P}(R))$. Hence for every $p_1 \in \mathscr{P}(S)$, $p_1 \supseteq p_0$, and every $q_1 \in \mathscr{P}(R)$ there is a $p_2 \in \mathscr{P}(S)$ and $q_2 \in \mathscr{P}(R)$ such that $p_2 \supseteq p_1$, $q_2 \supseteq q_1$ and $p_2 \Vdash q_2 \in \dot{\mathscr{D}}$. But this is the same as to say that

$$\{p \,|\, p \in \mathscr{P}(S \cup R) \;\&\; p | S \supseteq p_0 \;\&\; p | S \Vdash p | R \in \dot{\mathscr{D}}\} = \mathscr{D}'$$

is a dense subset of $\mathscr{P}(S \cup R)$. By the expressibility of forcing in $\mathfrak{M}_0$, $\mathscr{D}' \in \mathfrak{M}_0$. Since $\mathscr{G} | (S \cup R)$ is generic over $\mathfrak{M}_0$, for some $p \subseteq \mathscr{G} | (S \cup R)$, $p \in \mathscr{D}'$, i.e. $p | R \in \mathscr{D}$. Thus $\mathscr{G} | R$ includes an element from every dense set $\mathscr{D} \subseteq \mathscr{P}(R)$, $\mathscr{D} \in \mathfrak{M}_0(\mathscr{G} | S)$. This proves the lemma.

LEMMA 3.3.1. *Let $R \subseteq \lambda$, $S \subseteq \lambda$, $R \cap S = 0$, $S \in \mathfrak{M}_0$ and $R \in \mathfrak{M}_0(\mathscr{G} | S)$. Then $\mathscr{G} | R$ is generic over $\mathfrak{M}_0(\mathscr{G} | S)$ with respect to $\mathscr{P}(R)$.*

Proof. Put $\mathfrak{M}_0' = \mathfrak{M}_0(\mathscr{G} | S)$. By Lemma 3.3, $\mathscr{G} | (\lambda - S)$ is generic over $\mathfrak{M}_0$ with respect to $\mathscr{P}(\lambda - S)$. We are now confronted with a situation analogous to that of Lemma 3.2, but over $\mathfrak{M}_0'$ instead of over $\mathfrak{M}_0$. We consider Lemma 3.3.1 as proved.

The following lemma has been first proved by A. Levy.

LEMMA 3.4. *Let σ be a statement of the Cohen language. Then there is a set $S \subseteq \lambda$ such that $|S| \leqslant \aleph_0$, and for every $p \in \mathscr{P}$, if $p \Vdash \sigma$, then $p | S \Vdash \sigma$. This holds in $\mathfrak{M}_0$.*

Proof. Let P be a maximal set of conditions satisfying the following properties:

(1) $(\forall p)(p \in P \to p \Vdash \neg \sigma)$

(2) $(\forall p, p')(p \in P \;\&\; p' \in P \;\&\; p \neq p' \to p, p'$ are incompatible.)

The existence of such a set follows from Zorn lemma. Let $S = \bigcup_{p \in P} \operatorname{supp} p$. By (2), the countable chain condition and $|\operatorname{supp} p| < \aleph_0$, we get $|S| \leqslant \aleph_0$. Let $q \Vdash \sigma$. It is claimed that $q | S \Vdash \sigma$. Assume not. Then there is an $r \supseteq q | S$ such that $r \Vdash \neg \sigma$. By the maximality property of P, there is a $p \in P$ such that p, r are compatible. Thus $p, q | S$ are compatible by $r \supseteq q | S$. It follows that $p \cup q | S \in \mathscr{P}$, and $p \cup q | S \Vdash \neg \sigma$ by $p \in P$, (1), and $p \cup q | S \supseteq p$. Clearly, $\operatorname{supp}(p \cup q | S) \subset S$. Hence $p \cup (q | S) \cup \big(q | (\lambda - S)\big) = s \in \mathscr{P}$, and $s \Vdash \sigma$ by $s \supseteq q$. Also $s \Vdash \neg \sigma$, a contradiction. This proves the lemma.

Let $\Vdash \tau \colon \omega_0 \to \omega_0$. For every $i, j \in \omega_0$, let $S_{i,j}^{\tau} \subseteq \lambda$, $|S_{i,j}^{\tau}| \leqslant \aleph_0$ and for every p, if $p \Vdash \tau(i) = j$, then $p | S_{i,j}^{\tau} \Vdash \tau(i) = j$. The existence of a set $S^\tau_{i,j}$ with these properties follows from Lemma 3.4. Let $S^\tau = \bigcup_{i,j\in\omega_0} S^\tau_{i,j}$. Clearly, $|S^\tau| \leqslant \aleph_0$.

LEMMA 3.5. *Let $g \in (\omega_0^{\omega_0})^{\mathfrak{M}_1}$. Then there is a set $S \subseteq \lambda$ such that $|S| \leqslant \aleph_0$, $S \in \mathfrak{M}_0$ and $g \in \mathfrak{M}_0(\mathscr{G}|S)$.*

Proof. Let $\dot{g}$ denote g and $\Vdash \dot{g} \in \omega_0^{\omega_0}$. Take $S = S^{\dot{g}}$. Let $\bar{g}(m) = n \leftrightarrow \leftrightarrow (\exists p)\big(p \subset \mathscr{G}|S \,\&\, p \Vdash \dot{g}(m) = n\big)$. Clearly $\bar{g} \in \mathfrak{M}_0(\mathscr{G}|S)$. If $\bar{g}(m) = n$, then $g(m) = n$ since $\dot{g}$ denotes g. We prove that $\bar{g}$ is defined for all $m \in \omega_0$. For every $m \in \omega_0$, for some $q \subseteq \mathscr{G}$, $q \Vdash \dot{g}(m) = n$. By definition of $S^{\dot{g}}$, $q|S^{\dot{g}} \Vdash \dot{g}(m) = n$, i.e. $q|S \Vdash \dot{g}(m) = n$, and $q|S \subset \mathscr{G}|S$. Hence $\bar{g} = g$ and $g \in \mathfrak{M}_0(\mathscr{G}|S)$.

LEMMA 3.5.1. *Let $S \subseteq \lambda$, $S \in \mathfrak{M}_0$ and $g \in (\omega_0^{\omega_0})^{\mathfrak{M}_1}$. Then there is a set $R \subseteq \lambda - S$ such that $R \in \mathfrak{M}_0(\mathscr{G}|S) = \mathfrak{M}_0'$, $|R|^{\mathfrak{M}_0'} \leqslant \aleph_0$ and $g \in \mathfrak{M}_0(\mathscr{G}|S)(\mathscr{G}|R)$.*

Proof. We consider forcing for $\mathscr{P}(\lambda - S)$ over $\mathfrak{M}_0'$ and proceed as in Lemmas 3.4, 3.5.

LEMMA 3.6. *Let $\{x_\xi\}_{\xi\in\beta} \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}_1}$, $\{x_\xi\}_{\xi\in\beta} \in \mathfrak{M}_1$. Then there is a set $S \subseteq \lambda$ such that $S \in \mathfrak{M}_0$, $|S|^{\mathfrak{M}_0} \leqslant \beta$ and $\{x_\xi\}_{\xi\in\beta} \in \mathfrak{M}_0(\mathscr{G}|S)$.*

Proof. Let τ denote $\{x_\xi\}_{\xi\in\beta}$ in $\mathfrak{M}_1$. Let $\tau(\xi)$ be the natural term which denotes the ξ-th element of τ in $\omega_0^{\omega_0}$ if such exists, and $\tau(\xi) = 0$ otherwise for all $\xi \in \beta$. Let $S = \bigcup_{\xi\in\beta} S^{\tau(\xi)}$. As in Lemma 3.5, we get $x_\xi(m) = n \leftrightarrow (\exists p)\big(p \subseteq \mathscr{G}|S \,\&\, p \Vdash \tau(\xi)(m) = n\big)$ for all $\xi \in \beta$. This shows $\{x_\xi\}_{\xi\in\beta} \in \mathfrak{M}_0(\mathscr{G}|S)$. $|S|^{\mathfrak{M}_0} \leqslant \beta$ follows from $|S^{\tau(\xi)}|^{\mathfrak{M}_0} \leqslant \aleph_0$ for all $\xi \in \beta$.

LEMMA 3.7. (See [15]). *Let $a \in \mathfrak{M}_0$, $(a^{\aleph_0})^{\mathfrak{M}_0} = a$, $S \in \mathfrak{M}_0$, $S \subseteq \lambda$ and $|S|^{\mathfrak{M}_0} = a$. Then $2^{\aleph_0} = a$ holds in $\mathfrak{M}_0(\mathscr{G}|S) = \mathfrak{M}_0'$.*

We give only the main idea of the proof.

Let τ, τ' be two terms such that $\Vdash\big(\tau \in (\omega_0^{\omega_0})^{\mathfrak{M}_0'}\big)$ and similarly for τ'. Assume that $S^\tau = S^{\tau'}$ and, for all $p \in \mathscr{P}(S^\tau)$ and all $m, n \in \omega_0$, $p \Vdash \tau(m) = n \leftrightarrow p \Vdash \tau'(m) = n$. Then τ, τ' denote the same element of $\omega_0^{\omega_0}$ in $\mathfrak{M}_0'$. The rest is now a consequence of $a^{\aleph_0} = a$.

DEFINITION 3.8. For all $\xi \in \lambda$ and $m, n \in \omega_0$, let

$$f_\xi(m) = n \quad \text{iff} \quad \langle \xi, m, n\rangle \in \mathscr{G}.$$

LEMMA 3.9. *For every $g \in (\omega_0^{\omega_0})^{\mathfrak{M}_1}$,*

$$\Big|\big\{f_\xi \mid \xi \in \lambda \,\&\, (\forall i \in \omega_0)\big(f_\xi(i) \leqslant g(i)\big)\big\}\Big|^{\mathfrak{M}_1} \leqslant \aleph_0.$$

Proof. Let $|S|^{\mathfrak{M}_0} \leqslant \aleph_0$ and $g \in \mathfrak{M}_0(\mathscr{G}|S)$. Such an S exists by Lemma 3.5. By Lemma 3.3, $\mathscr{G}|(\lambda - S)$ is generic over $\mathfrak{M}_0(\mathscr{G}|S)$. We shall consider forcing for $\mathscr{P}(\lambda - S)$ over $\mathfrak{M}_0(\mathscr{G}|S)$. Let $\dot{f}_\xi$ be the term denoting f_ξ in $\mathfrak{M}_0(\mathscr{G}|S)\big(\mathscr{G}|(\lambda - S)\big)$ built on the basis of Definition 3.8. It is claimed that for every $\xi \in \lambda - S$, $(\exists i)\big(f_\xi(\dot{i}) > g(i)\big)$.

Let $\xi \in \lambda - S$ and $p \in \mathscr{P}(\lambda - S)$. Let $i_0 \in \omega_0$ be the smallest number such that, for all $j \in \omega_0$, $\langle \xi, i_0, j\rangle \notin p$. Let $j_0 > g(i_0)$. Put $p' = p \cup \{\langle \xi, i_0, j_0\rangle\}$. Hence $p' \in \mathscr{P}(\lambda - S)$ and $p' \Vdash f_\xi(i_0) = j_0 > g(i_0)$. It follows that $\{g \mid (\exists i \in \omega_0)(q \Vdash f_\xi(i) > g(i))\}$ is a dense subset of $\mathscr{P}(\lambda - S)$ for every $\xi \in \lambda - S$. This proves the claim and at the same time the lemma, since $|S|^{\mathfrak{M}_0} \leqslant \aleph_0$.

DEFINITION 3.10. Let $s \in \bigcup_{k \in \omega_0} \omega_0^k$. Then $B_s = \{g \mid g \supset s \;\&\; g \in \omega_0^{\omega_0}\}$. If $\mathfrak{M}$ is a transitive model of ZF, then $B_s^{\mathfrak{M}} = B_s \cap \mathfrak{M}$.

The following lemma is due to R. Solovay. The proof given here is due to J. Silver.

LEMMA 3.11. *Suppose $\mathfrak{M}$ is a countable transitive model of* ZF, *$(\mathscr{P}, \leqslant) \in \mathfrak{M}$, $|\mathscr{P}|^{\mathfrak{M}} = \aleph_0$ and $(\mathscr{P}, \leqslant)$ is a partial ordering. Suppose also $X \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}}$, $X \in \mathfrak{M}$ and X is of second category in $\mathfrak{M}$. Let $\mathscr{G}$ be generic over $\mathfrak{M}$ with respect to $\mathscr{P}$. Then X is of second category in $\mathfrak{N} = \mathfrak{M}(\mathscr{G})$.*

Proof. Assume that X is of first category in $\mathfrak{M}(\mathscr{G})$. Then there is a term τ such that

$$(1)\qquad \Vdash (\forall n \in \omega_0)\big(\tau_n \text{ is a dense open subset of } (\omega_0^{\omega_0})^{\mathfrak{N}}\big) \;\& \;\& \;(\forall x \in X)(\exists n \in \omega_0)\,(x \notin \tau_n).$$

So for every $p \in \mathscr{P}$ and every $x \in X$ there is a $q \in \mathscr{P}$, $q \leqslant p$ and $n \in \omega_0$, such that $q \Vdash x \notin \tau_n$. Let $X_{q,n} = \{x \mid x \in X \;\&\; g \Vdash x \notin \tau_n\}$. We have $\bigcup_{q \in \mathscr{P}, n \in \omega_0} X_{q,n} = X$ and $|\mathscr{P}|^{\mathfrak{M}} = \aleph_0$. Thus for some $q_0 \in \mathscr{P}$, $n_0 \in \omega_0$, X_{q_0,n_0} is of second category. Let $s \in \bigcup_{k \in \omega_0} \omega_0^k$ such that X_{q_0,n_0} is dense in B_s. Let $q_1 \leqslant q_0$, $r \supseteq s$ and

$$(*)\qquad q_1 \Vdash B_r^{\mathfrak{N}} \subseteq \tau_{n_0}.$$

The existence of q_1 and r as above follows from the first part of (1). Since $B_r \subseteq B_s$ and X_{q_0,n_0} is dense in B_s, there is an element $x_0 \in X_{q_0,n_0}$ such that $x_0 \in B_r$. By $(*)$, $q_1 \Vdash x_0 \in \tau_{n_0}$. By $x_0 \in X_{q_0,n_0}$ and the definition of $X_{q,n}$, $q_0 \Vdash x_0 \notin \tau_{n_0}$. It follows that $q_1 \Vdash x_0 \notin \tau_{n_0}$, since $q_1 \leqslant q_0$. Hence both $q_1 \Vdash x_0 \in \tau_{n_0}$, $q_1 \Vdash x_0 \notin \tau_{n_0}$, which is a contradiction.

LEMMA 3.12. *Let $\mathfrak{M}$, $\mathfrak{N}$ be transitive models of* ZF, *abbrev.* t.m. ZF, *$\mathfrak{M} \subseteq \mathfrak{N}$, $X \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}}$ and $X \in \mathfrak{M}$. Let X be of first category in $\mathfrak{N}$. Then there is a $g \in (\omega_0^{\omega_0})^{\mathfrak{N}}$ such that X is of first category in all* t.m. ZF *$\mathfrak{N}'$ such that $\mathfrak{M} \subseteq \mathfrak{N}'$ and $g \in \mathfrak{N}'$.*

Proof. If $u \subseteq \bigcup_{n \in \omega_0} \omega_0^n$ and $\mathfrak{M}'$ is a t.m. ZF such that $u \in \mathfrak{M}'$, let $\mathscr{O}^{\mathfrak{M}'}(u) = \bigcup_{s \in u} B_s^{\mathfrak{M}'}$. If $\mathfrak{M}''$ is another t.m. ZF and $f \in (\omega_0^{\omega_0})^{\mathfrak{M}'}, f \in (\omega_0^{\omega_0})^{\mathfrak{M}''}$, then clearly $f \in \mathscr{O}^{\mathfrak{M}'}(u) \leftrightarrow f \in \mathscr{O}^{\mathfrak{M}''}(u)$. Let $\mathscr{U} \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}'}$ and $\mathscr{U}$ be open in $\mathfrak{M}'$. Then $\mathscr{U} = \mathscr{O}^{\mathfrak{M}'}(u)$ for some $u \in \mathfrak{M}'$, $u \subseteq \bigcup_{n \in \omega_0} \omega_0^n$.

Let X be as above. There is a sequence $\{u_n\}_{n \in \omega_0} \in \mathfrak{N}$ such that, for all $n \in \omega_0$, $u_n \subseteq \bigcup_{k \in \omega_0} \omega_0^k$, $\mathscr{O}^{\mathfrak{N}}(u_n)$ is a dense open subset of $(\omega_0^{\omega_0})^{\mathfrak{N}}$ and

$X \cap \bigcap_{n \in \omega_0} \mathcal{O}^{\mathfrak{N}}(u_n) = 0$. There is a function $g \in (\omega_0^{\omega_0})^{\mathfrak{N}}$ from which $\{u_n\}_{n \in \omega_0}$ can be constructed using simple set theoretic operations. Hence if $g \in \mathfrak{N}'$, then $\{u_n\}_{n \in \omega_0} \in \mathfrak{N}'$; and it is easy to see that $X \cap \bigcap_{n \in \omega_0} \mathcal{O}^{\mathfrak{N}'}(u_n) = 0$, i.e. X is of first category in $\mathfrak{N}'$.

LEMMA 3.13. *Suppose $S \subseteq \lambda$, $S \in \mathfrak{M}_0$, $X \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}_1}$, $X \in \mathfrak{M}_0(\mathscr{G}|S)$ and X is of second category in $\mathfrak{M}_0(G|S)$. Then X is of second category in $\mathfrak{M}_1$.*

Proof. Assume that X is of first category in $\mathfrak{M}_1$. We apply Lemma 3.12 to the pair $\mathfrak{M}_0(G|S)$, $\mathfrak{M}_1$. There is a $g \in (\omega_0^{\omega_0})^{\mathfrak{M}_1}$ such that X is of first category in all t.m. ZF $\mathfrak{N}'$ such that $\mathfrak{M}_0(\mathscr{G}|S) \subseteq \mathfrak{N}'$ and $g \in \mathfrak{N}'$. We choose such a g. By Lemma 3.3, $\mathscr{G}|(\lambda - S)$ is generic over $\mathfrak{M}_0(G|S)$. Let $\dot{g}$ denote g in Cohen language associated with $\mathscr{P}(\lambda - S)$ over $\mathfrak{M}_0(\mathscr{G}|S)$. By Lemma 3.5.1, there is a set $R \subseteq \lambda - S$ such that $R \in \mathfrak{M}_0(\mathscr{G}|S)$, $|R| \leqslant \aleph_0$ in $\mathfrak{M}_0(\mathscr{G}|S)$ and $g \in \mathfrak{M}_0(\mathscr{G}|S)(\mathscr{G}|R) = \mathfrak{N}'$. Hence X is of first category in $\mathfrak{N}'$ because of the choice of g.

By Lemma 3.3.1, $\mathscr{G}|R$ is generic over $\mathfrak{M}_0(\mathscr{G}|S)$ with respect to $\mathscr{P}(R)$. By $|R| \leqslant \aleph_0$, we get $|\mathscr{P}(R)| \leqslant \aleph_0$. It follows from Lemma 3.11 that X remains of second category in $\mathfrak{M}_0(\mathscr{G}|S)(\mathscr{G}|R) = \mathfrak{N}'$. But we have already asserted that X is of first category in $\mathfrak{N}'$. This is a contradiction. Hence X is of second category in $\mathfrak{M}_1$. This proves the lemma.

We shall now prove several lemmas which contain no reference to models of set theory. This will bring us into a position to see that the following proposition C holds in $\mathfrak{M}_1$.

C: *Every set $X \subseteq \omega_0^{\omega_0}$ of second category can be decomposed into uncountably many disjoint sets of second category.*

DEFINITION 3.14. Let $X \subseteq \omega_0^{\omega_0}$. X is called *uniform* if, for every dense G_δ-set $G \subseteq \omega_0^{\omega_0}$, $|X \cap G| = 2^{\aleph_0}$.

LEMMA 3.15. *Let $X \subseteq \omega_0^{\omega_0}$ be uniform. Then X can be decomposed into $2^{\aleph_0}$ disjoint sets $\{X_\eta\}_{\eta \in \mathfrak{c}}$ such that for all $\eta \in \mathfrak{c}$ and, for every dense G_δ-set $G \subseteq \omega_0^{\omega_0}$, $G \cap X_\eta \neq 0$. Here $\mathfrak{c}$ denotes the least ordinal of cardinality continuum.*

Proof. Let $\{G_\xi\}_{\xi \in \mathfrak{c}}$ be an enumeration of all dense G_δ-sets $\subseteq \omega_0^{\omega_0}$. We define $a_\xi^\eta \in \omega_0^{\omega_0}$ for $\eta \leqslant \xi \in \mathfrak{c}$ by transfinite induction as follows. Let a_0^0 be an arbitrary element of $X \cap G_0$. Assume that $a_{\xi'}^\eta$ was defined for each η, ξ' such that $\eta \leqslant \xi' < \xi$ for some $\xi \in \mathfrak{c}$. We choose $a_\xi^\eta \in G_\xi \cap X$ for each $\eta \leqslant \xi$ in such a way that $a_\xi^\eta \neq a_{\xi'}^{\eta'}$ for all η, η', ξ' such that $\eta \leqslant \xi$, $\eta' \leqslant \xi'$, $\xi' \leqslant \xi$ and $\langle \eta, \xi \rangle \neq \langle \eta', \xi' \rangle$. This can be done for every $\xi \in \mathfrak{c}$, since $|G_\xi \cap X| = 2^{\aleph_0}$ and $|\{a_{\xi'}^{\eta'} | \eta' \leqslant \xi' < \xi\}| < \mathfrak{c}$ and $|\xi| < \mathfrak{c}$. For every $\eta \in \mathfrak{c}$, we put $X_\eta = \{a_\xi^\eta | \xi < \mathfrak{c}\}$. Let $\eta \in \mathfrak{c}$ and $G \subseteq \omega_0^{\omega_0}$ be a dense G_δ-set. Then for some $\xi \in \mathfrak{c}$, $G = G_\xi$. We have $a_\xi^\eta \in G_\xi \cap X_\eta$. Hence $G \cap X_\eta \neq 0$. This proves the lemma.

LEMMA 3.16. *Suppose $X \subseteq \omega_0^{\omega_0}$ is of second category, $X = 2^{\aleph_0}$ and*

$$(*) \qquad (\forall Y \subseteq X)(|Y| < 2^{\aleph_0} \to Y \quad \text{is of first } \mathit{category}).$$

Then X can be decomposed into $2^{\aleph_0}$ disjoint sets of second category.

Proof. Let G be an arbitrary G_δ-set, $G \subseteq \omega_0^{\omega_0}$. Then $X = (X \cap G) \cup \big(X \cap (\omega_0^{\omega_0} - G)\big)$. Since $\omega_0^{\omega_0} - G$ is of first category, $X \cap G$ must be of second category. Thus by $(*)$, $|X \cap G| = 2^{\aleph_0}$. It follows that X is uniform (Definition 3.14). Let $\{X_\xi\}_{\xi \in \mathfrak{c}}$ be a decomposition into disjoint sets as in Lemma 3.15. For all $\xi \in \mathfrak{c}$, X_ξ is of second category. Indeed, every set of first category is disjoint with a dense G_δ-set.

Lemma 3.16 has the following

COROLLARY 3.17. *Assume that the proposition* C *fails. Let λ be the least cardinal such that there is a set $X \subseteq \omega_0^{\omega_0}$, $|X| = \lambda$, X is of second category and X can not be decomposed into uncountably many disjoint sets of second category. Then $\lambda < \mathfrak{c}$.*

Proof. Let λ and X be as in the hypothesis of the corollary. Obviously, no subset of X can be decomposed into uncountably many disjoint sets of second category. Hence, for every $Y \subseteq X$, if Y is of second category, then $|Y| = \lambda$. If $\lambda = \mathfrak{c}$, then X satisfies the assumptions of Lemma 3.16. Thus X can be decomposed into uncountably many disjoint sets of second category. This contradiction proves the corollary.

Assume that the GCH holds in $\mathfrak{M}_0$. Let $X \subseteq (\omega_0^{\omega_0})^{\mathfrak{M}_1}$ be of second category in $\mathfrak{M}_1$. We prove that X can be decomposed into uncountably many disjoint sets of second category in $\mathfrak{M}_1$. Consider the set $\Sigma = \{\langle X', S'\rangle \mid X'$ is of second category in $\mathfrak{M}_0' = \mathfrak{M}_0(\mathscr{G} | S')$ & $S' \in \mathfrak{M}_0$ & $|S'|^{\mathfrak{M}_0} (= |S'|^{\mathfrak{M}_1}) = |X'|^{\mathfrak{M}_1}$ & $X' \subseteq X\}$. It follows from Lemmas 3.6, 3.12 that there is an $S \in \mathfrak{M}_0$ such that $\langle X, S\rangle \in \Sigma$. Choose an $\langle X', S'\rangle \in \Sigma$ so that $|S'|^{\mathfrak{M}_0}$ is minimal. It follows easily from the countable additivity of sets of first category that $cf|S'|^{\mathfrak{M}_0} \neq \omega_0$ in $\mathfrak{M}_0$. Since GCH holds in $\mathfrak{M}_0$, it follows that $(|S'|^{\mathfrak{M}_0})^{\aleph_0} = |S'|^{\mathfrak{M}_0}$ in $\mathfrak{M}_0$. Hence by Lemma 7, $(2^{\aleph_0})^{\mathfrak{M}_0'} = |S'|^{\mathfrak{M}_0}$. Also $|S'|^{\mathfrak{M}_0} = |S'|^{\mathfrak{M}_0'} = |X'|^{\mathfrak{M}_0'}$. For all $Y \subseteq X'$, if $Y \in \mathfrak{M}_0'$ and $|Y|^{\mathfrak{M}_0'} < |X'|^{\mathfrak{M}_0'}$, then Y is of first category in $\mathfrak{M}_0'$. This follows easily from the choice of $\langle X', S'\rangle$. By Lemma 3.16, in $\mathfrak{M}_0'$, X' can be decomposed into uncountably many disjoint sets $\{X_\xi' \mid \xi \in \omega_1^{\mathfrak{M}_0'}\}$ of second category. By Lemma 3.13, the X_ξ' are of second category in $\mathfrak{M}_1$ for all $\xi \in \omega_1^{\mathfrak{M}_0'}$. Also $\omega_1^{\mathfrak{M}_0'} = \omega_1^{\mathfrak{M}_1}$. This gives

LEMMA 3.18. *If the* GCH *holds in $\mathfrak{M}_0$, then* C *holds in $\mathfrak{M}_1$.*

Let D be the following statement:

(D) *For every uncountable cardinal $\mathfrak{a} \leqslant 2^{\aleph_0}$ there is a set $X \subseteq \omega_0^{\omega_0}$ such that $|X| = \mathfrak{a}$ and X is of second category.*

LEMMA 3.18.1. *If the* GCH *holds in* $\mathfrak{M}_0$, *then* D *holds in* $\mathfrak{M}_1$.

Proof. We proceed essentially the same way as in Lemma 3.13. The GCH enables us to assert that there is a set $X \subseteq \omega_0^{\omega_0}$, such that $|X| = \aleph_1$ and X is of second category in $\mathfrak{M}_0$.

LEMMA 3.19. *In* $\mathfrak{M}_1$ *there is no measure defined on* $\mathscr{P}(a)$ *for every* $a \leqslant \lambda$. (For definition of measure see introduction.)

Proof. It is a theorem of S. Banach and C. Kuratowski [1], that if there is a set $A \subseteq \omega_0^{\omega_0}$, $|A| = a$, such that for every $g \in \omega_0^{\omega_0}$,

$$\big|\{f \mid (\forall i \in \omega_0)\,(f(i) \leqslant g(i))\ \&\ f \in A\}\big| \leqslant \aleph_0,$$

then there is no measure on $\mathscr{P}(a)$. Take $A = \{f_\xi \mid \xi \in a\}$ and apply Lemma 3.9.

2. THEOREM 3.20. *Let* $\mathfrak{M}$ *be a countable transitive model of* ZF. *Let* $\nu, \varkappa$ *be cardinals of* $\mathfrak{M}$, $\nu < \varkappa$ *and* ν *regular. Let* $\mathscr{I}$ *be a* ν*-sat.* $\varkappa$*-comp. ideal in* $\mathscr{P}^{\mathfrak{M}}(\varkappa)$, $\mathscr{I} \in \mathfrak{M}$. *Let* $(\mathscr{P}, \leqslant) \in \mathfrak{M}$ *be a partial ordering such that for every* $S \subseteq \mathscr{P}$, *if* $S \in \mathfrak{M}$ *and for all* $p, q \in S$, $p \neq q$, p, q *are incompatible, then* $|S|^{\mathfrak{M}} < \nu$. *Let* $\mathfrak{N}$ *be a Cohen extension of* $\mathfrak{M}$ *with respect to* $(\mathscr{P}, \leqslant)$. *Then there is in* $\mathfrak{N}$ *a* ν*-sat.* $\varkappa$*-comp. ideal in* $\mathscr{P}^{\mathfrak{N}}(\varkappa)$.

Proof. Let $\overline{\mathscr{I}} = \{X \mid X \in \mathscr{P}^{\mathfrak{N}}(\varkappa)\ \&\ (\exists Y)\,(Y \in \mathscr{I}\ \&\ X \subseteq Y)\}$. It is claimed that $\overline{\mathscr{I}}$ is $\varkappa$-comp. in $\mathfrak{N}$. Suppose $\lambda < \varkappa$ and τ is a term such that $\Vdash \tau\colon \lambda \to \overline{\mathscr{I}}$. For every $\xi \in \lambda$ let P_ξ be a maximal family of pairwise incompatible conditions with the property: for every $p \in P_\xi$, there is a set $X_\xi^p \in \mathscr{I}$ such that $p \Vdash \tau(\xi) \subseteq X_\xi^p$. The existence of a family P_ξ as above follows from Zorn lemma. By a basic property of forcing, see e.g. [15], if $p \Vdash (\exists Y)\,(Y \in \mathscr{I}\ \&\ Y \supset \tau')$ for some term τ', then there is a $q \leqslant p$ and a set $Y_0 \in \mathscr{I}$ such that $q \Vdash Y_0 \supseteq \tau'$. It follows that, for every $\xi \in \lambda$ and every $p \in \mathscr{P}$, there is a $q \in P_\xi$ such that p, q are compatible. By one of the assumptions, $|P_\xi| < \nu < \varkappa$ for every $\xi \in \lambda$. Hence if $X_\xi = \bigcup_{p \in P_\xi} X_\xi^p$, then $X_\xi \in \mathscr{I}$ by $\varkappa$-completeness. For the same reason, $\bigcup_{\xi \in \lambda} X_\xi = X \in \mathscr{I}$. It is now easy to see that $\Vdash \bigcup \tau \subseteq X$, and thus $\Vdash \bigcup \tau \in \overline{\mathscr{I}}$. In fact, let $p \Vdash a \in \bigcup \tau$ for some $a \in \varkappa$ and $p \in \mathscr{P}$. The set $\{p' \mid p' \leqslant p\ \&\ p' \Vdash a \in \tau_\xi$, for some $\xi \in \lambda\} = P'$ is dense below p. Let $p' \in P'$ and $p' \Vdash a \in \tau_\xi$, $\xi \in \lambda$. p' is compatible with some $p'' \in P_\xi$. Let $p''' \leqslant p'$, $p''' \leqslant p''$. We have $p''' \Vdash \tau_\xi \subseteq X_\xi^{p''}$, $p''' \Vdash a \in \tau_\xi$. Hence $a \in X_\xi^{p''} \subset X$. This proves $\Vdash \bigcup \tau \subseteq X$. The rest follows from the fact that every sequence $\{t_\xi\}_{\xi \in \lambda} \in \mathfrak{N}$ such that $t_\xi \in \overline{\mathscr{I}}$ for all $\xi \in \lambda$ is denoted by a term τ as above. Hence $\overline{\mathscr{I}}$ is $\varkappa$-comp. in $\mathfrak{N}$.

It remains to be proved that $\overline{\mathscr{I}}$ is ν-saturated. We proceed much as in the earlier part of the proof. Let

$$\Vdash \big[\tau\colon \lambda \to \mathscr{P}(\varkappa) - \overline{\mathscr{I}}\ \&\ (\forall \xi, \eta \in \lambda)\,\big(\xi \neq \eta \to \tau(\xi) \cap \tau(\eta) = 0\big)\big].$$

We prove that assumption $\lambda = \nu$ (and hence obviously $\lambda \geqslant \nu$) leads to a contradiction. Let $X_\xi = \{\alpha \mid (\exists p \in \mathscr{P})\,(p \Vdash \alpha \in \tau(\xi))\}$ for every $\xi \in \nu$. We observe that $\Vdash \tau_\xi \subseteq X_\xi$. If not, then for some $p \in \mathscr{P}$, $(p \Vdash \alpha \in \tau_\xi)$, and $\alpha \notin X_\xi$. But this is impossible.

If $X_\xi \in \mathscr{I}$, then $\Vdash \tau_\xi \in \overline{\mathscr{I}}$, contradicting our assumptions on τ. Thus $X_\xi \notin \mathscr{I}$ for all $\xi \in \nu$. By Theorem 1.45 there is a set $U \subseteq \nu$, $|U| = \nu$, such that $\bigcap_{\xi \in U} X_\xi = X \neq 0$. Let $x \in X$. For every $\xi \in U$ let $p_\xi \in \mathscr{P}$ be such that $p_\xi \Vdash x \in \tau(\xi)$. Since $|U| = \nu$, there are $\xi, \eta \in \nu$, $\xi \neq \eta$ such that p_ξ, p_η are compatible. Let $p \leqslant p_\xi$, $p \leqslant p_\eta$. Then $p \Vdash (x \in \tau(\xi) \,\&\, x \in \tau(\eta))$. This contradicts $\Vdash (\xi \neq \eta \rightarrow \tau(\xi) \cap \tau(\eta) = 0)$. Hence $\lambda < \nu$ and $\overline{\mathscr{I}}$ is ν-saturated.

If we assume in Theorem 3.20 that $\varkappa$ is a measurable cardinal and $\mathscr{I}$ is a maximal $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$, then we can take $U = \nu$ in the above proof. This follows from the fact that the intersection of less than $\varkappa$ sets of measure 1, has measure 1. Thus Theorem 1.45 is not needed in this case.

By a result due to A. Tarski and S. Ulam, every measurable cardinal $\varkappa$ is strongly inaccessible. The proof of this is an easy variation of the one given in [28]. If we replace the ν-chain condition for $\mathscr{P}$ by $|\mathscr{P}| < \varkappa$ (the defining property of "mild Cohen extension" with respect to $\varkappa$), then $\varkappa$ remains strongly inaccessible in $\mathfrak{N}$. This follows from results of [15]. By Theorems 3.20 ans 1.47, $\varkappa$ is measurable in $\mathfrak{N}$. This provides a different proof of a theorem due to R. Solovay and A. Levy [11] that measurability is preserved under "mild Cohen extensions". An analogous proof also shows that strong compactness is preserved under mild Cohen extensions, a theorem of K. McAloon.

Suppose $\varkappa$ is measurable in $\mathfrak{M}$, the GCH holds in $\mathfrak{M}$ and δ is a regular cardinal of $\mathfrak{M}$, $\delta < \varkappa$. By a theorem of J. Silver [20], Con(ZF + MC) $\rightarrow$ Con(ZF + MC + GCH). It is well known, see e.g. [5], that in $\mathfrak{M}$ there is a system of conditions $(\mathscr{P}, \leqslant)$ such that $2^\nu = \nu^+$ for all $\nu < \delta$, $2^\delta \geqslant \varkappa$ and $2^\nu = \nu^+$ for $\nu \geqslant 2^\delta$ holds in $\mathfrak{N}$. This $(\mathscr{P}, \leqslant)$ can be taken to satisfy δ^+-chain condition. Hence there is in $\mathfrak{N}$ a δ^+-sat. $\varkappa$-comp. ideal in $\mathscr{P}^{\mathfrak{N}}(\varkappa)$. $\varkappa$ can not be measurable, since every measurable cardinal is strongly inaccessible.

We shall now give special attention to the case $\delta = \omega_0$. We impose an additional assumption on the $\mathfrak{M}_0$ investigated in Section 1 of this chapter. Namely, let $\lambda \geqslant \varkappa$ where $\varkappa$ is measurable in $\mathfrak{M}_0$. If $\lambda = \varkappa$, then (I) "*there is an ω_1-sat. $2^{\aleph_0}$-complete ideal in $\mathscr{P}(2^{\aleph_0})$*" holds in $\mathfrak{M}_1$. This follows from Lemma 3.1, Theorem 3.20 and Lemma 3.7. If $\lambda > \varkappa$, then (II) "*there is a cardinal $\varkappa < 2^{\aleph_0}$ such that there is an ω_1-sat. $\varkappa$-complete ideal in $\mathscr{P}(\varkappa)$*" holds in $\mathfrak{M}_1$. This is analogous to (I).

Combining these results with Lemmas 3.18, 3.18.1 and 3.19 we can state

THEOREM 3.21. *If* ZF + MC *is consistent, then* ZF + (I) *and* ZF + (II) *are both consistent and they remain consistent, if we add "for all ν, if $\nu \leqslant 2^{\aleph_0}$, then there is no measure on $\mathscr{P}(\nu)$"* + C + D.

Remark 3.21.1. That our results are best possible, i.e. that the assumption of the consistency of the existence of a measurable cardinal can not be weakened, follows from a theorem of R. Solovay [23]: If ZF + "*there are cardinals ν, $\varkappa$, $\nu < \varkappa$ such that there is a ν-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$*" is consistent, then ZF + MC is consistent.

Remark 3.21.2. R. Solovay proved that if ZF + MC is consistent, then ZF + "*there is a $2^{\aleph_0}$-addit. measure on $\mathscr{P}(2^{\aleph_0})$*" is consistent. (See [23]). K. Kunen showed that if there is a $2^{\aleph_0}$-addit. measure on $\mathscr{P}(2^{\aleph_0})$, then every subset of $\omega_0^{\omega_0}$ of cardinality $< 2^{\aleph_0}$ is of first category. Hence by Corollary 3.17, the existence of such a measure implies that C holds.

The following questions are stated in ([10], p. 260).

(1) Is it true that there is no ω_1-sat. $2^{\aleph_0}$-comp. ideal in $\mathscr{P}(2^{\aleph_0})$?

(2) Is it true that there is no ω_1-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$ for every $\varkappa < 2^{\aleph_0}$?

(7) If there is a ν-sat. $\varkappa$-comp. ideal in $\mathscr{P}(\varkappa)$ for some $\nu < \varkappa$, is $\varkappa$ a measurable cardinal?

A different terminology is used in [10]. The questions have been translated in terminology used in this thesis. Theorem 3.21 gives an answer to all these questions. In a sense, these answers are optimal (compare Remark 3.21.1).

We mention some questions about saturated ideals that remain open. By saturatedness of an ideal $\mathscr{I}$ in $\mathscr{P}(\varkappa)$ we mean the least cardinal number ν such that for every disjointed $S \subseteq \mathscr{P}(\varkappa) - \mathscr{I}$, $|S| < \nu$. The saturatedness of the ideals constructed above is always a successor cardinal. The author does not know how to get an ideal whose saturatedness is equal to a regular limit cardinal. Theorem 1.47 gives a partial explanation for this state of affairs. Thus, for example, it is a consequence of Theorem 1.47 that, if $\nu < \varkappa$ and ν is strongly inaccessible, then ν can not be the saturatedness of any $\varkappa$-complete ideal $\mathscr{I} \subseteq \mathscr{P}(\varkappa)$. We also remark, at this point, that the saturatedness of any $\varkappa$-complete ideal in $\mathscr{P}(\varkappa)$ is never equal to a singular cardinal. Hence the assumption that $\mathscr{I}$ is ν-saturated for some regular ν made in all theorems concerning saturated ideals does not cause a loss of generality.

The following two questions remain open. Is it consistent (say relative to ZF + MC) that there is a weakly (strongly) inaccessible cardinal $\varkappa$ such that there is a $\varkappa$-complete ideal $\mathscr{I} \subseteq \mathscr{P}(\varkappa)$ of saturatedness $\varkappa$? These questions are rephrasements of questions (3), (4), ([10], p. 260), after taking into account results of R. Solovay [23], [24]. According to these results there is no $\varkappa$-complete ideal $\mathcal{I}$ in $\mathcal{P}(\varkappa)$ of saturatedness $\varkappa$, if $\varkappa$ is the first weakly (strongly) inaccessible cardinal.

Let $\varkappa$ be a strongly inaccessible cardinal such that there is a $\varkappa$-complete ideal $\mathcal{I}$ of saturatedness $\varkappa$ in $\mathcal{P}(\varkappa)$. J. Silver proved that such a cardinal cannot be weakly compact. We state a slight strengthening of this result. Namely, there is a Suslin tree of cardinality $\varkappa$. A Suslin tree of cardinality $\varkappa$ is a tree $(T, <^T)$ such that $|T| = \varkappa$ and every chain and every anti-chain of T has cardinality $< \varkappa$. This is proved essentially in the same way as Theorem 1.47. Hence we shall give only the main idea of the proof. The elements of T are sets $S \subseteq \mathcal{P}(\varkappa)$ such that $S \notin \mathcal{I}$. If S, $R \in T$, $S \neq R$, then either $S \cap R = 0$ or $S \subset R$, or $R \subset S$. If either $R \subset S$ or $S \subset R$, then $S - R \notin \mathcal{I}$ or $R - S \notin \mathcal{I}$, respectively. We put $R <^T S$ if R, $S \in T$ and $R \supset S$. It follows easily from the $\varkappa$-saturatedness that there can be neither a chain nor an anti-chain of cardinality $\varkappa$ in $(T, <^T)$. This observation is in a contrast with Theorem 1.46.

Finally, we make a remark concerning proposition C. The problem whether C holds has been investigated by W. Sierpiński ([16], pp. 173-177.) He makes the following observation. Let $X \subseteq \omega_0^{\omega_0}$ be of second category and $|X| = \lambda$. Moreover, assume that X cannot be decomposed into uncountably many disjoint sets of second category. Then there is an ω_1-complete ω_1-sat. ideal $\mathcal{I}$ in $\mathcal{P}(\lambda)$. It is enough to see that there is such an ideal in $\mathcal{P}(X)$. Let $\mathcal{I}$ be the ideal of subsets of X of first category. It is well known that $\mathcal{I}$ is ω_1-complete. However, the assumption of indecomposability of X implies that $\mathcal{I}$ is ω_1-saturated as well. It follows that C holds unless $2^{\aleph_0}$ is an extremely large cardinal. Compare [28], ([16], pp. 173-177), and [23], [24]. Theorem 3.21 shows that it is consistent that there is an ω_1-complete ω_1-saturated ideal $\mathcal{I}$ in $\mathcal{P}(\lambda)$ where $\lambda = 2^{\aleph_0}$ or $\lambda < 2^{\aleph_0}$ and C holds (provided ZF + MC is consistent). Observe also that the conjunction of C and D appears in the statement of Theorem 3.21. This is of some importance in the light of Corollary 3.17.

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