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A new proof of a theorem of Saxer

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Abstract. A theorem of Saxer, which generalizes the Classical Schottky theorem, gives an estimate for the modulus of a function which is regular on a disk and assumes there the values zero and one a finite number of times. In this paper we show how a simple proof of Saxer's theorem can be based on a certain geometric proposition and in the process we obtain a slightly better result.

1. Introduction. In [2], W. Saxer established the following generalization of Schottky's theorem. Let Δ denote the open disk with center at the origin and radius one. Now let $f(z)$ be regular in Δ and suppose that $f(z)$ assumes the values zero and one p and q times respectively in Δ. Suppose further that

$$f(z) = a_0 + a_1 z + \ldots + a_p z^p + \ldots ,$$

sor $z \in \Delta$. Then W. Saxer essentially showed that there exist positive conftants A and B which depend only on p and q such that

$$\log^+ |f(z)| < A \frac{1+|z|}{1-|z|} (\log^+ \mu + B) \tag{1.1}$$

for $z \in \Delta$, where $\mu = |a_0| + |a_1| + \ldots + |a_p|$.

W. Saxer established the above result by means of Schottky's theorem and mathematical induction on $(p+q)$. Although the basic outline of the author's proof is clear enough, the author's argument is technically difficult and involves the interplay of geometric and analytic considerations in a complicated fashion. It is therefore of some interest to inquire not only as to whether a simple proof of Saxer's theorem is available but also whether a proof of Saxer's theorem can be fashioned in which the geometric and analytic aspects of the argument can be separated. A careful analysis of Saxer's argument shows that the author is really using a certain simple geometric proposition (Theorem (2.1)), which is of some interest in itself. As we shall see this proposition which deals with a certain geometric extremal problem gives us the best possible result in a certain sense. Once this proposition is established, Saxer's theorem follows very easily from Schottky's theorem and in the process we obtain a slightly better result (Corollary (3.1)). In addition the use of this geometric proposition allows us to establish some modest variants of Saxer's theorem with equal ease (Theorems (3.1) and (3.2)). As a further application of Theorem (2.1) in another direction we establish a Picard type theorem for functions polyanalytic on a disk (Theorem (4.1)).

2. A geometric proposition. If c is any finite complex number and if $0 < r < +\infty$, then $\Delta = \Delta(c, r)$ will denote the open disk with center c and radius r. Also $C_r = C(c, r)$ will denote the circumference with center c and radius r.

Now let G be any non-empty open connected subset of the finite complex plane Γ. Let w and z be two arbitrary points of G. Suppose there is some disk $\Delta(w, r) \subseteq G$ and $z \in \Delta(w, ar)$ for some $0 < a < 1$. We denote this fact by writing $wR_a z$. This relation R_a on G is not in general symmetric.

Now by a chain $C = \{z_1, z_2, \ldots, z_n\}$ in G with initial point z_1 and terminal point z_n we mean a sequence of $n \geqslant 2$ points $z_1, z_2, \ldots, z_n$ in G such that $z_k R_{a_k} z_{k+1}$ for some $0 < a_k < 1$ for $k = 1, 2, \ldots, n-1$.

For such a chain C it will be convenient to let $\beta_k = (1+a_k)/(1-a_k)$ for $k = 1, 2, \ldots, n-1$. We now define the length $L(C) = L(C; G)$ of this chain $C = \{z_1, z_2, \ldots, z_n\}$ in G by the condition that

$$L(C) = \prod_{k=1}^{n-1} (1+\beta_k). \tag{2.1}$$

Next let w and z be any two points of G. Since G is polygonally connected there is at least one chain C in G with initial point w and terminal point z. We now define the distance $L(w, z) = L(w, z; G)$ from w to z in G to be the greatest lower bound of the lengths $L(C) = L(C; G)$ of all chains C in G with initial point w and terminal point z. Note that in general that $L(w, z) \neq L(z, w)$. Somewhat more generally if A and B are two non-empty subsets of G we define $L(A, B) = L(A, B; G)$ to be the least upper bound of the quantities $L(a, b)$, where $a \in A$ and $b \in B$.

If w and z are two points of G which are not too close to the boundary of G we might expect the distance $L(w, z)$ from w to z in G to be small in some sense. This observation suggests the general problem of obtaining upper bounds for the distance $L(w, z)$ from w to z in G in terms say of the Euclidean distances of w and z from the boundary of G. For the applications that we have in mind we are only interested in a restricted version of this problem. First we are only interested in the case when G is a disk $\Delta(c, R)$ from which a finite number of points have been deleted. For such a G we wish to obtain upper bounds to $L(w, z)$, where z is arbitrary but w is restricted to lie on some suitable circumference $C(c, r)$ in G whose choice will depend in general upon z.

We now gather together a number of simple facts.

LEMMA (2.1). *Let $G_0 \subseteq G$ be two non-empty open connected subsets of the finite complex plane Γ. Then $L(a, b; G_0) \geqslant L(a, b; G)$ for $a, b \in G_0$. Somewhat more generally if A and B are two non-empty subsets of G_0, then $L(A, B; G_0) \geqslant L(A, B, G)$.*

If $C_1 = \{z_1, z_2, \ldots, z_n\}$ and $C_2 = \{z_n, z_{n+1}, \ldots, z_m\}$ are two chains in G, then $C = C_1C_2 = \{z_1, z_2, \ldots, z_m\}$ is a chain in G. In view of equation (2.1) it is easy to see that $L(C) = L(C_1)L(C_2)$. From this observation we obtain the following result.

LEMMA (2.2). *For $a, b, c \in G$ we have that $L(a, c) \leqslant L(a, b)L(b, c)$. More generally if A, B, and C are non-empty subsets of G, then $L(A, C) \leqslant L(A, B)L(B, C)$.*

We next have the following result.

LEMMA (2.3). *Let $G = \{z\colon \varrho-\sigma < |z| < \varrho+\sigma\}$, where $0 < \sigma \leqslant \varrho$. Then for any two points w and z with $|w| = |z| = \varrho$ we have that $L(w, z; G) \leqslant 4^n$, where n is some positive integer such that $n < 1+(2\pi\varrho)/\sigma$.*

Proof. For simplicity suppose that $w = \varrho$ and $z = \varrho e^{i\theta}$ with $0 \leqslant \theta \leqslant \pi$. Let φ be the angle of a sector of the disk $\Delta(0, \varrho)$ which subtends a chord of length $\sigma/2$. Thus $\varphi > \sigma/(2\varrho)$. Now let $z_0 = w = \varrho$, $z_1 = \varrho e^{i\varphi}$, and in general let $z_k = \varrho e^{ik\varphi}$ for $k = 0, 1, 2, \ldots$ Clearly $z_k R_{a_k} z_{k+1}$ with $a_k = \frac{1}{2}$ and $\beta_k = (1+a_k)/(1-a_k) = 3$ for $k = 0, 1, 2, \ldots$ Now let n be the largest positive integer such that $(n-1)\varphi \leqslant \theta \leqslant \pi$. Clearly $z_{n-1}R_{a_{n-1}}z$ with $0 < a_{n-1} \leqslant \frac{1}{2}$ and $\beta_{n-1} = (1+a_{n-1})/(1-a_{n-1}) \leqslant 3$. Consequently $C = \{w, z_1, \ldots, z_{n-1}, z\}$ is a chain C in G with initial point w and terminal point z for which $L(C) = (1+\beta_0)(1+\beta_1)\ldots(1+\beta_{n-1}) \leqslant 4^n$. Now $(n-1)\varphi \leqslant \pi$ and $\varphi > \sigma/(2\varrho)$ so that $n < 1+(2\pi\varrho)/\sigma$ as claimed.

Somewhat more generally we have the following result.

LEMMA (2.4). *Let $G = \{z\colon \varrho-\sigma < |z| < \varrho+\sigma\}$, where $0 < \sigma \leqslant \varrho$. Let Σ be any arc on the circumference $C(0, \varrho)$ with arclength $\varrho\varphi$ for some $0 < \varphi < 2\pi$. Then $L(\Sigma, \Sigma; G) \leqslant 4^n$, where n is some positive integer with $n < 1+(2\varrho\varphi)/\sigma$.*

Next let $\Delta(c, r)$ be a disk with center c and radius R and let $d_1, d_2, \ldots, d_n$ be $n \geqslant 0$ arbitrary distinct finite complex numbers which need not necessarily belong to $\Delta(c, R)$. Let $G = \Delta(c, R)-\{d_1, d_2, \ldots, d_n\}$. Thus G is the disk $\Delta(c, R)$ from which at most n points have been removed. Now for $z \in \Delta(c, R)$ we define $\delta(z) = \delta(z, G)$ by the condition that

$$\delta(z) = \min\{|z-d_1|, \ldots, |z-d_n|, R-|z-c|\}. \tag{2.2}$$

It should be noted that the determination of the quantity $\delta(z)$ is independent of those points $d_1, d_2, \ldots, d_n$ which do not lie in $\Delta(c, R)$. Note that for each $z \in \Delta(c, R)$, that $\delta(z)$ gives us a measure of how close the point z is to the points $d_1, d_2, \ldots, d_n$ and the boundary of $\Delta(c, R)$.

In the discussion to follow it will be necessary to compare the quantities $\delta(z, G)$ for various deleted disks G. We have the following result.

LEMMA (2.5). *Let* $G = \Delta(c, R) - \{d_1, d_2, \ldots, d_n\}$ *and* $G_0 = \Delta(c_0, R_0) - \{d_1, d_2, \ldots, d_n\}$, *where* $|c_0 - c| \leqslant R - R_0$. *Let* $\delta(z) = \delta(z, G)$ *and* $\delta_0(z) = \delta(z, G_0)$. *Then* $\delta(z) \geqslant \delta_0(z)$ *for all* $z \in \Delta(c_0, R_0)$.

Proof. From equation (2.2) recall that

$$\delta(z) = \min\{|z-d_1|, \ldots, |z-d_n|, R-|z-c|\} \quad \text{for } z \in \Delta(c, R)$$

and

$$\delta_0(z) = \min\{|z-d_1|, \ldots, |z-d_n|, R_0-|z-c_0|\} \quad \text{for } z \in \Delta(c_0, R_0).$$

Now $|z-c| \leqslant |z-c_0| + |c_0-c| \leqslant |z-c_0| + R - R_0$ so that $R - |z-c| \geqslant R_0 - |z-c_0|$. Hence $\delta(z) \geqslant \delta_0(z)$ for $z \in \Delta(c_0, R_0)$ as claimed.

We are now in a position to state and prove the following fundamental geometric proposition which we regard as our key result.

THEOREM (2.1). *Let* $G = \Delta(c, R) - \{d_1, d_2, \ldots, d_n\}$, *where* $n \geqslant 0$ *and let* $\delta(z) = \delta(z, G)$. *There exist constants* λ, μ, *and* A, *with* $0 < \lambda \leqslant 1 \leqslant \mu$, A, *depending only on* n *with the following property. For each* $z \in G$ *there is some circumference* $C_r = C(c, r)$ *in* G *with* $\lambda|z-c| \leqslant r \leqslant |z-c|$ *and* $\delta(C_r) \geqslant \delta(z)/\mu$ *such that* $L(C_r, z) \leqslant AR/\delta(z)$.

It is important to emphasize the fact that the constants λ, μ, and A of Theorem (2.1) depend only on n and not on G. Thus for example these constants are independent of the placement of the points $d_1, d_2, \ldots, d_n$.

We would also like to point out that the above estimate $L(C_r, z) \leqslant AR/\delta(z)$ in Theorem (2.1) is sharp except for the actual determination of the constant A as a function of n. This observation will become clear when we indicate how Saxer's theorem follows readily from the above estimate. For suppose that instead of the above estimate we had the estimate $L(C_r, z) \leqslant A[R/\delta(z)]^a$ for some $0 < a < 1$. We would then obtain a formulation of Saxer's theorem, where in equation (1.1) the factor $(1+r)/(1-r)$ is replaced by the factor $[(1+r)/(1-r)]^a$ and of course such a result is false.

Before embarking on a proof of Theorem (2.1) some minor comments are in order. First the condition that $\delta(C_r) \geqslant \delta(z)/\mu$, which is not utilized in the applications of Theorem (2.1) to follow, is incorporated in the statement of the above theorem only for the purpose of expediting the proof of this theorem by mathematical induction on $n \geqslant 0$. Next note that it clearly suffices to establish Theorem (2.1) only in the case when $c = 0$. Finally it will be convenient to indicate the dependence of the constants λ, μ, and A on n by using such notation as $\lambda = \lambda_n$, $\mu = \mu_n$, and $A = A_n$, when such need arises.

We now proceed to establish a proof of Theorem (2.1) by mathematical induction on $n \geqslant 0$. The argument is somewhat lengthy and consequently for organizational purposes will be decomposed into several propositions.

PROPOSITION (2.1). *We have that Theorem* (2.1) *is true for* $G = \Delta(c, R)$ *or* $G = \Delta(c, R) - \{c\}$ *with* $\lambda = \frac{1}{2}$, $\mu = 1$, *and* $A = 4^7$.

Proof. First suppose that $G = \Delta(c, R)$. It is only necessary to consider the case when $c = 0$. Note that $\delta(z) = \delta(z, G) = R - |z|$ for $z \in G = \Delta(0, R)$.

There are three cases to consider. First if $z = 0$, we take the circumference $C_r = C(0, r)$ with $r = 0$. Clearly $\lambda |z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ and $L(C_r, z) \leqslant AR/\delta(z)$ with $\lambda = \mu = 1$ and $A = 2$.

Next consider the case when $0 < |z| \leqslant R/2$. In this case consider the circumference $C_r = C(0, r)$ with $r = |z|$. Clearly $\lambda |z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ with $\lambda = \mu = 1$. If we now apply Lemma (2.3) with $\varrho = \sigma = r$ we see that $L(C_r, z) \leqslant 4^7$. Hence in this case we see that $L(C_r, z) \leqslant AR/(R - |z|) = AR/\delta(z)$ with $A = 4^7$.

Finally consider the case when $R/2 < |z| < R$. Here we consider the circumference $C_r = C(0, r)$ with $r = R/2$. Clearly $\lambda |z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ with $\lambda = \frac{1}{2}$ and $\mu = 1$. If we now apply Lemma (2.3) with $\varrho = \sigma = r$ we see that $L(C_r, C_r) \leqslant 4^7$. Now write $z = |z| e^{i\varphi}$, $-\pi < \varphi \leqslant \pi$, and set $w = e^{i\varphi} R/2$. Clearly $L(w, z) \leqslant R/(R - |z|)$. Now in view of Lemma (2.2) we see that $L(C_r, z) \leqslant L(C_r, w) L(w, z) \leqslant 4^7 R/(R - |z|)$. Hence in this case we see that $L(C_r, z) \leqslant AR/\delta(z)$ with $A = 4^7$. This establishes the proposition for $G = \Delta(c, R)$. There is a similar proof for $G = \Delta(c, R) - \{c\}$.

From Proposition (2.1) we see that Theorem (2.1) is true for $n = 0$. Now let $(p-1) \geqslant 0$ be fixed and assume the validity of Theorem (2.1) for all $n \leqslant (p-1)$. Let $G = \Delta(c, R) - \{d_1, d_2, \ldots, d_p\}$ and suppose that $|d_1 - c| \leqslant |d_2 - c| \leqslant \ldots \leqslant |d_p - c| = d < R$. If $d = 0$, then $G = \Delta(c, R) - \{c\}$ and the truth of Theorem (2.1) for G follows from Proposition (2.1). Henceforth assume that $0 < d < R$. We now wish to establish the validity of Theorem (2.1) for this G. It suffices to consider the case when $c = 0$ so that $G = \Delta(0, R) - \{d_1, d_2, \ldots, d_p\}$ with $|d_1| \leqslant |d_2| \leqslant \ldots \leqslant |d_p| = d < R$ and $d > 0$.

PROPOSITION (2.2). *We have that Theorem* (2.1) *is true for all* $z \in G$ *with* $|z| \leqslant d$. *Furthermore in this case we may take* $\lambda = \lambda_{p-1}/2$, $\mu = 6\mu_{p-1}$, *and* $A = 15 A_{p-1}$.

Proof. Let $G_d = \Delta(0, d) - \{d_1, d_2, \ldots, d_{p-1}\}$ *and let* $\delta_d(z) = \delta(z, G_d)$. Of course $\delta(z) = \delta(z, G)$. From Lemma (2.5) we see that $\delta(z) \geqslant \delta_d(z)$ for $|z| < d$. Now $d - |z| \geqslant |d_p - z|/3 \geqslant \delta(z)/3$ for $|z| \leqslant d/2$. Hence it is not difficult to be persuaded that $\delta_d(z) \geqslant \delta(z)/3$ for $|z| \leqslant d/2$.

First we consider the case when $z \in G$ and $|z| \leqslant d/2$. Clearly $z \in G_d$. Now by the inductive hypothesis, Theorem (2.1) is true for G_d. Hence there exist constants $0 < \lambda_{p-1} \leqslant 1 \leqslant \mu_{p-1}, A_{p-1}$, depending only on $(p-1)$ with the following property. There is some circumference $C_r = C(0, r)$ in G_d with $\lambda_{p-1}|z| \leqslant r \leqslant |z|$ and $\delta_d(C_r) \geqslant \delta_d(z)/\mu_{p-1}$ such that $L(C_r, z; G_d) \leqslant A_{p-1}d/\delta_d(z)$. From Lemma (2.1) we see that $L(C_r, z) = L(C_r, z; G) \leqslant A_{p-1}d/\delta_d(z)$. Recall that $\delta(z) \geqslant \delta_d(z) \geqslant \delta(z)/3$ for $|z| \leqslant d/2$. Hence for each $z \in G$ with $|z| \leqslant d/2$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta_d(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda_{p-1}$, $\mu = 3\mu_{p-1}$, and $A = 3A_{p-1}$.

Next consider the case when $z \in G$ and $d/2 < |z| \leqslant d$. Now $\delta(z) \leqslant |z-d_p| \leqslant |z|+d$ so that $\delta(z) < 3|z|$ for $d/2 < |z| \leqslant d$. Now for each $z \in G$ with $d/2 < |z| \leqslant d$ write $z = |z|e^{i\varphi}$, $-\pi < \varphi \leqslant \pi$, and introduce the auxiliary point $w = [|z| - \delta(z)/6]e^{i\varphi}$. Now $\Delta(z, \delta(z)) \subseteq G$ and $|w-z| = \delta(z)/6$. Hence $\Delta(w, 5\delta(z)/6) \subseteq G$ and $|w-z| = \delta(z)/6$. Consequently $L(w, z) \leqslant \frac{5}{2}$. Clearly $|z|/2 < |w| < |z|$ and $w \in G_d$. Also it is easily seen that $\delta_d(w) \geqslant \delta(z)/6$. Now by the inductive hypothesis, Theorem (2.1) is valid for G_d. Hence there exist constants $0 < \lambda_{p-1} \leqslant 1 \leqslant \mu_{p-1}, A_{p-1}$, depending only on $(p-1)$ with the following property. There is some circumference $C_r = C(0, r)$ in G_d with $\lambda_{p-1}|w| \leqslant r \leqslant |w|$ and $\delta_d(C_r) \geqslant \delta_d(w)/\mu_{p-1}$ such that $L(C_r, w; G_d) \leqslant A_{p-1}d/\delta_d(w)$. From Lemma (2.1) we see that $L(C_r, w) = L(C_r, w; G) \leqslant A_{p-1}d/\delta_d(w)$. In view of Lemma (2.2) we see that $L(C_r, z) \leqslant L(C_r, w) \times \times L(w, z)$. Hence for all $z \in G$ with $d/2 < |z| \leqslant d$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda_{p-1}/2$, $\mu = 6\mu_{p-1}$, and $A = 15A_{p-1}$. This completes the proof of the proposition.

PROPOSITION (2.3). *Let* $0 < d \leqslant 2R/3$. *Then Theorem* (2.1) *is true for all* $z \in G$ *for which* $|z| = (R+d)/2$. *Furthermore in this case we may take* $\lambda = \mu = 1$ *and* $A = 4^{32}$.

Proof. Let z be any point on the circumference $|z| = (R+d)/2$. Consider the circumference $C_r = C(0, r)$ with $r = (R+d)/2$. Clearly $\lambda|z| \leqslant r \leqslant |z|$ with $\lambda = 1$. Also $\delta(C_r) \geqslant \delta(z)/\mu$ with $\mu = 1$ since $\delta(C_r) = \delta(z) = R - |z| = (R-d)/2$. If now in Lemma (2.3) we take $\varrho = (R+d)/2$ and $\sigma = (R-d)/2$ we deduce that $L(C_r, z) \leqslant A$ with $A = 4^{32}$. This establishes the proposition.

We now come to the key proposition in the sequence of propositions leading to a proof of Theorem (2.1).

PROPOSITION (2.4). *Let* $2R/3 < d < R$. *Then Theorem* (2.1) *is true for all* $z \in G$ *for which* $|z| = (R+d)/2$. *Furthermore in this case we may take* $\lambda = \lambda_{p-1}/6$, $\mu = 6\mu_{p-1}^2$, *and* $A = 4^{27}\mu_{p-1}A_{p-1}^2$.

Proof. Recall that $G = \Delta(0, R) - \{d_1, d_2, \ldots, d_p\}$ with $|d_1| \leqslant |d_2| \leqslant \ldots \ldots \leqslant |d_p| = d < R$ and $d > 0$. There is no loss of generality in supposing that $d_p = d$.

Let z be an arbitrary point on the circumference $|z| = (R+d)/2$.

Write $z = e^{i\varphi}(R+d)/2$, $-\pi < \varphi \leqslant \pi$. For such a point z we introduce the auxiliary point $c = c(z) = (2d-R)e^{i\varphi}$. Also let $R_0 = 2(R-d)$. We next introduce the auxiliary disk $\Delta_0 = \Delta(c, R_0) = \Delta(c(z), R_0)$ with center $c(z)$ and radius R_0. There are now two cases to consider; namely: $d \notin \Delta_0$ or $d \in \bar{\Delta}_0$. Here $\bar{\Delta}_0$ denotes the closure of the open disk Δ_0. Note that these two cases are not mutually exclusive.

Let us first consider the case when z is a point on the circumference $|z| = (R+d)/2$ such that $d \notin \Delta_0 = \Delta(c, R_0) = \Delta(c(z), R_0)$. We refer the reader to Fig. 1 for the construction that follows. Let $G_0 = \Delta(c(z), R_0) -$

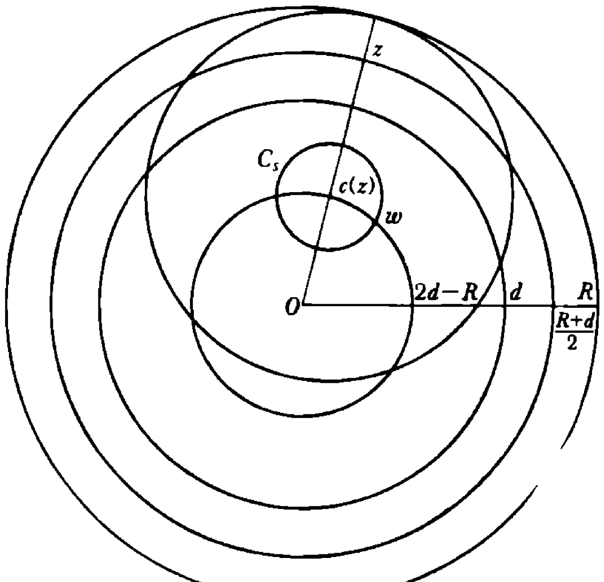

Fig. 1

$-\{d_1, d_2, \ldots, d_p\}$. Now there is a langest integer q with $0 \leqslant q \leqslant p-1$ such that $d_1, d_2, \ldots, d_q \in \Delta_0 = \Delta(c(z), R_0)$. Consequently $G_0 = \Delta(c(z), R_0) - \{d_1, d_2, \ldots, d_q\}$. Also let $\delta_0(w) = \delta(w, G_0)$ for $w \in \Delta_0 = \Delta(c(z), R_0)$. Clearly $z \in G_0$ and $\delta_0(z) = \delta(z) = (R-d)/2$. By the inductive hypothesis, Theorem (2.1) is valid for G_0. Hence there exist constants $0 < \lambda_{p-1} \leqslant 1 \leqslant \mu_{p-1}$, A_{p-1}, depending only on $(p-1)$ with the following property. There is some circumference $C_s = C(c, s)$ in G_0 with $\lambda_{p-1}|z-c| \leqslant s \leqslant |z-c|$ and $\delta_0(C_s) \geqslant \delta_0(z)/\mu_{p-1}$ such that $L(C_s, z; G_0) \leqslant A_{p-1}R_0/\delta_0(z)$. From Lemma (2.1) we see that $L(C_s, z) = L(C_s, z; G) \leqslant A_{p-1}R_0/\delta_0(z)$. Recall that $|z-c| = 3(R-d)/2$, $R_0 = 2(R-d)$, and $\delta_0(z) = \delta(z) = (R-d)/2$. From Lemma (2.5) we see that $\delta(C_s) \geqslant \delta_0(C_s)$. Thus we obtain that $3\lambda_{p-1}(R-d)/2 \leqslant s \leqslant 3(R-d)/2$, $\delta(C_s) \geqslant \delta(z)/\mu_{p-1}$, and $L(C_s, z) \leqslant 4A_{p-1}$.

Now there are two points w on the circumference $C_s = C(c, s)$ such that $|w| = |c|$. Let w be one of these points. Now $|w| = 2d-R$ whence $|z|/3 < R/3 < |w| < d < |z|$. Now w is a point of G with $|z| < d$. From Proposition (2.2) we see that there is some circumference $C_r = C(0, r)$ in G with

$(\lambda_{p-1}/2)|w| \leqslant r \leqslant |w|$ and $\delta(C_r) \geqslant \delta(w)/(6\mu_{p-1})$ such that $L(C_r, w) \leqslant 15A_{p-1}R/\delta(w)$. Now $|z|/3 < |w| < |z|$ and so $(\lambda_{p-1}/6)|z| \leqslant r \leqslant |z|$. Also $w \in C_s$ and $\delta(C_s) \geqslant \delta(z)/\mu_{p-1}$ so that $\delta(w) \geqslant \delta(z)/\mu_{p-1}$. Hence $\delta(C_r) \geqslant \delta(z)/(6\mu_{p-1}^2)$ and $L(C_r, w) \leqslant 15\mu_{p-1}A_{p-1}R/\delta(z)$. However, $w \in C_s$ and $L(C_s, z) \leqslant 4A_{p-1}$ so that $L(w, z) \leqslant 4A_{p-1}$. From Lemma (2.2) we have that $L(C_r, z) \leqslant L(C_r, w)L(w, z) \leqslant 60\mu_{p-1}A_{p-1}^2R/\delta(z)$. Thus for each point z on the circumference $|z| = (R+d)/2$ such that $d \notin \Delta_0 = \Delta(c(z), R_0)$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda_{p-1}/6$, $\mu = 6\mu_{p-1}^2$, and $A = 60\mu_{p-1}A_{p-1}^2$.

Next let us consider the case when z is a point on the circumference $|z| = (R+d)/2$ such that $d \in \bar{\Delta}_0$, See Fig. 2 for the construction to follow.

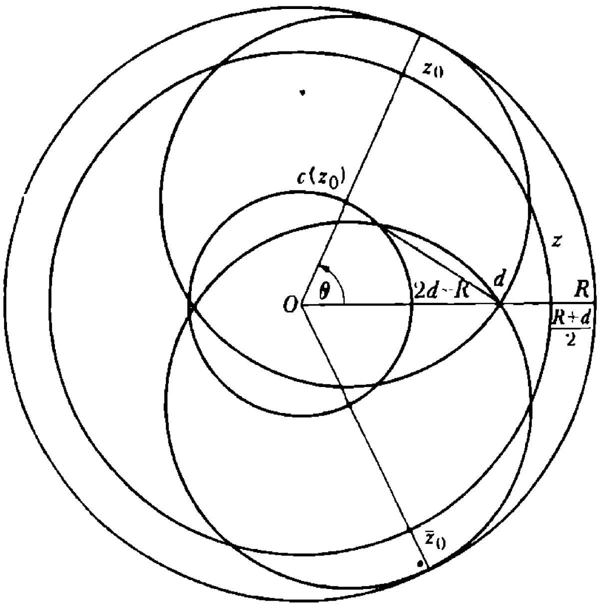

Fig. 2

It is clear that these points z lie on a closed arc S of the circumference $|z| = (R+d)/2$ which is symmetric with respect to the real axis. There is some $0 < \theta < \pi$ so that $z_0 = e^{i\theta}(R+d)/2$ and $\bar{z}_0$ are the end-points of this arc S. Note that z_0 is a point on the circumference $|z_0| = (R+D)/2$ such that $d \notin \Delta_0 = \Delta(c(z_0), R_0)$ and $d \in \bar{\Delta}_0$. The same remark also holds for the conjugate point $\bar{z}_0$. Now the arc S clearly has arc length $2\theta[(R+d)/2]$. If we apply the law of cosines to the triangle with vertices at the points $0, d$, and $c(z_0)$, we see that $(2R-2d)^2 = d^2+(2d-R)^2-2d(2d-R)\cos\theta$. Hence $\sin^2(\theta/2) = 3(R-d)^2/[4d(2d-R)]$. Now $\theta \leqslant \pi\sin(\theta/2)$ for $0 \leqslant \theta \leqslant \pi$. Hence we deduce that $\theta^2 \leqslant 3\pi^2(R-d)^2/[4d(2d-R)]$. Thus in view of the assumption that $d > 2R/3$ we find that $\theta \leqslant 6(R-d)/R$. Now there is no loss of generality in assuming that z is a point on the arc S with non-negative imaginary part. If we now apply Lemma (2.4) with

$\varrho = (R+d)/2$, $\sigma = (R-d)/2$, and $\varphi = 6(R-d)/R$ we deduce that $L(z_0, z) \leqslant 4^{24}$. As we have already mentioned before z_0 is a point on the circumference $|z_0| = (R+d)/2$ for which $d \in \Delta_0 = \Delta(c(z_0), R_0)$. Hence according to the results of the first case already established there is some circumference $C_r = C(0, r)$ in G with $\lambda|z_0| \leqslant r \leqslant |z_0|$ and $\delta(C_r) \geqslant \delta(z_0)/\mu$ such that $L(C_r, z_0) \leqslant AR/\delta(z_0)$, where $\lambda = \lambda_{p-1}/6$, $\mu = 6\mu_{p-1}^2$, and $A = 60\mu_{p-1}A_{p-1}^2$. Of course $\delta(z_0) = \delta(z) = (R-d)/2$. Also from Lemma (2.2) we have that $L(C_r, z) \leqslant L(C_r, z_0)L(z_0, z)$. Hence for each point z on the circumference $|z| = (R+d)/2$ such that $d \in \Delta_0$, where $\Delta_0 = \Delta(c(z), R_0)$, we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda_{p-1}/6$, $\mu = 6\mu_{p-1}^2$, and $A = 60 \times 4^{24}\mu_{p-1}A_{p-1}^2$. This completes the proof of the second case. Proposition (2.4) now follows.

PROPOSITION (2.5). *We have that Theorem* (2.1) *is valid for all points z of G which lie on the annulus $d < |z| < (R+d)/2$. In this case we may take $\lambda = \lambda_{p-1}/6$, $\mu = 12\mu_{p-1}^2$, and $A = 4^{33}\mu_{p-1}A_{p-1}^2$.*

Proof. There are two cases to consider according as $\delta(z) \leqslant 2(|z|-d)$ or $\delta(z) > 2(|z|-d)$.

Consider first the case when z is a point on the annulus $d < |z| < (R+d)/2$ for which $\delta(z) \leqslant 2(|z|-d)$. Let $S = 2|z|-d$. Clearly $d < S < R$. Let $G_1 = \Delta(0, S) - \{d_1, d_2, \ldots, d_p\}$ and let $\delta_1(w) = \delta(w, G_1)$ for $w \in \Delta(0, S)$. Now $|z| = (S+d)/2$. From Propositions (2.3) and (2.4) we see that there are constants $\lambda^* = \lambda_{p-1}/6$, $\mu^* = 6\mu_{p-1}^2$, and $A^* = 4^{32}\mu_{p-1}A_{p-1}^2$, depending only on p, with the following property. There is some circumference $C_r = C(0, r)$ in G_1 with $\lambda^*|z| \leqslant r \leqslant |z|$ and $\delta_1(C_r) \geqslant \delta_1(z)/\mu^*$ such that $L(C_r, z; G_1) \leqslant A^*S/\delta_1(z)$. From Lemma (2.5) we see that $\delta(w) \geqslant \delta_1(w)$ for all $|w| < S$. Clearly, $\delta_1(z) = |z|-d \geqslant \delta(z)/2$. Thus for each point z in the annulus $d < |z| < (R+d)/2$ such that $\delta(z) \leqslant 2(|z|-d)$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda^* = \lambda_{p-1}/6$, $\mu = 2\mu^* = 12\mu_{p-1}^2$, and $A = 2A^* = 2 \times 4^{32}\mu_{p-1}A_{p-1}^2$.

Next consider the case when z is a point of the annulus $d < |z| < (R+d)/2$ for which $\delta(z) > 2(|z|-d)$. Write $z = |z|e^{i\varphi}$, $-\pi < \varphi \leqslant \pi$, and set $w = de^{i\varphi}$. Now $\Delta(z, \delta(z)) \subseteq G$ and $|w-z| = |z|-d < \delta(z)/2$. Hence it is easy to see that $L(w, z) \leqslant 12$. Since $w \in G$ and $|w| \leqslant d$ we see from Proposition (2.2) that there is some circumference $C_r = C(0, r)$ in G with $(\lambda_{p-1}/2)|w| \leqslant r \leqslant |w|$ and $\delta(C_r) \geqslant \delta(w)/(6\mu_{p-1})$ such that $L(C_r, w) \leqslant 15A_{p-1}R/\delta(w)$. Clearly $|w| = d < |z|$. Now $2(|z|-d) < \delta(z) \leqslant |z-d_p| \leqslant |z|+d$ whence $|w| = d > |z|/3$. Also it is easy to see that $\delta(w) \geqslant \delta(z)/2$. Also from Lemma (2.2) we see that $L(C_r, z) \leqslant L(C_r, w)L(w, z) \leqslant 12L(C_r, w)$. Thus for each point z in the annulus $d < |z| < (R+d)/2$ with $\delta(z) > 2(|z|-d)$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z$

$\leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda_{p-1}/6$, $\mu = 12\mu_{p-1}$, and $A = 360A_{p-1}$. This completes the proof of the proposition.

PROPOSITION (2.6). *We have that Theorem* (2.1) *is valid for all points z of G which lie in the annulus $(R+d)/2 < |z| < R$. Furthermore in this case we may take $\lambda = \lambda_{p-1}/12$, $\mu = 6\mu_{p-1}^2$, and $A = 4^{33}\mu_{p-1}A_{p-1}^2$.*

Proof. Let z be such that $(R+d)/2 < |z| < R$. Write $z = |z|e^{i\varphi}$, $-\pi < \varphi \leqslant \pi$, and set $w = e^{i\varphi}(R+d)/2$. From Propositions (2.3) and (2.4) we see that there are constants $\lambda^* = \lambda_{p-1}/6$, $\mu^* = 6\mu_{p-1}^2$, and $A^* = 4^{32}\mu_{p-1}A_{p-1}^2$, depending only on p, with the following property. There is some circumference $C_r = C(0, r)$ in G with $\lambda^*|w| \leqslant r \leqslant |w|$ and $\delta(C_r) \geqslant \delta(w)/\mu^*$ such that $L(C_r, w) \leqslant A^*R/\delta(w)$. Now $\delta(w) = (R-d)/2$ and $\delta(z) = R-|z|$ so that $\delta(w) > \delta(z)$ and $L(C_r, w) \leqslant 2A^*R/(R-d)$. Now $\Delta(w, (R-d)/2) \subseteq G$ and $|z-w| = |z|-(R+d)/2 < (R-d)/2$. Hence we see that $L(w, z) \leqslant (R-d)/(R-|z|)$. From Lemma (2.2) we see that $L(C_r, z) \leqslant L(C_r, w)L(w, z) \leqslant 2A^*R/(R-|z|) = 2A^*R/\delta(z)$. Thus for each point z in the annulus $(R+d)/2 < |z| < R$ we have found a circumference $C_r = C(0, r)$ in G with $\lambda|z| \leqslant r \leqslant |z|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that $L(C_r, z) \leqslant AR/\delta(z)$, where $\lambda = \lambda^*/2 = \lambda_{p-1}/12$, $\mu = \mu^* = 6\mu_{p-1}^2$, and $A = 2A^* = 2\times 4^{32}\mu_{p-1}A_{p-1}^2$. This completes the proof of the proposition.

In view of the foregoing propositions, we see that Theorem (2.1) is true for $G = \Delta(c, R) - \{d_1, d_2, \ldots, d_p\}$. Hence by the principle of mathematical induction we see that Theorem (2.1) is true for all $n \geqslant 0$.

A perusal of the above argument shows that we can obtain recursive estimates for the constants $\lambda = \lambda_n$, $\mu = \mu_n$, and $A = A_n$ which appear in the formulation of Theorem (2.1). We can therefore obtain explicit bounds for λ_n, μ_n, and A_n in terms of n. First from Proposition (2.1) we see that we may take $\lambda_0 = \frac{1}{2}$, $\mu_0 = 1$, and $A_0 = 4^7$. Next from Propositions (2.2), (2.3), (2.4), (2.5), and (2.6) we see that we may assume that $\lambda_n \leqslant \lambda_{n-1}/12$, $\mu_n \geqslant 12\mu_{n-1}^2$, and $A_n \geqslant a\mu_{n-1}A_{n-1}^2$ for $n = 1, 2, \ldots$, where $a = 4^{33}$. From the above recursive inequalities we see that Theorem (2.1) is valid if we assume that $\lambda = \lambda_n$, $\mu = \mu_n$, and $A = A_n$ are given by means of the expressions $\lambda_n = 1/(12)^{n+1}$, $\log\mu_n = 2^n\log 12$, and $\log A_n = 4^n\log a$ for $n = 0, 1, 2, \ldots$, where $a = 4^{33}$.

As a simple consequence of Theorem (2.1) we mention the following result which is perhaps of some interest in itself.

COROLLARY (2.1). *Let $G = \Delta(c, R) - \{d_1, d_2, \ldots, d_n\}$, where $n \geqslant 0$ and let $\delta(z) = \delta(z, G)$. There exist constants λ, μ, and B, with $0 \leqslant \lambda \leqslant 1 \leqslant \mu$ and $B > 0$, depending only on n with the following property. For each $z \in G$ there is some circumference $C_r = C(c, r)$ in G with $\lambda|z-c| \leqslant r \leqslant |z-c|$ and $\delta(C_r) \geqslant \delta(z)/\mu$ such that for each point $w \in C_r$ there is some polygonal arc γ in G*

joining w to z for which

$$\int_{\gamma} \frac{ds}{\delta} < B + \ln\left(\frac{R}{\delta(z)}\right).$$

Proof. Let $z \in G$ be given. Now assume the conclusion of Theorem (2.1). Let $w \in C_r$. If $\varepsilon \in 0$ is given, there is some chain $C = \{z_1, z_2, \ldots, z_m\}$ in G with $w = z_1$ and $z = z_m$ such that $L(C) < \varepsilon + AR/\delta(z)$. Hence for $k = 1, 2, \ldots, m-1$ there are constants $r_k > 0$ and $0 < a_k < 1$ such that $\Delta(z_k, r_k) \subseteq G$ and $z_{k+1} \in \Delta(z_r, a_r r_r)$. Let $\beta_k = (1+a_k)/(1-a_k)$ for $k = 1, 2, \ldots$ $\ldots, m-1$. Then of course

$$L(C) = \prod_{k=1}^{m-1} (1+\beta_k).$$

Now for $k = 1, 2, \ldots, m-1$, let γ_k denote the directed line segment from z_k to z_{k+1}. Also let $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$. Clearly γ is a polygonal arc in G joining w to z. Now for $t \in \gamma_k$ it is clear that $\delta(t) \geqslant r_k - |t - z_k|$. If we now let $z(t) = z_k + t(z_{k+1} - z_k)$, $0 \leqslant t \leqslant 1$, be the parametric equation of γ_k we see that $\delta\big(z(t)\big) \geqslant r_k - |z_{k+1} - z_k| t \geqslant r_k(1 - a_k t)$ and $ds(t) = |\dot{z}(t)|\, dt = |z_{k+1} - z_k|\, dt \leqslant a_k r_k dt$ for $0 \leqslant t \leqslant 1$. Consequently

$$\int_{\gamma_k} \frac{ds}{\delta} \leqslant \int_0^1 \frac{a_k dt}{1 - a_k t} = \ln \frac{1}{1 - a_k} \leqslant \ln \beta_k$$

for $k = 1, 2, \ldots, m-1$. Hence we obtain that

$$\int_{\gamma} \frac{ds}{\delta} < \ln A + O(\varepsilon) + \ln\left(\frac{R}{\delta(z)}\right).$$

Thus we see that Corollary (2.1) is true with $B = \ln A$. This establishes the result.

3. Saxer's theorem. In this section we will indicate how a simple proof of Saxer's theorem can be fashioned from the geometric proposition formulated in Theorem (2.1).

First recall that if $f(z)$ is regular in $\Delta(0, 1)$ and does not assume the values zero and one there, then

$$\log^+ |f(z)| \leqslant \frac{1 + |z|}{1 - |z|} \big(\log^+ |f(0)| + \pi\big),$$

for all $z \in \Delta(0, 1)$, where π is the best possible. This version of Schottky's theorem was established by W. K. Hayman in [1] by means of the modular function and the hyperbolic metric.

We first have the following result.

THEOREM (3.1). *Let $f(z)$ be meromorphic in $\Delta(0,1)$ and suppose that $f(z)$ assumes the values zero, one, and infinity at most n distinct points $d_1, d_2, \ldots, d_n$ of $\Delta(0,1)$. Let $G = \Delta(0,1)-\{d_1, d_2, \ldots, d_n\}$ and $\delta(z) = \delta(z, G)$. Let $\mu > 0$ and suppose that for each circumference $C_r = C(0,r)$ in G there is at lease one point $z \in C_r$ for which $|f(z)| \leqslant \mu$. Then there is some constant A depending only on n such that*

$$\log^+|f(z)| \leqslant \frac{A}{\delta(z)}(\log^+\mu+\pi)$$

for all $z \in \Delta(0,1)$ such that $f(z) \neq 0, 1, \infty$.

Proof. Let $z \in \Delta(0,1)$ be a fixed point for which $f(z) \neq 0, 1, \infty$. Thus $z \in G = \Delta(0,1)-\{d_1, d_2, \ldots, d_n\}$. According to Theorem (2.1) there is some constant A depending only on n with the following property. There is some circumference $C_r = C(0,r)$ in G for which $L(C_r, z) \leqslant A/\delta(z)$. There is some $w \in C_r$ for which $|f(w)| \leqslant \mu$. Now $L(w, z) \leqslant A/\delta(z)$. Hence if $\varepsilon > 0$ is given we can find a chain $C = \{z_1, z_2, \ldots, z_{m+1}\}$ in G with $z_1 = w$ and $z_{m+1} = z$ such that $L(C) < \varepsilon + A/\delta(z)$. Hence for $k = 1, 2, \ldots, m$ there exist constants $r_k > 0$ and $0 < a_k < 1$ such that $\Delta(z_k, r_k) \leqslant G$ and $z_{k+1} \in \Delta(z_k, a_k r_k)$. Let $\beta_k = (1+a_k)/(1-a_k)$ for $k = 1, 2, \ldots, m$. Consequently,

$$L(C) = \prod_{k=1}^{m}(1+\beta_k) < \varepsilon + \frac{A}{\delta(z)}.$$

Now for each $k = 1, 2, \ldots, m$ we see that $f(z)$ is regular in $\Delta(z_k, r_k)$ and does not assume the values zero and one there. Also $z_{k+1} \in \Delta(z_k, a_k r_k)$. Hence by Schottky's theorem we deduce that

$$\log^+|f(z_{k+1})| \leqslant \beta_k\big(\log^+|f(z_k)|+\pi\big)$$

for $k = 1, 2, \ldots, m$. Now taking into account the fact that $|f(z_1)| \leqslant \mu$ and $z_{m+1} = z$, we deduce from the foregoing recursive inequalities that

$$\log^+ |f(z)| \leqslant (\beta_1\beta_2\ldots\beta_m)\log^+\mu + \pi\sum_{k=1}^{m}(\beta_k\beta_{k+1}\ldots\beta_m) \leqslant L(C)(\log^+\mu+\pi).$$

Consequently,

$$\log^+|f(z)| \leqslant \left[\varepsilon + \frac{A}{\delta(z)}\right][\log^+\mu+\pi]$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

On the basis of the above result we are easily led to the following variant of Saxer's theorem.

COROLLARY (3.1). *Let $f(z)$ be regular in $\Delta(0,1)$. Assume that $f(z)$ assumes the value zero at most p times in $\Delta(0,1)$, counting multiplicities. Assume further that $f(z)$ assumes the values zero and one at most n distinct* *points of $\Delta(0,1)$. Finally suppose that*

$$f(z) = a_0 + a_1 z + \ldots + a_p z^p + \ldots$$

for $z \in \Delta(0,1)$. Then there is a constant A depending only on n such that

$$\log^+ |f(z)| \leqslant A \frac{1+|z|}{1-|z|} (\log^+ \mu + \pi)$$

for $z \in \Delta(0,1)$, where $\mu = |a_0| + |a_1| + \ldots + |a_p|$.

Proof. Let $P(z) = a_0 + a_1 z + \ldots + a_p z^p$ for $z \in \Delta(0,1)$. Clearly, $|P(z)| \leqslant \mu$ for all $z \in \Delta(0,1)$. Suppose, if possible, that there is some circumference $C_r = C(0,r)$ in $\Delta(0,1)$ on which $|f(z)| > \mu$. Thus $r > 0$ and $|f(z)| > |P(z)|$ for all $z \in C_r$, Hence by Rouche's theorem we deduce that $f(z)$ and $f(z) - P(z)$ have the same number of zeros in $\Delta(0,r)$. This is impossible. Hence for each circumference $C_r = C(0,r)$ in $\Delta(0,1)$ there is at least one point $z \in C_r$ for which $|f(z)| \leqslant \mu$.

Next let $d_1, d_2, \ldots, d_n$ be the n distinct points of $\Delta(0,1)$ at which the function $f(z)$ assumes the values zero or one. Let $G = \Delta(0,1) - \{d_1, d_2, \ldots, d_n\}$ and $\delta(z) = \delta(z, G)$.

Now let $z \in \Delta(0,1)$ be fixed and consider the annulus $\{w\colon |z| < |w| < (1+|z|)/2\}$. Let $\sigma_k = |z| + k[1-|z|]/[2(n+1)]$ for $k = 0, 1, \ldots, (n+1)$. There is some $k = 0, 1, \ldots, n$ such that $f(w) \neq 0, 1$ for $\sigma_k < |w| < \sigma_{k+1}$. Let $\sigma = (\sigma_k + \sigma_{k+1})/2$. Clearly $\delta(w) \geqslant [1-|z|]/[4(n+1)]$ for all $w \in C_\sigma = C(0, \sigma)$.

Hence from Theorem (3.1) we see that

$$\log^+ |f(w)| \leqslant \frac{4(n+1)A}{1-|z|} (\log^+ \mu + \pi),$$

for all $w \in C_\sigma$. Since $|z| < \sigma$, the result follows.

As a further application of Theorem (2.1) we offer the following result.

THEOREM (3.2). *Let $f(z)$ and $g(z)$ be regular in $\Delta(0,1)$ and suppose that $f(z)$ has at most p zeros in $\Delta(0,1)$, counting multiplicities. Suppose further that the total number of distinct zeros of the functions $f(z)$, $f(z) - g(z)$, and $g(z)$ in $\Delta(0,1)$ is at most n. Finally let the Laurent series expansion of $f(z)/g(z)$ with center at the origin be*

$$\frac{f(z)}{g(z)} = a_{-m} z^{-m} + \ldots + a_p z^p + \ldots,$$

and let $\mu = |a_{-m}| + \ldots + |a_p|$. Then there exist constants $0 < \lambda \leqslant 1 \leqslant A$ depending only on n such that

$$\log M(r,f) \leqslant \log M(R,g) + \frac{A}{R-r}\left[\log^+ \left(\frac{\mu}{\lambda^m R^m}\right) + \pi\right],$$

for $0 \leqslant r < R < 1$.

Proof. There are n distinct points $d_1, d_2, \ldots, d_n$ of $\Delta(0,1)$ such that the functions $f(z), f(z)-g(z)$, and $g(z)$ never vanish on $G = \Delta(0,1)-\{d_1, d_2, \ldots, d_n\}$. Let $\delta(z) = \delta(z, G)$. Consequently $f(z)/g(z) \neq 0, 1, \infty$ in G. If we let $P(z) = a_{-m}z^{-m}+ \ldots +a_p z^p$ we see that for every circumference $C_r = C(0,r)$ in G there is at least one point $z \in C_r$ for which $|f(z)/g(z)| \leqslant |P(z)|$. But $|P(z)| \leqslant \mu/|z|^m$ for $0<|z|<1$. Hence for every circumference $C_r = C(0,r)$ in G we see that there is at least one point $z \in C_r$ for which $|f(z)/g(z)| \leqslant \mu/|z|^m$. Then by reasoning similar to the proof of Theorem (3.1) we see that there are constants $0<\lambda \leqslant 1 \leqslant A$ depending only on n such that

$$\log|f(z)| \leqslant \log|g(z)| + \frac{A}{\delta(z)}\left[\log^+\left|\frac{\mu}{\lambda^m z^m}\right| + \pi\right]$$

for all $z \in G = \Delta(0,1)-\{d_1, d_2, \ldots, d_n\}$.

Now let $0 \leqslant r < R < 1$ be fixed. There is some $(R+r)/2 < \sigma < (3R+r)/4$ such that for the circumference $C_\sigma = C(0,\sigma)$ we have that $\delta(C_\sigma) \geqslant [R-r]/[8(n+1)]$. Consequently,

$$\log|f(z)| \leqslant \log|g(z)| + \frac{8(n+1)A}{R-r}\left[\log^+\left(\frac{\mu}{\lambda^m \sigma^m}\right) + \pi\right]$$

or $|z| = \sigma$. The result now follows.

4. Functions polyanalytic on a disk.

In this section we offer an application of Theorem (2.1) in another direction by establishing a Picard type theorem for functions polyanalytic on a disk.

For purposes of completeness we give the following definitions. Le G be a non-empty open connected subset of the finite complex plane Γ. A function $f\colon G \to \Gamma$ is said to be *polyanalytic on G* or *n-analytic on G* if and only if there exist $(n+1) \geqslant 1$ functions $f_0, f_1, \ldots, f_n$ analytic on G such that

$$f(z) = \sum_{k=0}^{n} \bar{z}^k f_k(z), \tag{4.1}$$

for all $z \in G$, where $\bar{z}$ denotes the complex conjugate of z. It is not difficult to be persuaded that this representation of f on G is unique.

Now let f be polyanalytic on $\Delta = \Delta(0,1)$. Then for $0 \leqslant r \leqslant 1$ we define $M(r,f)$ to be the maximum of $|f(z)|$ for $|z| = r$. We shall say that f is *admissible on Δ* if and only if

$$\limsup_{r\to 1}(1-r)\log M(r,f) = +\infty.$$

Again if f is polyanalytic on $\Delta = \Delta(0,1)$ and if a is any finite complex number, then a is said to be an *exceptional value for f on Δ* if and only if there is some $0<r<1$ such that $f(z) \neq a$ for all $r<|z|<1$.

If f is polyanalytic on $\Delta = \Delta(0, 1)$ and is represented on Δ by means of (4.1), then for $0 \leqslant \varrho < 1$ it will be convenient to introduce the auxiliary function $f(z, \varrho)$ defined on Δ by the condition that

$$f(z, \varrho) = \sum_{k=0}^{n} \varrho^{2k} z^{n-k} f_k(z)$$

for $z \in \Delta$. Note that for $0 \leqslant \varrho < 1$ that $f(z, \varrho)$ is regular on Δ and that $f(z, \varrho) = z^n f(z)$ for $|z| = \varrho$.

Next if f is a finite complex valued function which is continuous and never zero on some circumference $C(0, \varrho)$ we define $\Delta_\varrho \arg f$ to be $1/(2\pi)$ times the change in the argument of f around the positively oriented circumference $C(0, \varrho)$.

We now have the following result.

THEOREM (4.1). *Let f be polyanalytic on $\Delta = \Delta(0, 1)$ and admit the exceptional values zero and one on Δ. Then f is not admissible on Δ.*

Proof. Let f be represented on Δ by means of equation (4.1). There is some $0 < r_0 < 1$ so that $f(z) \neq 0, 1$ for $r_0 < |z| < 1$. Hence there are integers s and t so that $\Delta_\varrho \arg f = s$ and $\Delta_\varrho \arg(f-1) = t$ for $r_0 < \varrho < 1$. Consequently $\Delta_\varrho \arg f(z, \varrho) = n+s$ and $\Delta_\varrho \arg[f(z, \varrho) - z^n] = n+t$ for $r_0 < \varrho < 1$. Let $p = \max\{n+s, n+t, n\}$ and $m = 3p$. Thus if $r_0 < \varrho < 1$ we see that $f(\varrho z, \varrho)$ is regular in Δ and has at most p zeros there. Also for $r_0 < \varrho < 1$ we see that the total number of distinct zeros of the functions $f(\varrho z, \varrho)$, $f(\varrho z, \varrho) - (\varrho z)^n$, and $(\varrho z)^n$ in Δ is at most m.

Finally let

$$\frac{f(\varrho z, \varrho)}{(\varrho z)^n} = \frac{a_{-n}(\varrho)}{z^n} + \ldots + a_p(\varrho) z^p + \ldots,$$

be the Laurent series expansion of $f(\varrho z, \varrho)/(\varrho z)^n$ with center zero. Now set $\mu(\varrho) = |a_{-n}(\varrho)| + \ldots + |a_p(\varrho)|$. It is easy to see that there is some positive constant μ independent of ϱ such that $\mu(\varrho) < \mu$ for $r_0 < \varrho < 1$.

From Theorem (3.1) we see that there are constants $0 < \lambda \leqslant 1 \leqslant A$ depending only on m such that $\log|f(\varrho z, \varrho)| \leqslant A[\log^+(\mu/\lambda^n) + \pi]/(1-|z|)$ for $0 \leqslant |z| < 1$ and $r_0 < \varrho < 1$. Hence there is some positive constant B independent of ϱ such that

$$\log|f(z, \varrho)| \leqslant \frac{B}{\varrho - |z|}$$

for $0 \leqslant |z| < \varrho$ and $r_0 < \varrho < 1$.

We now wish to show by virtue of the above estimate that the functions $f_0, f_1, \ldots, f_n$ are not admissible on Δ. To this end let $r_0 < r < 1$ be fixed. Now let $\varrho_0, \varrho_1, \ldots, \varrho_n$ be $(n+1)$ distinct real numbers subject for the moment only to the restriction that $r < \varrho_0 < \varrho_1 < \ldots < \varrho_n < 1$. Hence

$$f(z, \varrho_\nu) = \sum_{k=0}^{n} \varrho_\nu^{2k} z^{n-k} f_k(z) \qquad (\nu = 0, 1, \ldots, n).$$

The above equations can be regarded as a system of $(n+1)$ linear equations in the $(n+1)$ unknown functions $z^{n-k} f_k(z)$ $(k = 0, 1, \ldots, n)$. The determinant of the coefficients of these unknowns is the Alternant $D = \Pi(\varrho_j^2 - \varrho_i^2)$, where $i, j = 0, 1, \ldots, n$ and $i < j$. From Cramer's rule we see that there is some positive constant K depending only on n such that

$$|z^{n-k} f_k(z)| \leqslant \frac{K}{D} \sum_{\nu=0}^{n} |f(z, \varrho_\nu)| \qquad (k = 0, 1, \ldots, n).$$

Consequently

$$\log M(r, f_k) \leqslant O(1) + O\left(\log \frac{1}{r}\right) + \log^+ \frac{1}{D} + \sum_{\nu=0}^{n} \log^+ M\big(r, f(z, \varrho_\varrho)\big)$$

for $k = 0, 1, \ldots, n$. Hence

$$\log M(r, f_k) \leqslant O(1) + O\left(\log \frac{1}{r}\right) + \log^+ \frac{1}{D} + B \sum_{\nu=0}^{n} \frac{1}{(\varrho_\nu - r)}$$

for $k = 0, 1, \ldots, n$. If we now suppose that

$$\varrho_\nu = r + (\nu+1)(1-r)/[2(n+1)] \quad \text{for } \nu = 0, 1, \ldots, n,$$

it follows that

$$\log M(r, f_k) < O(1) + O\left(\log \frac{1}{r}\right) + O\left(\frac{1}{1-r}\right)$$

for $r_0 < r < 1$ and $k = 0, 1, \ldots, n$. We have therefore shown that the functions $f_0, f_1, \ldots, f_n$ are not admissible on Δ.

However, from equation (4.1) one readily sees that

$$\log M(r, f) \leqslant \sum_{k=0}^{n} \log^+ M(r, f_k) + \log(n+1)$$

for $0 \leqslant r < 1$. Hence f is not admissible on Δ as claimed.

References

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