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INDEPENDENT SETS OF TRANSITIVE POINTS

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Let X be a separable complete metric space which is dense in itself and $T\colon X \to X$ be a continuous mapping. A point $x \in X$ is called *transitive* if its forward orbit, $\{T^j x\colon j \geqslant 0\}$, is dense in X. The transitive points form a G_δ set and if x is transitive then so is any $T^j x$ for $j \geqslant 0$. Therefore the set of transitive points is either empty or residual. Two transitive points x, y will be called *independent* if (x, y) is transitive in the product flow $(X \times X, T)$. If such two points exist then (X, T) is said to be (*topologically*) *weakly mixing* ([3], [7], [10]). More generally, we have

DEFINITION. A subset E of X is called *independent* if any finite sequence $(x_1, \ldots, x_n)$ of distinct elements from E is a transitive point in the product flow (X^n, T).

It is clear that independent sets with at least two elements can only exist in weakly mixing flows. The aim of this note is to produce an uncountable independent set in every weakly mixing flow. In fact, in Section 1 we apply a general theorem of Mycielski [8] in order to obtain a dense uncountable independent Borel set. If in addition X is compact and there exists an invariant probability measure μ with $\operatorname{supp}\mu = X$ then we construct an uncountable independent set by transfinite induction (Section 2). Under an additional set theoretic assumption (the continuum hypothesis will suffice) this set intersects every Borel set of the second category. In Section 3 we prove that if, additionally, (X, μ, T) is mildly mixing (in particular if it is strongly mixing) then, assuming e.g. the continuum hypothesis, there exists an independent set of outer measure one.

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1. Independent Borel sets

For every $n \in N$ we define an n-argument relation by letting $R_n(x_1, \ldots, x_n)$ iff there exist distinct elements $y_1, \ldots, y_k \in \{x_1, \ldots, x_n\}$ such that $(y_1, \ldots, y_k)$ is not a transitive point in (X^k, T). Since

$$R_{n+1}(x_1, \ldots, x_i, x_j, x_{i+1}, \ldots, x_n) = R_n(x_1, \ldots, x_n)$$

for every $1 \leqslant j \leqslant n$, the relational structure $\mathfrak{R} = (X, R_1, R_2, \ldots)$ is closed under identification of variables.

THEOREM 1. *The following conditions are equivalent*:

(1) *(X, T) is topologically weakly mixing,*

(2) *there exists a transitive point in each (X^n, T), $n \in N$,*

(3) *there exists an infinite independent set in (X, T),*

(4) *there exists an uncountable independent Borel set in (X, T).*

(5) *there exists an independent set which is a dense countable union of Cantor sets.*

Proof. Each condition is stronger than its predecessor, so it suffices to prove (1) $\Rightarrow$ (5). First note that (1) $\Leftrightarrow$ (2), which is the content of Prop. II.3 in Furstenberg [3] (the proof in a noninvertible noncompact case is essentially the same). This implies that the set of transitive points is residual in each X^n, so each R_n is a first category set in X^n. The relational structure $\mathfrak{R}$ satisfies the assumptions of Mycielski's theorem [8], so there exists an $\mathfrak{R}$-independent set E which is a dense countable union of Cantor sets. The $\mathfrak{R}$-independence means that if $x_1, \ldots, x_n$ are distinct elements from E then $R_n(x_1, \ldots, x_n)$ does not hold, or $(x_1, \ldots, x_n)$ is a transitive point in (X^n, T). Therefore E is an independent set of transitive points in (X, T).

Remark 1. The equivalence of the first four conditions can also be obtained by a theorem of Kuratowski. In fact, if $C(X)$ denotes the family of all nonempty compact subsets of X endowed with Hausdorff distance then the set of those $E \in C(X)$ which are independent Cantor sets is residual in $C(X)$ (see [5], § 6).

2. An inductive construction

In this section X is additionally assumed to be compact. This implies the existence of a T-invariant probability (Borel) measure μ. Recall that $\operatorname{supp} \mu = X$ means $\mu(V) > 0$ for every nonempty open set V.

LEMMA. *Let (X, T) be topologically weakly mixing and X be compact. Assume there exists an invariant probability measure μ with* $\operatorname{supp} \mu = X$. *Then for every transitive $x \in X$,*

$$\{y \in X\colon (x, y) \text{ is transitive in } X \times X\}$$

is a dense G_δ set in X.

Proof. This is essentially Thm. 1.1 in McMahon and Wu [7]. The only difference lies in the fact that now T is not necessarily invertible. This requires some changes in the theory of weakly mixing flows developed by Westerbeck and Wu [10] on which the result in [7] depends. We indicate that if the abelian group of [10] is replaced by the additive semigroup of natural numbers then Lemmas 2.1–2.6 as well as Theorems 4.1 (a), 5.2, and 6.3 (a) $\Rightarrow$ (b) in [10] remain valid with almost the same proofs. We only have to use the fact that if an equicontinuous flow has a recurrent transitive point then it is homeomorphic.

Let $\mathscr{I}$ be a σ-ideal of subsets of X. We define $\operatorname{cov}(\mathscr{I})$ to be the minimal cardinal number γ such that there exists a family $\{A_\lambda\colon \lambda \in \Lambda\} \subset \mathscr{I}$ with $\operatorname{card}\Lambda = \gamma$ and $\bigcup A_\lambda = X$. If $\mathscr{I}$ is the σ-ideal $\mathscr{M}$ of the first category sets or the σ-ideal $\mathscr{N}$ of null sets for some nonatomic measure then clearly

$$\aleph_1 \leqslant \operatorname{cov}(\mathscr{I}) \leqslant 2^{\aleph_0}$$

(for these and related cardinal numbers see e.g. [2]). We note that for all dense in itself separable complete metric spaces X the numbers $\operatorname{cov}(\mathscr{M})$ are the same (see [9], Ch. II, § 35 Example (e), and 32.5). Likewise, the numbers $\operatorname{cov}(\mathscr{N})$ coincide for all finite nonatomic measures μ on X. In particular no Borel set of the second category can be covered by less than $\operatorname{cov}(\mathscr{M})$ first category sets and no Borel set of positive measure μ can be covered by less than $\operatorname{cov}(\mathscr{N})$ μ-null sets.

THEOREM 2. *Let (X, T) be topologically weakly mixing. Assume in addition that X is compact and* $\operatorname{supp}\mu = X$ *for some invariant probability measure μ. If $\mathscr{C} \subset \mathscr{B} \setminus \mathscr{M}$ and $\operatorname{card}\mathscr{C} \leqslant \operatorname{cov}(\mathscr{M})$ then there exists an independent set E which intersects each C in $\mathscr{C}$ (here $\mathscr{B}$ denotes the Borel σ-algebra).*

Proof. We may order $\mathscr{C}$ into a transfinite sequence $\{C_\alpha\colon \alpha < \operatorname{cov}(\mathscr{M})\}$ with $C_\alpha = X$ whenever α is a limit ordinal. Using transfinite induction we shall construct an ascending chain $\{E_\alpha\colon \alpha < \operatorname{cov}(\mathscr{M})\}$ of independent sets such that $\operatorname{card} E_\alpha \leqslant \alpha + 1$ and $E_\alpha \cap C_\alpha \neq \emptyset$. The union $\bigcup E_\alpha$ will clearly satisfy the assertion. The construction bears comparison with that of Bernstein set.

Since the set of transitive points is residual, there exists a transitive point x in C_0 and we let $E_0 = \{x\}$. Now let $0 < \beta < \operatorname{cov}(\mathscr{M})$ and suppose the sets E_α, $\alpha < \beta$, have already been defined. We consider two cases.

1. β is a nonlimit ordinal. For any distinct elements $x_1, \ldots, x_n$ in $E_{\beta-1}$ consider the set of those $y \in X^n$ such that $(x_1, \ldots, x_n, y)$ is transitive in X^{2n}. It is a dense G_δ set in X^n (apply (1) $\Leftrightarrow$ (2) of Theorem 1 and the Lemma), so its projection $Z(x_1, \ldots, x_n)$ into the first coordinate is residual in X ([6], § 24, VI, Cor. 2). The cardinality of the family

$$\mathscr{Z} = \{Z(x_1, \ldots, x_n)\colon x_1, \ldots, x_n \text{ are distinct elements in } E_{\beta-1},\ n \in N\}$$

is less than $\operatorname{cov}(\mathscr{M})$ hence the union of the complements cannot cover C_β. Therefore there exists $z \in C_\beta$ contained in the intersection of the family $\mathscr{Z}$. We let $E_\beta = E_{\beta-1} \cup \{z\}$. It is now clear that $\operatorname{card} E_\beta \leqslant \beta+1$ and E_β is independent.

2. β is a limit ordinal. We let $E_\beta = \bigcup_{\alpha<\beta} E_\alpha$. The independence is obvious. We also have $\operatorname{card} E_\beta \leqslant \sup\{\alpha+1\colon \alpha<\beta\} < \beta+1$.

COROLLARY 1. *If in addition* $\operatorname{cov}(\mathscr{M}) = 2^{\aleph_0}$ *then there exists an independent set* E *such that* $E \cap C \neq \emptyset$ *for every second category Borel set* C.

Proof. Let $\mathscr{C} = \mathscr{B} \setminus \mathscr{M}$ in Theorem 2.

3. Independent sets in mildly mixing flows

In this section X is again any dense in itself separable complete metric space. The orbit of x in (X, T) is the set of all $y \in X$ such that $T^j x = T^k y$ for some $j, k \geqslant 0$. If E is independent then it is not hard to see that E intersects each orbit at most one point. This implies that the sets $T^{-j}E$, $j = 0, 1, \ldots$, are pairwise disjoint. In particular, if μ is an invariant probability measure then the inner measure of E is equal to zero, so either $\mu(E) = 0$ or E is nonmeasurable. Can both cases occur? A partial answer to this question is given in Corollary 2.

First we recall the definition of mild mixing. A subset S of N is called an *IP-set* if there exists an infinite sequence $p_1, p_2, \ldots$ in N such that

$$S = \{p_{i_1} + \ldots + p_{i_n}\colon i_1 < \ldots < i_n,\ n \in N\}.$$

A set that intersects all IP-sets is called an IP^*-set. The IP^*-sets form a filter in N. A numerical sequence a_n IP^*-converges to a limit a if for every $\varepsilon > 0$ the set

$$\{n\colon |a_n - a| < \varepsilon\}$$

is an IP^*-set. We then write $IP^*\text{-}\lim a_n = a$. A measure preserving transformation T of a probability space $(X, \mathscr{B}, \mu)$ is called *mildly mixing* if

$$IP^*\text{-}\lim \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$$

for all $A, B \in \mathscr{B}$ (see [4], Ch. 9, § 4). By $\mathscr{N}$ we shall always denote the σ-ideal of μ-null sets.

THEOREM 3. *Suppose there exists an invariant probability measure* μ *such that* $\operatorname{supp}\mu = X$ *and* $(X, \mathscr{B}, \mu, T)$ *is mildly mixing. If* $\mathscr{C} \subset \mathscr{B} \setminus \mathscr{M} \cap \mathscr{N}$ *and* $\operatorname{card}\mathscr{C} \leqslant \min(\operatorname{cov}(\mathscr{M}), \operatorname{cov}(\mathscr{N}))$ *then there exists an independent set* E *such that* $E \cap C \neq \emptyset$ *for any* $C \in \mathscr{C}$.

Proof. As in the previous proof, the construction runs by transfinite induction. We denote $\gamma = \min(\operatorname{cov}(\mathscr{M}), \operatorname{cov}(\mathscr{N}))$ and let $\mathscr{C} = \{C_\alpha\colon \alpha < \gamma\}$. As

before, we define a chain of independent sets E_α, $\alpha < \gamma$. For $\alpha = 0$ we let $E_0 = \{x_0\}$ where x_0 is a transitive point contained in C_0. Note that such a point always exists since the set of transitive points is a dense G_δ of measure one (ergodic theorem) while C_0 is either of positive measure or of the second category. Suppose the sets E_α, $\alpha < \beta$, have already been defined and $\operatorname{card} E_\alpha = \alpha + 1$, $E_\alpha \cap C_\alpha \neq \emptyset$. The limit case is handled in the same manner as before, so we may assume that β is a nonlimit ordinal. We pick any distinct points $x_1, \ldots, x_n$ from $E_{\beta-1}$ and any neighborhoods $U_1, \ldots, U_n$ from a countable neighborhood basis $\mathscr{U}$ in X. Now denote $x = (x_1, \ldots, x_n)$, $U = U_1 \times \ldots \times U_n$, and $N(x, U) = \{j\colon T^j x \in U\}$. Since x is transitive, there exists a natural number p such that $T^p x = z \in U$. Now z is still a transitive point so $N(z, U)$ contains an IP-set S ([4], Thm. 2.17). We want to prove that the set

$$Y(x, U, V) = \{y \in X\colon N(y, V) \cap N(x, U) \neq \emptyset\}$$

is residual and has measure one for any $V \in \mathscr{U}$. It suffices to show that the set

$$\bigcup_{j \in p+S} T^{-j} V$$

has the two properties. Since it is open, we only have to show that it has measure one. Suppose to the contrary that its complement B has positive measure. Then

$$\{j\colon \mu(B \cap T^{-(p+j)} V) > 0\} = \{j\colon \mu(B \cap T^{-j}(T^{-p} V)) > 0\}$$

is an IP^*-set. Consequently, there exists $j \in S$ such that

$$B \cap T^{-(p+j)} V \neq \emptyset,$$

which is impossible. Now denote by Y_β the intersection of the sets $Y(x, U, V)$, where x, U, V are as above. Since the sets are less than γ in number and each of them is large in both measure theoretic and topological sense, Y_β must intersect C_β. Choose any $y \in Y_\beta \cap C_\beta$ and define $E_\beta = E_{\beta-1} \cup \{y\}$. Finally, let $E = \bigcup_{\alpha<\gamma} E_\alpha$.

COROLLARY 2. *Assume in addition* $\operatorname{cov}(\mathscr{M}) = \operatorname{cov}(\mathscr{N}) = 2^{\aleph_0}$. *Then there exists an independent set E such that $E \cap C \neq \emptyset$ for every $C \in \mathscr{B} \setminus \mathscr{M}$ and $E \cap B \neq \emptyset$ for every $B \in \mathscr{B} \setminus \mathscr{N}$* $(\mu^*(E) = 1)$.

Remark 2. It is clear that the assumption $\operatorname{cov}(\mathscr{N}) = 2^{\aleph_0}$ is sufficient for the existence of an independent set of outer measure one in a mildly mixing system with $\operatorname{supp} \mu = X$. We also note that by a result of Bartoszyński, the set theoretic assumption of Corollary 2 can be replaced by a single and stronger one: $\operatorname{add}(\mathscr{N}) = 2^{\aleph_0}$, where $\operatorname{add}(\mathscr{N})$ is the minimal cardinal number γ such that a union of γ sets of (Lebesgue) measure zero is not of measure zero (see [1] or [2]). All these set theoretic assumptions follow trivially from the continuum hypothesis.

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