COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 1

ON THE SOLVABILITY OF INFINITE SYSTEMS OF BOOLEAN POLYNOMIAL EQUATIONS

BY

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The aim of this paper is to prove the Fundamental Lemma 3 and to show how its consequence Theorem 1 paves the way to proving Theorem 2 in a natural manner. Theorem 2 is also proved in [2], and can be obtained in still another way by using results of [1] and [2], yet our proof is direct and elementary in contradistinction to those mentioned above.

We observe that Lemma 3 as well as Theorem 1 are proved without the use of the Axiom of Choice or any of its equivalent statements.

First we prove the following lemmas.

LEMMA 1. *Let a and c be elements of a Boolean ring B. Then the equation $ax = c$ has a solution in B if and only if $ac = c$.*

Proof. If $am = c$ for some $m \in B$, then $am = aam = ac = c$. Conversely, if $ac = c$, then clearly, $x = c$ is a solution of $ax = c$.

LEMMA 2. *If the system of equations*

$$a_1 x = c_1 \quad \textit{and} \quad a_2 x = c_2$$

over a Boolean ring B has a solution in B, then

$$x = c_1 + c_2 + c_1 c_2$$

is a solution of the system.

Proof. If $a_1 m = c_1$ and $a_2 m = c_2$ for some $m \in B$, then $(a_1 + a_2)m = c_1 + c_2$ and therefore by Lemma 1

$$a_1 c_1 = c_1, \quad a_2 c_2 = c_2, \quad (a_1 + a_2)(c_1 + c_2) = c_1 + c_2$$

from which it follows that

$$a_1 c_1 + a_1 c_2 + a_1 c_1 c_2 = c_1 \quad \text{and} \quad a_2 c_1 + a_2 c_2 + a_2 c_1 c_2 = c_2$$

implying the conclusion of the Theorem.

Let us recall that if for elements x and y of a Boolean ring B we write $x \leqslant y$ whenever $xy = x$, then $(B, \leqslant)$ is a partially ordered set. Moreover, in $(B, \leqslant)$

$$(1) \qquad \sup\{a, c\} = a + c + ac \quad \text{and} \quad a(\sup_i c_i) = \sup_i (ac_i),$$

where the second equality holds provided $\sup_i c_i$ exists in B (even if B has no unit element). As expected, a Boolean ring is called *complete* if every subset of B has a supremum, in which case, $\sup B$ (or for that matter any upper bound of B) is the unit element of B.

LEMMA 3. *Let B be a complete Boolean ring and I an index set. Then the system of equations $(a_i x = c_i)_{i \in I}$ over B has a solution in B if every subsystem consisting of two equations has a solution in B, in which case $x = \sup_{i \in I} c_i$ is a solution of the system.*

Proof. Let $j \in I$. Since every subsystem consisting of two equations has a solution in B, in view of (1) and Lemma 2, we have

$$a_j(\sup\{c_j, c_i\}) = c_j$$

and since B is complete

$$\sup_{i \in I} \big(a_j(\sup\{c_j, c_i\})\big) = \sup_{i \in I} c_j = c_j$$

which in view of (1) yields

$$a_j\big(\sup_{i \in I}(\sup\{c_j, c_i\})\big) = a_j(\sup_{i \in I} c_i) = c_j$$

implying that $\sup_{i \in I} c_i$ is a solution of the equation $a_j x = c_j$. Since j is an arbitrary element of I, we see that Lemma 3 is proved.

LEMMA 4. *Let B be a Boolean ring such that every system of equations of the form $(a_i x = c_i)_{i \in I}$ over B has a solution in B provided every subsystem consisting of two equations has a solution in B. Then B is complete.*

Proof. Let S be a subset of B. Consider the system of equations

$$(2) \qquad sx = s \quad \text{for every } s \in S.$$

Clearly, every subsystem of (2) consisting of two equations $sx = s$ and $tx = t$ has a solution, in B, say, $x = s + t + st$ and therefore, in view of the hypothesis of the Lemma, system (2) has a solution in B. Moreover, the same proof, with S replaced by B, show, that B has a unit e. Let U be the set of all solutions of system (2). Clearly, each $u \in U$ is an upper bound of S. Next, consider the system of equations

$$(3) \qquad (sx = s)_{s \in S} \quad \text{and} \quad (ux = x)_{u \in U},$$

where each $ux = x$ can be replaced by $(u - e)x = 0$.

Again, it is easy to verify that every subsystem of (3) consisting of two equations has a solution in B. For instance, a solution of $sx = s$ and $ux = x$ with $s \in S$ and $u \in U$ is $x = u$. On the other hand, a solution of $ux = x$ and $vx = x$ with $u \in U$ and $v \in U$ is $x = uv$. Therefore, system (3) has a solution in B, say, $x = p$ with $p \in B$. But then, as (3) shows, $p = \sup S$. Since S is an arbitrary subset of B we see that the Lemma is proved.

Combining Lemmas 3 and 4 we obtain:

THEOREM 1. *A Boolean ring B is complete if and only if every (finite or infinite) system Γ of Boolean polynomial equations in one unknown with coefficients in B has a solution in B provided every subsystem of Γ consisting of two equations has a solution in B.*

Motivated by Theorem 1 and based on the proof of Lemma 3 we prove the following theorem where I and J are (finite or infinite) index sets and where, naturally, each polynomial $P_i(\ldots, x_j, \ldots)$ in the unknowns x_j contains finitely many unknowns.

THEOREM 2. *A Boolean ring B is complete if and only if every (finite or infinite) system $\big(P_i(\ldots, x_j, \ldots) = c_i\big)_{i \in I}$ with $j \in J$ of Boolean polynomial equations over B has a solution in B provided every finite subsystem has a solution in B.*

Proof. Let us suppose that B is complete and that every finite subsystem of $\big(P_i(\ldots, x_j, \ldots) = c_i\big)_{i \in I}$ has a solution in B. Consider the Boolean ring Φ which is generated by B and the Boolean polynomials $P_i(\ldots, x_i, \ldots)$ with $i \in I$ and $j \in J$. In view of our supposition there exists a homomorphism φ from Φ into B such that

$$\varphi\big(P_i(\ldots, x_j, \ldots)\big) = c_i \quad \text{and} \quad \varphi(b) = b \tag{4}$$

for every $i \in I$ and every $b \in B$.

By virtue of Zorn's Lemma it can be easily shown that there exists a maximal (with respect to the set-theoretical inclusion) Boolean ring Ψ of polynomials (in the unknowns x_j with $j \in J$) over B and a homomorphism ψ from Ψ into B such that $\varphi \subset \psi$.

To prove the Theorem it is enough to show that for every $j \in J$ the polynomial x_j is an element of Ψ, in which case, clearly, $\big(x_j = \psi(x_j)\big)_{j \in J}$ is a solution of the system under consideration.

Assume on the contrary that for some $j \in J$ the polynomial x_j is not an element of Ψ. Consider x_j and all the elements P_m and P_n of Ψ such that for some elements P_u and P_v of Ψ

$$x_j P_u + P_m = x_j P_v + P_n.$$

But then, since ψ is a homomorphism, it follows from Lemma 3 that the system

$$\big(\psi(P_u + P_v)\big) x_j = \psi(P_m + P_n)$$

has a solution, say, $x_j = r_j$ with $r_j \in B$. Thus,

(5) $x_j P_u + P_m = x_j P_v + P_n$ implies $r_j \psi(P_u) + \psi(P_m) = r_j \psi(P_v) + \psi(P_n)$.

Next, consider the Boolean ring Ψ^* generated by Ψ and the polynomial x_j. By virtue of (5) the mapping ψ^* given by

$$\psi^*(x_j P_u + P_m) = r_j \psi(P_u) + \psi(P_m)$$

is a homomorphism from Ψ^* into B. But this contradicts the maximality of Ψ. Thus, our assumption is false and the necessity is proved. We omit the proof of the sufficiency since it is analogous to the proof of Lemma 4.

Let us call a system of equations as being *consistent* if the system has a solution. Then in view of Theorem 2 we have the following

COROLLARY. *A system of polynomial equations over a complete Boolean ring B is consistent if every finite subsystem has a solution in B.*

REFERENCE

[1] J. Mycielski and C. Ryll-Nardzewski, *Equationally compact algebras*, Fundamenta Mathematicae 61 (1968), p. 272–298.

[2] B. Węglorz, *Equationally compact algebras*, ibidem 59 (1966), p. 289–298.

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Reçu par la Rédaction le 2. 9. 1968