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ON SELF-CONJUGATE BANACH SPACES

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In this note we solve the following problem of Mazur from the "Scottish Book" (Problem 91):

A convex body W with a center is given in the n-dimensional Euclidean space. It is affine to its conjugate body. Is W then an ellipsoid?

The answer is negative in the case where n is an even number, and for odd n the problem is not solved. It is equivalent to the following:

If an n-dimensional Banach space is isometric to its conjugate space, is it then isometric to the Euclidean space?

The answer is given by the following

THEOREM 1. *For each $n \geqslant 2$ there exists an n-dimensional self-conjugate Banach space which is not isometric to the Euclidean space E^n.*

First we prove the following

LEMMA. *If $X = (R^n, \|\cdot\|_1)$ and $Y = (R^m, \|\cdot\|_2)$ are Banach spaces, then $Z = (R^n \times R^m, \|\cdot\|)$ is a Banach space ($\|z\| = (\|x\|_1^2 + \|y\|_2^2)^{1/2}$ for $x \in R^n$, $y \in R^m$, $z = (x, y) \in R^n \times R^m$). Moreover, Z^* and $\big(R^n \times R^m, [(\|\cdot\|_1^*)^2 + (\|\cdot\|_2^*)^2]^{1/2}\big)$ are isometric.*

Proof. It is obvious that $\|\cdot\|$ is a norm. To prove the second part, we evaluate the norm in Z^*.

Let $\xi \in Z^*$. Then $\xi(z) = \xi(x, y) = \xi(x, 0) + \xi(0, y) = \xi_1(x) + \xi_2(y)$, where $\xi_1 \in X^*$, $\xi_2 \in Y^*$. We have

$$\|\xi\|^* = \sup_{\substack{z \in Z \\ z \neq 0}} \frac{|\xi(z)|}{\|z\|} = \sup_{\substack{x \in X, y \in Y \\ \|x\|_1 + \|y\|_2 > 0}} \frac{|\xi_1(x) + \xi_2(y)|}{(\|x\|_1^2 + \|y\|_2^2)^{1/2}}$$

$$\leqslant \sup_{\substack{x \in X, y \in Y \\ \|x\|_1 + \|y\|_2 > 0}} \frac{\|\xi_1\|_1^* \|x\|_1 + \|\xi_2\|_2^* \|y\|_2}{(\|x\|_1^2 + \|y\|_2^2)^{1/2}}$$

$$\leqslant [(\|\xi_1\|_1^*)^2 + (\|\xi_2\|_2^*)^2]^{1/2}.$$

We can easily see that there are $x_0 \in X$ and $y_0 \in Y$ such that

$$\xi_1(x_0) = \|\xi_1\|_1^* \|x_0\|_1, \qquad \|x_0\|_1 = \|\xi_1\|_1^*,$$

$$\xi_2(y_0) = \|\xi_2\|_2^* \|y_0\|_2, \qquad \|y_0\|_2 = \|\xi_2\|_2^*.$$

If $\xi \neq 0$, then $\|x_0\|_1 + \|y_0\|_2 > 0$, so we have

$$\sup_{\substack{x\in X, y\in Y\\ \|x\|_1+\|y\|_2>0}} \frac{|\xi_1(x)+\xi_2(y)|}{(\|x\|_1^2+\|y\|_2^2)^{1/2}} \geqslant \frac{|\xi_1(x_0)+\xi_2(y_0)|}{(\|x_0\|_1^2+\|y_0\|_2^2)^{1/2}}$$

$$= \frac{\|\xi_1\|_1^*\|x_0\|_1 + \|\xi_2\|_2^*\|y_0\|_2}{(\|x_0\|_1^2+\|y_0\|_2^2)^{1/2}} = [(\|\xi_1\|_1^*)^2 + (\|\xi_2\|_2^*)^2]^{1/2}.$$

This implies (with the preceding inequality) that

$$\|\xi\|^* = [(\|\xi_1\|_1^*)^2 + (\|\xi_2\|_2^*)^2]^{1/2},$$

which completes the proof.

Proof of Theorem 1 (by induction). 1. If $n = 2$, then the space $(R^2, \|\cdot\|)$, where $\|x\| = \max(|x_1|, |x_2|)$ for $x = (x_1, x_2) \in R^2$, has the desired properties.

2. Suppose that X is an n-dimensional Banach space satisfying the assertion of the theorem. Then $Y = (X \times E^1, \|\cdot\|)$, where $\|y\| = (\|x\|^2 + t^2)^{1/2}$ for $y = (x, t) \in X \times E^1$, is an $(n+1)$-dimensional Banach space. To prove that Y is self-conjugate, observe that if U is an isometry of X onto X^* and $V(y) = (U(x), t)$ for $y = (x, t) \in Y$, then V is, by the Lemma, an isometry of Y onto Y^*. Moreover, Y is not isometric to the Euclidean space, since

$$Y_0 = \{y \in Y: y = (x, 0),\ x \in X\}$$

is isometric to X.

Theorem 1 can considerably be strengthened in view of the following

THEOREM 2. *For each $n \geqslant 2$ there is a class $\mathfrak{A}_n$ ($\overline{\overline{\mathfrak{A}_n}} = \mathfrak{c}$) of n-dimensional self-conjugate Banach spaces such that any space in $\mathfrak{A}_n$ is isometric neither to the Euclidean space nor to any other space in $\mathfrak{A}_n$.*

Proof. Let $p > 1$, $f(p) = p/(p-1)$, $x = (x_1, x_2, \ldots, x_n) \in R^n$ and put

$$\|x\|_p = \begin{cases} \Big[(|x_1|^p + |x_2|^p)^{2/p} + \sum_{i=3}^n x_i^2\Big]^{1/2} & \text{for } x_1x_2 \geqslant 0, \\ \Big[(|x_1|^{f(p)} + |x_2|^{f(p)})^{2/f(p)} + \sum_{i=3}^n x_i^2\Big]^{1/2} & \text{for } x_1x_2 < 0. \end{cases}$$

Then $\|\cdot\|_p$ is a norm in R^n and, consequently, $X_{n,p} = (R^n, \|\cdot\|_p)$ is an n-dimensional Banach space.

First we prove that $\|\cdot\|_p$ is a norm in R^2. Let $S_p = \{x \in R^2: \|x\|_p \leqslant 1\}$, $x, y \in S_p$, $x = (x_1, x_2), y = (y_1, y_2)$. It suffices to show that $(x+y)/2 \in S_p$. Assuming that $x_1, x_2 > 0$, we have two non-trivial cases.

1. $y_1 \leqslant 0$ and $y_2 \leqslant 0$.

Then

$$\left|\frac{x_1+y_1}{2}\right| \leqslant \frac{1}{2} \quad \text{and} \quad \left|\frac{x_2+y_2}{2}\right| \leqslant \frac{1}{2},$$

whence

$$\left|\frac{x_1+y_1}{2}\right|^r + \left|\frac{x_2+y_2}{2}\right|^r \leqslant 1 \quad \text{for all } r \geqslant 1.$$

2. $y_1 \leqslant 0$ and $y_2 > 0$.

Then the segment joining x and y has a common point $z = (0, z_2)$ with the axis $x_1 = 0$ and $0 < z_2 \leqslant 1$. The middle point $(x+y)/2$ of this segment lies between x and z or between z and y, so $(x+y)/2 \in S_p$, since $S_p \cap \{x \in R^2: x_1, x_2 \geqslant 0\}$ and $S_p \cap \{x \in R^2: x_1 \leqslant 0, x_2 > 0\}$ are convex. Hence S_p is convex. Now it is easily seen that $\|\cdot\|_p$ is a norm in R^2, and the Lemma implies that $\|\cdot\|_p$ is a norm in R^n.

If we prove that $(X_{2,p})^*$ is isometric to $X_{2,f(p)}$, then, by the Lemma, $(X_{n,p})^*$ is isometric to $X_{n,f(p)}$.

Let $\xi \in (X_{2,p})^*$, $\xi(x) = \xi_1 x_1 + \xi_2 x_2$ for $x = (x_1, x_2) \in X_{2,p}$. Then

$$\|\xi\|_p^* = \sup_{\|x\|_p \leqslant 1} |\xi_1 x_1 + \xi_2 x_2| = \max\{\sup_{\substack{x_1 x_2 \geqslant 0 \\ \|x\|_p \leqslant 1}} |\xi_1 x_1 + \xi_2 x_2|, \sup_{\substack{x_1 x_2 < 0 \\ \|x\|_p \leqslant 1}} |\xi_1 x_1 + \xi_2 x_2|\}.$$

Let $|\cdot|_p$ be the l_p-norm in R^2. If $\xi_1 \xi_2 \geqslant 0$, then the last expression is equal to

$$\sup_{\substack{x_1 x_2 \geqslant 0 \\ \|x\|_p \leqslant 1}} |\xi_1 x_1 + \xi_2 x_2| = \sup_{|x|_p \leqslant 1} |\xi_1 x_1 + \xi_2 x_2| = (|\xi_1|^{f(p)} + |\xi_2|^{f(p)})^{1/f(p)}.$$

Analogously, for $\xi_1 \xi_2 < 0$ we have $\|\xi\|_p^* = (|\xi_1|^p + |\xi_2|^p)^{1/p}$. Hence

$$\|\xi\|_p^* = \begin{cases} (|\xi_1|^{f(p)} + |\xi_2|^{f(p)})^{1/f(p)} & \text{for } \xi_1 \xi_2 \geqslant 0, \\ (|\xi_1|^p + |\xi_2|^p)^{1/p} & \text{for } \xi_1 \xi_2 < 0, \end{cases}$$

which completes the proof, since the identity is an isometry of $(X_{2,p})^*$ onto $X_{2,f(p)}$.

Now, for $x = (x_1, x_2, \ldots, x_n) \in R^n$ we define $F: R^n \to R^n$ by

$$F(x) = (x_1, -x_2, x_3, \ldots, x_n).$$

F is an isometry of $X_{n,f(p)}$ onto $X_{n,p}$. Hence $(X_{n,p})^*$ and $X_{n,p}$ are isometric, which means that $X_{n,p}$ is self-conjugate. Obviously, $X_{n,p}$ is not isometric to the Euclidean space.

It remains to prove that $X_{n,p}$ and $X_{n,r}$ are isometric only if $p = r$ or $p = f(r)$ (1).

Suppose that $T\colon X_{n,p} \to X_{n,r}$ is an isometry and let $[a_{ij}]_{1\leqslant i,j\leqslant n}$ be the matrix of T. We can assume that $p < 2$. For $x \in X_{n,p}$ such that $x_1 x_2 \geqslant 0$, we have

$$(1)\qquad (|x_1|^p+|x_2|^p)^{2/p}+\sum_{m=3}^{n} x_m^2 = \Big(\Big|\sum_{j=1}^{n} a_{1j}x_j\Big|^{a(x)}+\Big|\sum_{j=1}^{n} a_{2j}x_j\Big|^{a(x)}\Big)^{2/a(x)}+\sum_{m=3}^{n}\Big(\sum_{j=1}^{n} a_{mj}x_j\Big)^2,$$

where $a(x) = r$ or $a(x) = f(r)$.

Put $x = (\lambda, 1, 0, \ldots, 0)$ in (1). Then we obtain

$$(2)\qquad (\lambda^p+1)^{2/p} = \big(F_1(\lambda)+F_2(\lambda)\big)^{2/q}+A+B\lambda+C\lambda^2,$$

where $F_i(\lambda) = |a_{i1}\lambda+a_{i2}|^q$ for $i = 1, 2$, A, B, C are constants, and $q = a(x)$. It is easily seen that there is a $\delta > 0$ such that, for $0 < \lambda < \delta$, q is a constant.

If $a_{12} \neq 0$ or $a_{22} \neq 0$, then, by the Taylor theorem, for these F_i for which $a_{i2} \neq 0$ we obtain

$$F_1(\lambda)+F_2(\lambda) = a+b\lambda+c\lambda^2+o(\lambda^2)+d\lambda^q, \qquad a \neq 0,$$

and, by the binomial theorem, we get

$$(3)\qquad \big(F_1(\lambda)+F_2(\lambda)\big)^{2/q} = a'+b'\lambda+c'\lambda^2+d'\lambda^q+o(\lambda^2).$$

If $a_{12} = a_{22} = 0$, then, simply,

$$\big(F_1(\lambda)+F_2(\lambda)\big)^{2/q} = (|a_{11}|^q+|a_{21}|^q)^{2/q}\lambda^2.$$

Moreover,

$$(4)\qquad (\lambda^p+1)^{2/p} = 1+\frac{2}{p}\lambda^p+o(\lambda^p).$$

If we put (3) and (4) in (2), we obtain

$$(5)\qquad 1+\frac{2}{p}\lambda^p+o(\lambda^p) = A'+B'\lambda+C'\lambda^2+d'\lambda^q+o(\lambda^2).$$

We have $1 < p < 2$. If we take $\lambda \to 0^+$, we can see that $A' = 1$. So, dividing both sides of (5) by λ and taking $\lambda \to 0^+$, we deduce that $B' = 0$ and we obtain

$$(6)\qquad \frac{2}{p}\lambda^p+o(\lambda^p) = C'\lambda^2+d'\lambda^q+o(\lambda^2).$$

(1) This part of the proof is due to A. Rek. The first version given by the authors was more complicated.

Now we divide both sides by λ^p. It is easily seen that

$$\lim_{\lambda\to 0+} d'\lambda^{q-p} = \frac{2}{p}.$$

So $p = q$, i.e. $p = r$ or $p = f(r)$.

Let $\mathfrak{A}_n$ be the class of n-dimensional Banach spaces $X_{n,p}$ for $1 < p < 2$. Then $\overline{\overline{\mathfrak{A}_n}} = \mathfrak{c}$ and, as we have shown, $X_{n,p}$ are self-conjugate, not isometric to the Euclidean space and not isometric one to the other. The proof is completed.

In case where $n = 3$ the unit ball in a Banach space satisfying the assertion of Theorem 1 can be even a polyhedron.

Example. Let $X = (R^3, \|\cdot\|)$ and

$$\|x\| = \max(|x_2|+|x_3|, |x_1|+\tfrac{1}{2}|x_3|) \quad \text{for } x = (x_1, x_2, x_3) \in R^3.$$

The unit ball B is an intersection of the cuboid

$$C = \{x \in R^3: |x_1| \leqslant 1, \ |x_2|+|x_3| \leqslant 1\}$$

and the octahedron

$$W = \{x \in R^3: \max(|x_1|, |x_2|)+\tfrac{1}{2}|x_3| \leqslant 1\}.$$

The set $\mathrm{Ext}B$ of the extremal points of B consists of 8 points: $(\varepsilon_1, \varepsilon_2, 0)$, $(\frac{1}{2}\varepsilon_1, 0, \varepsilon_2)$, where $\varepsilon_i = \pm 1$ for $i = 1, 2$. If $\xi \in X^*$,

$$\xi(x) = \xi_1x_1+\xi_2x_2+\xi_3x_3 \quad \text{for } x = (x_1, x_2, x_3) \in R^3,$$

then

$$\begin{aligned}\|\xi\|^* &= \max_{x\in \mathrm{Ext}B} |\xi_1x_1+\xi_2x_2+\xi_3x_3| \\ &= \max(|\xi_1+\xi_2|, |\xi_1-\xi_2|, |\tfrac{1}{2}\xi_1+\xi_3|, |\tfrac{1}{2}\xi_1-\xi_3|) \\ &= \max(|\xi_1|+|\xi_2|, \tfrac{1}{2}|\xi_1|+|\xi_3|).\end{aligned}$$

Let $F: X \to X^*$ be defined by $F((x_1, x_2, x_3)) = (x_3, x_2, x_1)$. Then F is an isometry.

Remark. Analogous examples of n-dimensional Banach spaces for $n > 3$ are not known to the authors. **(P 1196)**

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