APPROXIMATION AND FUNCTION SPACES
BANACH CENTER PUBLICATIONS, VOLUME 22
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

APPROXIMATION ON THE SPHERE — A SURVEY

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In this lecture we are concerned with our recent investigations in approximation of functions on the unit sphere σ of the n-dimensional Euclidean space R^n by polynomials in shperical harmonics. We also comment on the results of other authors.

One of the approaches which we have been working out is based on the consideration of suitably averaged differences of the functions under study along geodesics on σ. With the aid of those differences the corresponding moduli of smoothness are constructed. An important point is the construction of the nearest polynomial in the form of spherical convolution with the Jackson kernel.

Here we restrict ourselves to the power moduli of smoothness and to the classes $H_p^r(\sigma)$. Our methods permit, however, more general moduli to be considered, leading to Besov $B_{p,\theta}^r(\sigma)$ spaces on the sphere.

1. Preliminaries

Let R^n be the n-dimensional space of points $x = (x_1, \ldots, x_n)$ and

$$\sigma = \{x: \sum_{j=1}^{n} x_j^2 = 1\}$$

the unit sphere in it. Let $L_p(\sigma)$ $(1 \leqslant p \leqslant \infty)$ be the space of functions defined on σ with the norm

$$\|f\|_p = \Big(\int_\sigma |f(\mu)|^p \, d\mu\Big)^{1/p} \quad (1 \leqslant p < \infty);$$

for $p = \infty$ we understand that $L_\infty(\sigma) = C(\sigma)$ is the space of continuous functions on σ with the norm

$$\|f\|_{C(\sigma)} = \max_{\mu \in \sigma} |f(\mu)|.$$

Further, let

$$P_N(x) = \sum_{|k| \leq N} a_k x^k, \quad k = (k_1, \ldots, k_n), \quad |k| = \sum_{j=1}^{n} k_j, \quad x^k = x_1^{k_1} \ldots x_n^{k_n},$$

be an algebraic polynomial of degree N.

Its trace on σ:

$$P_N(x)|_\sigma = T_N(\mu) \quad (\mu \in \sigma)$$

is a *spherical polynomial* of degree N.

It is known that if $T_N(\mu)$ is the trace of an algebraic polynomial $P_N(x)$ of degree N, then it is also the trace of a harmonic polynomial $U_N(x)$:

$$P_N|_\sigma = U_N|_\sigma \quad (\Delta U_N(x) \equiv 0, \ \Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2).$$

The trace $Y_k(\mu)$ of a homogeneous harmonic polynomial of degree k is called a *spherical harmonic*. It is known that spherical harmonics Y_k, Y_l of different orders are orthogonal on σ:

$$\int_\sigma Y_k(\mu)\, Y_l(\mu)\, d\mu = 0 \quad (k \neq l, \ k, l = 0, 1, 2, \ldots).$$

Every function $f \in L_p(\sigma)$ has an expansion with respect to spherical harmonics:

$$f(\mu) \sim \sum_{k=0}^{\infty} Y_k(\mu),$$

converging to f at least in a generalized sense.

We will produce spherical polynomials of degree N in the following way.

Suppose there is given an algebraic polynomial $Q_N(u)$ of degree N in one real variable u. Then for fixed $\mu' \in \sigma$

$$Q_N(x\mu') \quad (x \in \mathbf{R}^n, \ \mu' \in \sigma),$$

where $x\mu'$ is the scalar product of x and μ', is an algebraic polynomial in x of degree N. For $f \in L_p(\sigma)$

$$\Lambda_N(x) = \int_\sigma Q_N(x\mu') f(\mu')\, d\mu'$$

is also an algebraic polynomial in x of degree N, and the expression

$$(1) \qquad \Lambda_N(\mu) = \int_\sigma Q_N(\mu\mu') f(\mu')\, d\mu' = \int_\sigma Q_N(\cos(\mu\mu')\check{\ }) f(\mu')\, d\mu' \quad (\mu \in \sigma)$$

is its trace on σ, and hence a spherical polynomial of degree N. Here $(\mu\mu')\check{\ } = \gamma$ is the length of the orthodrome connecting μ and μ'.

(1) can also be rewritten as

$$\Lambda_N(\mu) = \int_\sigma K_N\big((\mu\mu')\breve{}\,\big) f(\mu')\, d\mu' \qquad (\mu \in \sigma),$$

where $K_N(\gamma)$ is a certain even trigonometric polynomial of degree N.

The notion of translate of a function $f \in L_p(\sigma)$ at $\mu \in \sigma$ by γ is known in the literature:

$$S_\gamma f(\mu) = \frac{1}{|\sigma_{n-2}| \sin^{n-2}\gamma} \int_{(\mu\mu')\breve{} = \gamma} f(\mu')\, d\mu'.$$

Here the integration is over the set of all $\mu' \in \sigma$ at arc distance γ from $\mu \in \sigma$; we take the mean value of $f(\mu')$ over this set whose $(n-2)$-dimensional measure is $|\sigma_{n-2}| \sin^{n-2}\gamma$, where $|\sigma_{n-2}|$ is the measure of the $(n-2)$-dimensional sphere σ_{n-2}.

The operation

$$\Delta_\gamma f = (S_\gamma - E) f,$$

where E is the identity operator, is viewed as the increment of $f \in L_p(\sigma)$ at $\mu \in \sigma$ with step $\gamma > 0$. If

$$f(\mu) \sim \sum_{k=0}^{\infty} Y_k(\mu) \qquad (f \in L_p(\sigma))$$

is an expansion with respect to spherical harmonics, then the corresponding expansion of $S_\gamma f(\mu)$ is

$$S_\gamma f(\mu) \sim \sum_{m=0}^{\infty} \frac{P_m^\lambda(\cos\gamma)}{P_m^\lambda(1)} Y_m(\mu),$$

where $\lambda = (n-2)/2$ and $P_m^\lambda(u)$ $(-\infty < u < \infty)$ are the Gegenbauer polynomials of order m, orthogonal on $[-1, 1]$ with weight $(1-u^2)^{\lambda-1/2}$:

$$\int_{-1}^{1} P_m^\lambda(u) P_l^\lambda(u) (1-u^2)^{\lambda-1/2}\, du = 0 \qquad (m \neq 1).$$

We also use the classical Laplace–Beltrami operator $\mathscr{D}$, acting on sufficiently smooth functions:

$$\mathscr{D} f(\mu) = \Delta f(x/|x|)_\sigma = \varphi(\mu) \qquad (\mu = x/|x| \in \sigma).$$

For every $r > 0$ we introduce the class $H_p^r(\sigma)$ of functions satisfying the inequality

$$\|(\Delta_\gamma)^k \mathscr{D}^l f(\mu)\|_p \leqslant M\gamma^{r-2l} \tag{2}$$

$(2k > r - 2l > 0,\ k = 1, 2, \ldots,\ l = 0, 1, 2, \ldots)$.

$H_p^r(\sigma)$ is a Banach space with the norm

$$\|f\|_{H_p^r(\sigma)} = \|f\|_p + M_f,$$

where M_f is the least constant in (2) (for all $\gamma > 0$).

Let us now formulate an assertion whose proof will be discussed below in a number of cases.

THEOREM 1. 1) *For every $f \in H_p^r(\sigma)$ there is a sequence of spherical polynomials T_N of degree N such that*

$$\|f - T_N\|_p \leqslant \frac{cM_f}{(N+1)^r} \quad (N = 0, 1, 2, \ldots).$$

2) *If for a function $f \in L_p(\sigma)$ there is a sequence of spherical polynomials T_N of degree N such that*

$$\|f - T_N\|_p \leqslant \frac{K}{(N+1)^r} \quad (N = 0, 1, 2, \ldots),$$

with K independent of N, then $f \in H_p^r(\sigma)$ and

$$\|f\|_{H_p^r(\sigma)} \leqslant c(\|f\|_p + K),$$

where c is independent of the neighbouring factor.

For $k = 1$, $n = 3$, $2 > r-2l > 0$ and $p = \infty$ this theorem was proved by G. G. Kushnirenko [2], and for any n and $1 \leqslant p \leqslant \infty$ by S. Pawelke [8].

Since l is a nonnegative integer, the condition $2k > r-2l > 0$ does not admit r even if we consider only the first order differences $(\Delta_\gamma)^1 = \Delta_\gamma$, i.e. if $k = 1$.

In the case of r even one has to consider the differences $(\Delta_\gamma)^k$ of order at least two ($k \geqslant 2$). Butzer and Johnen's paper [1] calls attention to the necessity of closing this gap.

The present paper is a survey of our investigations concerning the proof of Theorem 1 and similar ones for higher order differences. Along with the differences $(\Delta_\gamma)^k$ (which we will call power ones) we introduce the differences $^*\Delta_\gamma^k$ of a different kind (in some cases), which are more advantageous.

2. Case $p = 2$. Some comments

In our papers [3], [4], Theorem 1 is proved for any $k \geqslant 2$ and $p = 2$.

Let us comment on its proof. It suffices to prove the sufficiency, i.e. assertion 1), since we may consider the necessity to be already established in the above-mentioned papers.

We introduce the Jackson kernel

$$D_N(\theta) = \left(\frac{\sin \frac{1}{2} m\theta}{\sin \frac{1}{2}\theta}\right)^{2s}, \tag{1}$$

where s is a fixed (nonnegative) integer and m runs through the positive integers. The degree of the kernel (1) is $N = s(m-1)$.

We set

$$\varkappa_N \int_\sigma D_N\big((\mu\mu')\check{}\,\big)\,d\mu' = \varkappa_N \int_0^\pi D_N(\gamma) \int_{(\mu\mu')\check{} = \gamma} d\mu'\,d\gamma \tag{2}$$

$$= |\sigma_{n-2}|\,\varkappa_N \int_0^\pi D_N(\gamma) \sin^{n-2}\gamma\,d\gamma = 1.$$

We introduce the operator

$$Tf = T_N(f, \mu) = \varkappa_N \int_\sigma f(\mu') D_N\big((\mu\mu')\check{}\,\big)\,d\mu'$$

$$= \varkappa_N |\sigma_{n-2}| \int_0^\pi D_N(\gamma) \sin^{n-2}\gamma \left(\frac{1}{|\sigma_{n-2}| \sin^{n-2}\gamma} \int_{(\mu\mu')\check{} = \gamma} f(\mu')\,d\mu'\right) d\gamma$$

$$= \varkappa_N |\sigma_{n-2}| \int_0^\pi D_N(\gamma) \sin^{n-2}\gamma\, S_\gamma f(\mu)\,d\gamma.$$

For $i = 1, 2, \ldots$, put

$$^iTf = {}^iT_N(f, \mu) = \{-(E-T)^i + E\} f(\mu).$$

We show by induction that ^iTf is a spherical polynomial of degree N. indeed, for $i = 1$ this is so; and if ^{i-1}Tf is a spherical polynomial of degree N, then

$$^iTf = \{(E-T)[-(E-T)^{i-1} + E - E] + E\} f$$

$$= \{(E-T)(^{i-1}T - E) + E\} f$$

$$= \{^{i-1}T - T \cdot {}^{i-1}T + T\} f;$$

the right-hand side is clearly a spherical polynomial of degree N.

We also introduce the operator $^iU_N(f, \mu) = {}^iU_N f$ by the formula ($i = 1, 2, \ldots$)

$$f(\mu) - {}^iU_N(f, \mu) = \varkappa_N |\sigma_{n-2}| \int_0^\pi D_N(\gamma) \sin^{n-2}\gamma (E - S_\gamma)^i f(\mu)\,d\mu.$$

It turns out that if $f(\mu) \sim \sum_0^\infty Y_k(\mu)$, then

$$(E - {}^iT_N) f(\mu) = \sum_{k=0}^\infty \tau_k^i\, Y_k(\mu),$$

$$(E-{}^iU_N)f(\mu)=\sum_{k=0}^{\infty} u_k^i\, Y_k(\mu),$$

where

$$\tau_k^i=\left\{\varkappa_N|\sigma_{n-2}|\int_0^{\pi} D_N(\gamma)\sin^{n-2}\gamma\left(1-\frac{P_k^\lambda(\cos\gamma)}{P_k^\lambda(1)}\right)d\gamma\right\}^i,$$

$$u_k^i=\varkappa_N|\sigma_{n-2}|\int_0^{\pi} D_N(\gamma)\sin^{n-l}\gamma\left(1-\frac{P_k^\lambda(\cos\gamma)}{P_k^\lambda(1)}\right)^i d\gamma.$$

Moreover, using the properties of Gegenbauer polynomials one can prove that

$$\tau_k^i\leqslant u_k^i\qquad (k=0,1,2,\ldots).$$

But then the Parseval equality yields

$$\text{(3)}\qquad \begin{aligned}\|{}^iT_Nf-f\|_2&\leqslant\|f-{}^iU_Nf\|_2\\ &=\Big\|\int_0^{\pi} D_N(\gamma)\sin^{n-2}\gamma\,(S_\gamma-E)^if(\mu)\,d\mu\Big\|_2\\ &\leqslant|\sigma_{n-2}|\varkappa_N\int_0^{\pi} D_N(\gamma)\sin^{n-2}\gamma\,\|(\Delta_\gamma)^if(\mu)\|_2\,d\gamma.\end{aligned}$$

If $i=k+l$, if $f\in H_2^r(\sigma)$ and if s is suitably chosen, then the right-hand side of (3) may be estimated in such a way as to finally obtain

$$\|{}^{k+l}T_N(f,\mu)\|_2\leqslant\frac{cM_f}{(N+1)^r},$$

where the constant c is independent of N.

Thus assertion 1) of Theorem 1 is proved for positive integers of the form $N=s(m-1)$. It is well known how to extend it to all positive integers.

Remark. As already noted, the necessity part of Theorem 1 (i.e. assertion 2)) is proved for all $p\in[1,\infty]$ (in particular, in [4]). We suppose that the sufficiency also holds for all $1\leqslant p\leqslant\infty$ (and not for $p=2$ only), but this must still be considered an open problem. The attempts existing in the literature at giving an affirmative answer have turned out to be unfounded.

In this connection we should mention M. Wehrens' paper [9] where it is claimed that with the use of moduli constructed on the basis of the power differences $(\Delta_\gamma)^k$, $k>1$, it is impossible to obtain a characterization of H_p^r spaces in terms of best approximation. This cannot be true, at least because no mention is made of the case $p=2$ considered above.

Wehrens [9] obtained a theorem of the type of Theorem 1 by introducing the iterated moduli of smoothness

$$\sup_{0<\gamma_j\leqslant\delta}\|\Delta_{\gamma_1}\ldots\Delta_{\gamma_k}f\|_p.$$

3. Differences along geodesics (case $p = \infty$)

Let us take two points $\mu \equiv \mu^0$ and $\mu' \equiv \mu^1$ on σ and draw through them a great circle Γ. Moving along Γ from μ to μ' and farther, we mark on Γ the points

$$\mu = \mu^0, \mu^1, \mu^2, \mu^3, \dots$$

with equal distances

$$\gamma = (\mu^0 \mu^1)\check{} = (\mu^1 \mu^2)\check{} = (\mu^2 \mu^3)\check{} = \dots$$

We introduce the differences

$$\begin{aligned} &\Delta_{\mu'} f(\mu) = f(\mu') - f(\mu), \\ &\Delta_{\mu'}^2 f(\mu) = f(\mu^2) - 2f(\mu') + f(\mu), \\ &\dots\dots\dots\dots\dots\dots \\ &\Delta_{\mu'}^k f(\mu) = \sum_{j=0}^{k} (-1)^{k+j} C_k^j f(\mu^j). \end{aligned} \tag{1}$$

By definition, $f \in \bar{H}^r_\infty(\sigma)$ if

$$|\Delta_{\mu'}^k f(\mu)| \leqslant M\big((\mu\mu')\check{}\big)^r \quad (k > r > 0). \tag{2}$$

(Other definitions of this class are also possible, using linear combinations of the differences $\Delta_{\mu'}^k$, the Laplace–Beltrami operator or derivatives along Γ.) Let

$$\|f\|_{\bar{H}^r_\infty(\sigma)} = \|f\|_\infty + M_f,$$

where M_f is the least constant in (2).

THEOREM 2. 1) *For every $f \in \bar{H}^r_\infty(\sigma)$ there is a sequence of spherical polynomials $\{T_N\}$ such that*

$$\|f - T_N\|_C \leqslant \frac{cM_f}{(N+1)^r} \quad (N = 0, 1, 2, \dots),$$

where c is independent of M_f and N.

2) *Conversely, if $f \in L_p(\sigma)$ and there is a sequence of spherical polynomials $\{T_N\}$ such that for some K*

$$\|f - T_N\|_C \leqslant \frac{K}{(N+1)^r} \quad (N = 0, 1, 2, \dots), \tag{3}$$

then $f \in \bar{H}^r_\infty(\sigma)$ and

$$\|f\|_{\bar{H}^r_\infty(\sigma)} \leqslant c(K + \|f\|_C), \tag{4}$$

where c is independent of the neighbouring factor.

For n even and m divisible by $2k!$ we have the formula (see [5], [6])

$$T_N(\mu)-f(\mu)=(-1)^{k+1}\varkappa_N\int_\sigma D_N((\mu\mu')\check{})\Delta^k_{\mu'}f(\mu)\,d\mu' \tag{5}$$
$$=\int_\sigma K_N((\mu\mu')\check{})f(\mu')\,d\mu'-f(\mu),$$

where $T_N(\mu)$ is a spherical polynomial of degree N and

$$K_N(\gamma)=\frac{1}{\varkappa_N}\sum_{j=1}^{k}(-1)^{j+1}C_k^j\frac{1}{j}\left[\sum_{\nu=0}^{j-1}{}' D_N\left(\frac{\nu\pi+\theta}{j}\right)\sin^{n-2}\left(\frac{\nu\pi+\theta}{j}\right)\right.$$
$$\left.+\sum_{\nu=0}^{j-1}{}'' D_N\left(\frac{(\nu+1)\pi-\theta}{j}\right)\sin^{n-2}\left(\frac{(\nu+1)\pi-\theta}{j}\right)\right],$$

where the sum $\sum'$ is over ν even, and $\sum''$ over ν odd. The point here is that under the indicated assumptions $K_N(\gamma)$ is an even trigonometric polynomial in γ of degree N.

If $f\in\bar{H}^r_\infty(\sigma)$, then it is easily seen that

$$|f(\mu)-T_N(\mu)|\leqslant\varkappa_N|\sigma_{n-1}|M\int_0^\pi\gamma^r\sin^{n-2}\gamma\, D_N(\gamma)\,d\gamma \tag{6}$$
$$\leqslant\frac{cM}{(N+1)^r},$$

which proves 1). We do not dwell on the proof of 2) here.

For n odd, $T_N(\mu)$ in (5) is no longer a spherical polynomial. In this case, modifying our construction, A. P. Terekhin obtained a new representation for the approximating polynomial $T_N(\mu)$ and proved the estimate (3).

The converse estimate (4) is established by applying a suitable Bernstein type inequality for spherical polynomials.

4. Averaged differences ($1\leqslant p\leqslant\infty$)

The differences $\Delta^k_{\mu'}f(\mu)$ considered in the previous section can be used to define q-moduli of smoothness of order k for the $L_p(\sigma)$ norm by the formula

$$\omega_k(f,\delta)^q_p\overset{\text{def.}}{=}\sup_{0<\gamma\leqslant\delta}\left\|\left(\frac{1}{|\sigma_{n-2}|\sin^{n-2}\gamma}\int_{(\mu\mu')\check{}=\gamma}|\Delta^k_{\mu'}f(\mu)|^q\,d\mu'\right)^{1/q}\right\|_{L_p(\sigma)}, \tag{1}$$

where the inner integral can also be written as an integral over all directions issuing from μ (in the direction of μ'), i.e. as an integral over the unit sphere σ_{n-2}:

$$\frac{1}{|\sigma_{n-2}|}\int_{\sigma_{n-2}}|\Delta^k_{\mu'}f(\mu)|^q\,d\omega$$

($\omega\in\sigma_{n-2}$, $d\mu'=\sin^{n-2}\gamma\,d\omega$, $d\omega$ is the Lebesgue measure on σ_{n-2}). In the setting of Section 3, one takes $q = \infty$, $p = \infty$ and considers the case where

$$\omega_k(f, \delta)_\infty^\infty \leqslant M_f \delta^2, \quad k > r > 0.$$

We have not studied the construction (1) in the general case. In this section we present the results obtained for $1 \leqslant p \leqslant \infty$ and $q = 1$ in (1). However, dropping the absolute value sign in the integral of (1), we first consider a "pure" averaged difference $^*\Delta_\gamma^k f(\mu)$ (the absolute value may be introduced afterwards) by setting

$$(2) \qquad {}^*\Delta_\gamma^k f(\mu) = \frac{1}{|\sigma_{n-2}| \sin^{n-2}\gamma} \int_{(\mu\mu')^\smile = \gamma} \Delta_{\mu'}^k f(\mu)\, d\mu'.$$

A suitable computation shows that this difference may also be written as

$$(3) \qquad {}^*\Delta_\gamma^k f(\mu) = \Delta_\gamma^k S_t f(\mu)|_{t=0},$$

where the right-hand side is the usual kth difference with step γ of the function $S_t f(\mu)$ with respect to t at $t = 0$.

We will now write that $f \in {}^*H_p^r(\sigma)$ if $f \in L_p(\sigma)$ and

$$(4) \qquad \|{}^*\Delta_\gamma^k f(\mu)\|_p \leqslant M\gamma^r \quad (k > r > 0),$$

where M is independent of γ. Set

$$\|f\|_{{}^*H_p^r(\sigma)} = \|f\|_p + M_f,$$

where M_f is the least constant M in (4).

We will prove

THEOREM 3. *Theorem 2 remains true with $\bar{H}_\infty^r(\sigma)$ replaced by ${}^*H_p^r(\sigma)$.*

The proof of the first assertion (see Theorem 2) for n even is based on the equality (5) from Section 3. We rewrite it in the form

$$(5) \quad T_N(\mu) - f(\mu)$$
$$= (-1)^{k+1} \varkappa_N |\sigma_{n-2}| \int_0^\pi \sin^{n-2}\gamma\, D_N(\gamma) \left(\frac{1}{\sin^{n-2}\gamma\, |\sigma_{n-2}|} \int_{(\mu\mu')^\smile = \gamma} \Delta_{\mu'}^k f(\mu)\, d\mu' \right) d\gamma$$
$$= (-1)^{k+1} \varkappa_N |\sigma_{n-2}| \int_0^\pi \sin^{n-2}\gamma\, D_N(\gamma)\, {}^*\Delta_\gamma^k f(\mu)\, d\gamma$$
$$= (-1)^{k+1} \varkappa_N |\sigma_{n-2}| \int_0^\pi \sin^{n-2}\gamma D_N(\gamma)\, \Delta_\gamma^k S_t f(\mu)|_{t=0}\, d\gamma.$$

In the last equality we have used (3). From (4) we obtain for $f \in {}^*H_p^r(\sigma)$

$$\|T_N - f\|_p \leqslant \varkappa_N |\sigma_{n-2}| \int_0^\pi \sin^{n-2}\gamma\, D_N(\gamma)\, \|{}^*\Delta_\gamma^k f(\mu)\|_p\, d\gamma$$
$$\leqslant M \varkappa_N |\sigma_{n-2}| \int_0^\pi \sin^{n-2}\gamma\, D_N(\gamma)\, \gamma^r\, d\gamma \leqslant cM/(N+1)^r.$$

We have obtained the first assertion of the theorem for positive integers N of the form $N = s(m-1)$, where m is divisible by $2k!$. In the well-known manner it can be extended to all intermediate values of N. One only needs an estimate for $N = 0$:

$$\|T_0 - f\|_p \leqslant cM;$$

this can be obtained by defining

$$T_0 = \frac{1}{|\sigma|} \int_\sigma f(\mu)\, d\mu. \tag{6}$$

In the case of n odd the first assertion of the theorem can be obtained by using the representation (5) with $S_t f(\mu)$ replaced by ${}^{-}S_t f(\mu)$, the odd 2π-periodic extension of $S_t f(\mu)$ as a function of t from the interval $(0, \pi)$. The calculations get a little more involved in this case.

The second (converse) assertion of the theorem follows by applying a specific Bernstein type inequality for spherical polynomials [7].

We know several proofs of this inequality, coming from different considerations. We now present one of them which seems to be the most elegant.

Let

$$T_N(\mu) = \sum_{m=0}^{N} Y_m(\mu)$$

be the expansion of a spherical polynomial with respect to spherical harmonics. We have

$$S_t T_N(\mu) = \sum_{m=0}^{N} \frac{P_m^\lambda(\cos t)}{P_m^\lambda(1)} Y_m(\mu).$$

This is a trigonometric polynomial in t of degree N. We apply to it M. Riesz's formula:

$$\frac{d}{dt} S_t T_N(\mu) = \frac{1}{4N} \sum_{l=1}^{2N} (-1)^{l+1} \frac{1}{\sin^2(\theta_l/2)} S_{t+\theta_l} T_N(\mu),$$

$$\theta_l = \frac{2l-1}{2N}\pi,\ l = 1, \dots, 2N.$$

Since

$$\frac{1}{4N} \sum_{l=1}^{N} \frac{1}{\sin^2(\theta_l/2)} = N,$$

we hence obtain

$$\|(d/dt) S_t T_N(\mu)\|_p \leqslant N \max_t \|S_t T_N(\mu)\|_p.$$

By induction,

$$\|(d^k/dt^k)\, S_t\, T_N(\mu)\|_p \leqslant N^k \max_t \|S_t\, T_N(\mu)\|_p,$$

and since $\|S_t f\|_p \leqslant \|f\|_p$ (see [1]), we obtain

$$\|(d^k/dt^k)\, S_t\, T_N(\mu)\|_p \leqslant N^k \|T_N\|_p.$$

This obviously yields

$$\|{}^*\Delta_\gamma^k S_t\, T_N(\mu)\,|_{t=0}\|_p \leqslant (N\gamma)^k \|T_N\|_p,$$

which is exactly the inequality required for the proof of assertion 2) of the theorem.

Let us note finally that ${}^*H_p^r(\sigma)$ may be equivalently defined as the class of those $f \in L_p(\sigma)$ for which

$$\|{}^*\Delta_\gamma^k D^l f(\mu)\|_p \leqslant M\gamma^{r-2l},$$

where M is independent of γ $(k > r-2l > 0)$.

5. Symmetric averaged differences

If one means to apply the differences ${}^*\Delta_\gamma^k$ in numerical calculations, it is advisible to symmetrize them. In the symmetric form, just as the power differences $(\Delta_\gamma)^k$, they have the doubling property (i.e. they behave locally as differences of order $2k$). Furthermore, the symmetrized differences $\mathring{\Delta}_\gamma^k$ defined below need less computations.

The symmetric differences are defined by

$$\mathring{\Delta}_\gamma^k f(\mu) = \sum_{j=0}^{k} (-1)^{k+j} B_k^j S_{j\gamma} f(\mu), \qquad B_k^j = 2\frac{(k-j+1)\ldots k}{(k+1)\ldots(k+j)}.$$

With the use of these differences, one obtains a representation of the approximating polynomial (with S_t replaced by ${}^-S_t$ for n odd) and a theorem analogous to Theorem 2. In proving the second (= converse) assertion, one uses the Bernstein type inequality

$$\|\mathring{\Delta}_\gamma^k T_N(\cdot)\|_p \leqslant \frac{(k!)^2}{(2k)!}(N\gamma)^{2k} \|T_N(\cdot)\|_p.$$

This estimate reveals the above-mentioned doubling property.

We also note that the advantage of the operator $\mathring{\Delta}_\gamma^k$ over ${}^*\Delta_\gamma^k$ is that using the former one can detect the differential properties of H_p^r functions for $r < 2k$, while using the latter for $r < k$ only.

As already noted in the introduction, we have extended the results listed above to the case of the spaces $B_{p,\theta}^r(\sigma)$ for $1 \leqslant \theta < \infty$.

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Translated from the Russian
by Jerzy Trzeciak

Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986