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Exactness of expanding mappings

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Abstract. A new sufficient condition for exactness of piecewise expanding mappings on the d-dimensional cube $[0, 1]^d$ is proved. It generalizes the results of [6] concerning the piecewise expanding mappings on the unit interval.

1. Introduction. The purpose of this paper is to establish a sufficient condition for the exactness of piecewise expanding mappings on the d-dimensional cube. In the one-dimensional case, this problem is almost completely solved. The solution follows from the results of Rényi [10], Rochlin [11], Lasota and Yorke [7], and Lasota and Mackey [6]. In the d-dimensional case, this problem has been investigated in [1], [5] and [13]. In particular, K. Krzyżewski and W. Szlenk proved the exactness for expanding mappings of compact manifolds under assumption $T(A) = A$. G. Pianigiani and J. A. Yorke obtained a somewhat analogous result for expanding mappings in the case $A \subset T(A)$. In the present paper we assume $T(A) \subset A$.

We shall study our problem from a statistical point of view as transformations of probability densities. This leads immediately to the idea of considering the properties of the corresponding Frobenius–Perron operators. The proof of our theorem is based on idea due to [6]. Section 2 of the paper contains some preliminary notation. The main result is proved in Section 3.

2. Preliminaries. Let (X, Σ, m) be a σ-finite measure space and let $S: X \to X$ be a given transformation. In what follows, we shall assume that S is doubly measurable which means that $S(A) \in \Sigma$ and $S^{-1}(A) \in \Sigma$ for $A \in \Sigma$. We shall also assume that S is non-singular, i.e., $m(A) = 0$ implies $m(S^{-1}(A)) = 0$. We shall deal with the space $L^1 = L^1(X, \Sigma, m)$ and the norm $\|\cdot\| = \|\cdot\|_{L^1}$. By $D = D(X, \Sigma, m)$ we shall denote the set of all (normalized) densities on X, that is,

$$D = \{f \in L^1: f \geqslant 0 \text{ and } \|f\| = 1\}.$$

For any density g we shall denote by m_g the measure

$$m_g(A) = \int_A g\,dm \quad \text{for } A \in \Sigma.$$

An important role in our considerations is played by the Frobenius–Perron operator. For a given S, we define the Frobenius–Perron operator P corresponding to S by

$$(1)\qquad \int_A Pf\,dm = \int_{S^{-1}(A)} f\,dm \quad \text{for } A\in\Sigma \text{ and } f\in L^1.$$

From (1) it can be seen that P is linear and preserves the integral and is contractive in L^1 (that is, $\|Pf\| \leqslant \|f\|$).

To define the exactness of the system (S, μ) assume that (X, Σ, μ) is a normalized measure space and that $S\colon X \to X$ is a measure preserving transformation. If

$$\lim_n \mu(S^n(A)) = 1$$

for every $A\in\Sigma$ with $m(A) > 0$, then the system (S, μ) is called *exact*.

We shall use the following sufficient condition for exactness (see [8]).

THEOREM 1. *Let (X, Σ, m) be a σ-finite measure space and let $S\colon X \to X$ be doubly measurable and non-singular. Assume that there exists $h\in L^1$, $h \geqslant 0$, $\|h\| > 0$, such that*

$$(2)\qquad \lim_n \|(h-P^n f)^+\| = 0 \quad \textit{for } f\in D.$$

Then there is a unique density g such that the system (S, m_g) is exact.

In (2), z^+ denotes $\max(0, z)$. A non-negative function $h\in L^1$ satisfying (2) will be called a *lower bound function* for P.

We shall use some standard notions of the theory of differential inequalities. A function $f\colon (a, b) \to R$ is called *left lower semicontinuous* if

$$\liminf_{\varepsilon>0,\ \varepsilon\to 0} f(t-\varepsilon) \geqslant f(t) \quad \text{for } t\in(a, b).$$

For any function $f\colon (a, b)\to R$ we may define its right lower derivative by setting

$$\frac{d_+ f(t)}{dt} = \liminf_{\varepsilon>0,\ \varepsilon\to 0} \frac{f(t+\varepsilon)-f(t)}{\varepsilon} \quad \text{for } t\in(a, b).$$

It is well known (see [12]) that for every left lower semicontinuous function $f\colon (a, b)\to R$ the inequality

$$\frac{d_+ f(t)}{dt} \leqslant Kf(t) \quad \text{for } t\in(a, b)$$

implies

$$(3)\qquad f(t) \leqslant f(s)\, e^{K(t-s)} \quad \text{for } t\in[s, b).$$

Now we are going to define the expanding maps on R^d. We say that a matrix M is *λ-expansive* if $\inf\{|M\cdot v|: |v| = 1\} \geq \lambda$. Let $S: X \to R^d$, $X \subset R^d$ be a C^1-function and let $DS(x)$ denote the Jacobian matrix for S. We call the mapping S *λ-expansive* if $DS(x)$ is a λ-expansive matrix for all $x \in X$.

3. The main result. Denote by m the Lebesgue measure on the unit cube $X = [0, 1]^d$. We shall write $x = (x_1, \ldots, x_d)$ for $x \in X$. Let $X = \bigcup_{i=1}^{k} X_i$, where X_i, $i = 1, \ldots, k$, are closed cubes which can be written in the form

$$X_i = \{x \in X: a_j^i \leq x_j \leq b_j^i \text{ for } j = 1, \ldots, d\}.$$

Assume that $m(X_i \cap X_j) = 0$ for $i \neq j$. Consider a sequence of mappings $S^i: X_i \to X$, $i = 1, \ldots, k$, which satisfy the conditions:

(i) S^i is a C^2-diffeomorphism onto its image;

(ii) $\partial T_j^i/\partial x_l \geq 0$, $\det DT^i > 0$ for $j, l = 1, \ldots, d$, where $T^i = (S^i)^{-1}$ and $T^i = (T_1^i, \ldots, T_d^i)$;

(iii) there exists $\lambda > 1$ such that S^i is λ-expansive;

(iv) if $x \in X_i$ and $x_j \neq a_j^i$ for a given $1 \leq j \leq d$, then $S_j^i(x) \neq 0$;

(v) if $x \in Y_i = S^i(X_i)$ and $y_j \leq x_j$ for $j = 1, \ldots, d$, then $y \in Y_i$.

We define the mapping $S: X \to X$ by the condition

(vi) $S(x) = S^i(x)$ for $x \in \operatorname{int} X_i$, $i = 1, \ldots, k$.

THEOREM 2. *If S is given by formula* (vi) *and S^i satisfy conditions* (i)–(v), *then there exists a unique density g such that the system (S, m_g) is exact.*

Proof. Let Q be the set of all C^1-functions $q: \Delta \to X$ such that

$$dq_j/dt \geq 0 \quad \text{for } j = 1, \ldots, d \text{ and } t \in \Delta,$$

where Δ is a compact interval which depends in general upon q. Now let D_0 denote the subset of $D = D(X, \Sigma, m)$ consisting of all functions f satisfying for every $q \in Q$ the following two conditions:

(C1) The function $f \circ q$ is left lower semicontinuous;

(C2)
$$\frac{d_+ f \circ q}{dt} \leq K_f \left|\frac{dq}{dt}\right| f \circ q,$$

where $|\cdot|$ stands for the norm in R^d and the constant K_f depends in general on f but not on q. The proof will be given in two steps. First, we shall show that $P^n f \in D_0$ for $f \in D_0$ and that there exists a constant K such that $K_{P^n f} \leq K$ for every $f \in D_0$ and for $n \geq n(f)$. Then in Step II we shall prove the existence of a constant r such that for every $f \in D_0$

(4)
$$\int_{B_r} P^n f\, dm \leq 1/2, \qquad B_r = \{x \in X: x_j \leq r \text{ for some } j\}$$

for sufficiently large n. This allows to construct a non-trivial (different from zero) lower bound function for P.

Step I. Let $f \in D_0$ and $q \in Q$. A simple computation shows that the operator P can be written in the form

$$Pf(x) = \sum_{i=1}^{k} \mathbf{1}_{Y_i}(x) f \circ T^i(x) \det DT^i(x).$$

Hence

$$Pf \circ q(t) = \sum_{i=1}^{k} \mathbf{1}_{Y_i} \circ q(t) f \circ T^i \circ q(t) \det DT^i \circ q(t) \quad \text{for } t \in \Delta. \tag{5}$$

From conditions (v) and (ii) it follows that

$$\mathbf{1}_{Y_i} \circ q = \mathbf{1}_{[0,b_i]} \quad \text{and} \quad T^i \circ q|_{[0,b_i]} \in Q,$$

where $b_i = \sup\{t\colon q(t) \in Y_i\}$. The function $Pf \circ q$ as calculated from equation (5) is left lower semicontinuous. Differentiation of (5) gives

$$\frac{d_+ Pf \circ q}{dt} = \sum_{i=1}^{k} \mathbf{1}_{Y_i} \circ q \frac{d_+ f \circ T^i \circ q}{dt} \det DT^i \circ q + \sum_{i=1}^{k} \mathbf{1}_{Y_i} \circ q \frac{d(\det DT^i \circ q)}{dt} f \circ T^i \circ q.$$

By conditions (C2) and (iii) we obtain

$$\frac{d_+ f \circ T^i \circ q}{dt} \leqslant K_f \left|\frac{dT^i \circ q}{dt}\right| f \circ T^i \circ q \leqslant \beta K_f \left|\frac{dq}{dt}\right| f \circ T^i \circ q,$$

where $\beta = 1/\lambda$. Since Y_i are compact, there exists a constant L such that

$$\frac{\partial \det DT^i}{\partial x_j} \leqslant L \det DT^i \quad \text{for } i = 1, \ldots, k;\ j = 1, \ldots, d$$

so, as a consequence,

$$\frac{d(\det DT^i \circ q)}{dt} \leqslant L \det DT^i \circ q \sum_{j=1}^{d} \frac{dq_j}{dt} \leqslant dL \det DT^i \circ q \left|\frac{dq}{dt}\right|.$$

Thus, finally

$$\frac{d_+ Pf \circ q}{dt} \leqslant (\beta K_f + dL) \left|\frac{dq}{dt}\right| Pf \circ q,$$

and $Pf \in D_0$ with a constant $K_{Pf} \leqslant \beta K_f + dL$. An induction argument shows that $P^n f \in D_0$ and

$$K_{P^n f} \leqslant \beta^n K_f + dL/(1-\beta) \leqslant K$$

for n sufficiently large and $K = 1 + dL/(1-\beta)$.

Before passing to the Step II we shall proof three simple lemmas.

LEMMA 1. *Let $f \in D_0$ and $K_f \leqslant K$. If $x_j \leqslant y_j$ for $j = 1, \ldots, d$, then $f(y) \leqslant Mf(x)$, where $M = e^{Kd}$.*

Proof. Define $q\colon I \to X$ by the formula $q(t) = ty + (1-t)x$. Evidently $q \in Q$ and $|dq/dt| \leqslant d$. By the definition of D_0

$$d_+ f \circ q/dt \leqslant Kdf \circ q$$

and the function $f \circ q$ is left lower semicontinuous. From (3) it follows $f(y) \leqslant Mf(x)$.

LEMMA 2. *Assume that $f \in D_0$ and $K_f \leqslant K$. Let c, r be positive real numbers such that $c + r \leqslant 1$. Then*

$$\int_{C_j} f\,dm \leqslant rM/c \qquad \text{where } C_j = \{x \in X\colon\ c \leqslant x_j \leqslant c+r\}.$$

Proof. Let $B_j = \{x \in X\colon\ x_j \leqslant c\}$ and let

$$F_j(x) = (x_1, \ldots, x_{j-1}, c + x_j r/c, x_{j+1}, \ldots, x_d)$$

for $x \in B_j$. Then $\det DF_j = r/c$, and consequently, from Lemma 1 it follows that

$$\int_{C_j} f\,dm - \int_{B_j} f \circ F_j |\det DF_j|\,dm \leqslant (r/c) \int_{B_j} Mf\,dm \leqslant rM/c.$$

LEMMA 3. *Let j ($1 \leqslant j \leqslant d$) be a given number and let r be a positive real number. Then*

$$T^i(K_r \cap Y_i) \subset K^i_{\beta r} \qquad \text{for } i = 1, \ldots, k,$$

where

$$K_r = \{x \in X\colon\ x_j \leqslant r\}, \qquad K^i_{\beta r} = \{x \in X_i\colon\ x_j \leqslant a^i_j + \beta r\}.$$

Proof. First we shall show that if $x, y \in Y_i$ for a certain i, then $|T^i x - T^i y| \leqslant \beta |x - y|$. Define $q\colon I \to X$ by the formula $q(t) = ty + (1-t)x$. Using condition (iii) we may write

$$|T^i x - T^i y| \leqslant \int_0^1 \left|\frac{dT^i \circ q}{dt}\right| dt \leqslant \beta \int_0^1 \left|\frac{dq}{dt}\right| dt = \beta |x - y|.$$

For a given point $x \in T^i K_r$ and for $y = S^i x$ we have $y_j \leqslant r$. Set $z = (y_1, \ldots, y_{j-1}, 0, y_{j+1}, \ldots, y_d)$. From (iv) and (v) it follows that $z \in Y_i$ and

$T_j^i z = a_j^i$. Thus, since

$$x_j - a_j^i \leqslant |x - T^i z| \leqslant \beta |y - z| \leqslant \beta r,$$

we have $x \in K_{\beta r}^i$.

Step II. Define B_r as in formula (4). For a given j ($1 \leqslant j \leqslant d$), set

$$I = \{i\colon\ a_j^i = 0\}, \quad J = \{i\colon\ a_j^i \neq 0\} \quad \text{and} \quad b = \min\{a_j^i\colon\ i \in J\}.$$

Let K_r, $K_{\beta r}^i$ be as in Lemma 3. If $f \in D_0$, then there exists N such that $K_{P^n f} \leqslant K$ for $n \geqslant N$. Since $m(X_i \cap X_j) = 0$ for $i \neq j$, it follows that $m(K_{\beta r}^i \cap K_{\beta r}^j) = 0$ for $i \neq j$. From Lemma 3 and Lemma 2 we have

$$\int_{K_r} P^{N+1} f\, dm = \sum_{i=1}^{k} \int_{T^i(K_r)} P^N f\, dm \leqslant \sum_{i=1}^{k} \int_{K_{\beta r}^i} P^N f\, dm$$

$$\leqslant \sum_{i \in I} \int_{K_{\beta r}^i} P^N f\, dm + \sum_{i \in J} \int_{K_{\beta r}^i} P^N f\, dm \leqslant \int_{K_{\beta r}} P^N f\, dm + k\beta r M/b.$$

Again, using a simple induction argument it follows that

$$\int_{K_r} P^{N+n} f\, dm \leqslant \int_{K_{\beta^n r}} P^N f\, dm + k\beta r M/b(1-\beta) \leqslant 1/2d$$

and consequently

$$\int_{B_r} P^{N+n} f\, dm \leqslant 1/2$$

for $r < b(1-\beta)/2dk\beta M$ and n sufficiently large.

We are going to show that $h = (1/2M)\mathbf{1}_G$ with

$$G = \{x \in X\colon\ x_j \leqslant r \text{ for every } j = 1, \ldots, d\}$$

is a lower bound function for P. Let $f \in D_0$ and

$$K_{P^n f} \leqslant K \quad \text{and} \quad \int_{B_r} P^n f\, dm \leqslant 1/2 \quad \text{for } n \geqslant N.$$

From Lemma 1 for $x \in G$ we have

$$1/2 \leqslant \int_{X - B_r} P^n f\, dm \leqslant \int_{X - B_r} M P^n f(x)\, dm \leqslant M P^n f(x)$$

and $P^n f(x) \geqslant 1/2M$. Since the set D_0 is dense in D, this completes the proof.

Remark. The existence of absolutely continuous invariant measure for a certain class of piecewise differentiable mappings on the d-dimensional cube has been proved in [2]–[4]. The transformation

$$S(x, y) = \begin{cases} (4x-4xy, 4y) & \text{for } 0 \leqslant x, y \leqslant 1/4, \\ (4x, 4y) (\mathrm{mod}\, 1) & \text{elsewhere}, \end{cases}$$

on the unit square I^2 satisfies the assumptions of our Theorem 2 and it is not of the form considered in the mentioned papers.

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