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An inequality for the maximum of trigonometric polynomials

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Let

$$f_n(\vartheta) = \sum_{k=1}^{n} (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

be a trigonometric polynomial with real coefficients. Put

$$M = \max_{0 \leqslant \vartheta < 2\pi} |f_n(\vartheta)| .$$

It immediately follows from the Parseval relation that

$$M \geqslant \frac{1}{\sqrt{2}} \Big(\sum_{k=1}^{n} (a_k^2 + b_k^2) \Big)^{1/2} .$$

S. Bernstein [1] gave an example of a polynomial for which

$$M < C \Big(\sum_{k=1}^{n} (a_k^2 + b_k^2) \Big)^{1/2}$$

and (2) and (3) holds. I conjecture that there exists an absolute constant $c > 0$ so that

(1) $$M \geqslant \frac{1+c}{\sqrt{2}} \Big(\sum_{k=1}^{n} (a_k^2 + b_k^2) \Big)^{1/2} .$$

$c \leqslant \sqrt{2}-1$ as is shown by $f(\vartheta) = \cos\vartheta$. Perhaps $c = \sqrt{2}-1$. In this note I shall prove the following

THEOREM. *Assume that*

(2) $$\max_{1 \leqslant k \leqslant n} (\max |a_k|, |b_k|) = 1$$

and that

(3) $$\sum_{k=1}^{n} (a_k^2 + b_k^2) = An .$$

Then there exists a $c = c_A > 0$ *depending only on* A *for which* $\lim_{A\to 0} c_A = 0$ *and*

$$M > \frac{1+c_A}{\sqrt{2}}\Big(\sum_{k=1}^{n}(a_k^2+b_k^2)\Big)^{1/2}.$$

At present I cannot even prove that (1) holds for $b_k = 0$ and $a_k = 0$, or ± 1 (i.e. for the polynomials $\Sigma \varepsilon_k \cos m_k x$).

For rational polynomials one would conjecture that

$$\max_{|z|=1}\Big|\sum_{k=1}^{n}\varepsilon_k z^{m_k}\Big| > (1+c_1)n^{1/2}, \qquad |\varepsilon_k| = 1, \tag{4}$$

where $c_1 > 0$ is an absolute constant, but I cannot even prove this for $m_k = k$. In this direction D. Newman [2] (¹) proved certain preliminary results. His result implies $n^{1/2} + c_1/n^{1/2}$ instead of (4). The analogon of (1) is of course false here as can be seen by the polynomial z. The most that one could hope is that if $\max_{1\le k\le n} |a_k| = 1$ and $\sum_{k=1}^{n}|a_k|^2 = 1+B$ (i.e. if the sum of the squares of the coefficients is appreciably greater than the largest coefficient), then

$$\max_{|z|=1}\Big|\sum_{k=1}^{n}a_k z^k\Big|^{\vartheta} > (1+c_B)\Big(\sum_{k=1}^{n}|a_k|^2\Big)^{1/2}. \tag{5}$$

It seems likely that (5) holds.

To prove our theorem we need three lemmas. Assume that $f_n(\vartheta)$ is a trigonometric polynomial satisfying (2) for which

$$\max_{0\le\vartheta<2\pi}|f_n(\vartheta)| < \frac{1+\varepsilon}{\sqrt{2}}\Big(\sum_{k=1}^{n}(a_k^2+b_k^2)\Big)^{1/2} \qquad (0<\varepsilon<1). \tag{6}$$

LEMMA 1. *Let* $f_n(\vartheta)$ *satisfy* (3) *and* (6). *Then the measure of the set in* ϑ *for which*

$$|f_n(\vartheta)| < \frac{1-\varepsilon^{1/2}}{\sqrt{2}}\Big(\sum_{k=1}^{n}(a_k^2+b_k^2)\Big)^{1/2} = T \tag{7}$$

is less than $20\varepsilon^{1/2}$.

(¹) D. Newman proves in fact that if in (4) $m_k = k$ and $\varepsilon_k = \pm 1$ then

$$\int_{|z|=1}\Big|\sum_{k=1}^{n}\varepsilon_k z^k\Big|\,dz < \sqrt{n-c}\,.$$

A slight modification of our proof would show that if $f_n(\vartheta)$ satisfies (2) and (3) then

$$\int_{0}^{2\pi}|f_n(\vartheta)|\,d\vartheta < (1-c_A')\,n^{1/2}.$$

Denote by U the measure of the set satisfying (7). We evidently have for $\varepsilon < 1$

$$\int_0^{2\pi} f_n(\vartheta)^2 d\vartheta = \pi \sum_{k=1}^{n} (a_k^2+b_k^2) < UT^2 + (2\pi - U)\frac{(1+\varepsilon)^2}{2}\sum_{k=1}^{n}(a_k^2+b_k^2)$$

$$= \sum_{k=1}^{n}(a_k^2+b_k^2)\left[\pi + \pi\,\frac{2\varepsilon+\varepsilon^2}{2} - U\,\frac{2\varepsilon^{1/2}+\varepsilon+\varepsilon^2}{2}\right]$$

$$< \sum_{k=1}^{n}(a_k^2+b_k^2)[\pi + 3\varepsilon\pi - U\varepsilon^{1/2}]$$

or

$$U < 3\pi\varepsilon^{1/2} < 10\,\varepsilon^{1/2},$$

which proves the lemma.

LEMMA 2. *Assume that* (6) *holds. Then*

$$\max_{0\leqslant\vartheta<2\pi} |f_n'(\vartheta)| < \frac{n(1+\varepsilon)}{\sqrt{2}}\Big(\sum_{k=1}^{n}(a_k^2+b_k^2)\Big)^{1/2} = \frac{1+\varepsilon}{\sqrt{2}} A^{1/2} n^{3/2}.$$

This is a well-known theorem of S. Bernstein, which states that

$$\max_{0\leqslant\vartheta<2\pi} |f_n'(\vartheta)| \leqslant n \max_{0\leqslant\vartheta<2\pi} |f_n(\vartheta)|.$$

LEMMA 3. *Assume that* (2) *and* (3) *holds. Then*

$$\int_0^{2\pi} f_n'(\vartheta)^2 d\vartheta = \pi\sum_{k=1}^{n} k^2(a_k^2+b_k^2)$$

$$\geqslant \pi \sum_{1\leqslant k\leqslant[An/2]} 2k^2 + 2\pi\left(\left[\frac{An}{2}\right]+1\right)^2\left(\frac{An}{2}-\left[\frac{An}{2}\right]\right) > A^3\frac{n^3}{4}.$$

The proof of lemma 3 follows immediately from the elementary observation that if (2) and (3) are satisfied, then $\sum_{k=1}^{n} k^2(a_k^2+b_k^2)$ is the minimum if the a's and b's with the smallest possible indices are as large as possible. That is if $a_k = b_k = 1$ for $1 \leqslant k \leqslant [An/2]$.

Assume now that $f_n(\vartheta)$ satisfies (2) and (3). From lemmas 2 and 3 we evidently have

$$(8) \qquad \int_0^{2\pi} |f_n'(\vartheta)|\,d\vartheta \geqslant \int_0^{2\pi} f_n'(\vartheta)^2 d\vartheta \,\big(\max_{0\leqslant\vartheta<2\pi} |f_n'(\vartheta)|\big)^{-1} > \frac{A^{5/2}n^{3/2}}{2^{3/2}(1+\varepsilon)}.$$

$\int_0^{2\pi} |f_n'(\vartheta)|\,d\vartheta$ is the total variation of $f_n(\vartheta)$ in $(0,2\pi)$. $f_n(\vartheta)$ is a trigonometric polynomial of degree n, and thus it consists of at most $2n$ monotonic arcs. Hence its total variation on the set E for which $f_n(\vartheta)$ is in the intervals

$$(9)\quad \left(\frac{1-\varepsilon^{1/2}}{\sqrt{2}} A^{1/2} n^{1/2}, \frac{1+\varepsilon}{\sqrt{2}} A^{1/2} n^{1/2}\right) \quad \text{and} \quad \left(-\frac{1+\varepsilon}{\sqrt{2}} A^{1/2} n^{1/2}, -\frac{1-\varepsilon^{1/2}}{\sqrt{2}} A^{1/2} n^{1/2}\right)$$

is at most $4(\varepsilon^{1/2}+\varepsilon)A^{1/2}n^{3/2}$, or

$$(10)\qquad \int_E |f_n'(\vartheta)|\,d\vartheta \leqslant 4A^{1/2}(\varepsilon+\varepsilon^{1/2})n^{3/2}.$$

From (8) and (10) we have for $\varepsilon < A^4/1000$ ($\bar{E}$ is the complement of E)

$$(11)\qquad \int_{\bar{E}} |f_n'(\vartheta)|\,d\vartheta > \frac{A^{5/2}n^{3/2}}{2^{3/2}(1+\varepsilon)} - 4A^{1/2}(\varepsilon+\varepsilon^{1/2})n^{3/2} > \frac{A^{5/2}n^{3/2}}{10}.$$

From lemma 2 and (11) it follows that the measure of the set $\bar{E}$ (which has been denoted by U in lemma 1) is greater than

$$(12)\qquad U > \frac{A^{5/2}n^{3/2}}{10}\left(\frac{1+\varepsilon}{\sqrt{2}} A^{1/2} n^{3/2}\right)^{-1} > \frac{A^2}{10}.$$

By assumption $f_n(\vartheta)$ satisfies (2), (3) and (6). Thus from (12) and lemma 1

$$(13)\qquad 10\varepsilon^{1/2} > A^2/10 \quad \text{or} \quad \varepsilon > A^4/10\,000\,.$$

(13) implies our theorem with $c_A \geqslant A^4/10\,000$. It would be easy to improve this value of c_A, but at present I see no way to determine the best possible value of c_A.

References

[1] S. Bernstein, *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus 158 (1914), p. 1661-1663.

[2] D. J. Newman, *Norms of polynomials*, Amer, Math. Monthly 67 (1960), p. 778-779.

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