COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

SOME UNIQUENESS THEOREMS FOR SOLUTIONS OF PARABOLIC AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN UNBOUNDED REGIONS

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In paper [1] there were proved some theorems on uniqueness of regular and weak solutions of the Cauchy problem for a single linear parabolic equation with unbounded coefficients under the assumption that the solutions satisfy a certain growth condition determined by the convergence of an improper integral.

Section 1 of this paper deals with a generalization of those results, which concern the regular solutions (see Theorems 1 and 2 of [1] and also Appendix to [2]), to a system of semilinear equations. We also improve the result of [1] by requiring less restrictive assumptions concerning the growth of the coefficients and the solutions (Theorems 2 and 3).

In section 2 we extend the results of section 1 to solutions of the Dirichlet problem in unbounded regions for elliptic equations. In substance, the obtained theorems constitute generalizations of Krzyżański's results [3] in two directions. Firstly, the pointwise growth condition imposed on the solution in reference [3] is replaced here by a weaker integral growth condition, and, secondly, one linear equation is substituted by a system of semilinear equations. Moreover ,the coefficients are allowed to grow to infinity in various ways, whereas in [3] they were assumed to be bounded.

1. Let $S = E^n \times (0, T]$ and $\bar{S} = E^n \times [0, T]$, where E^n is a Euclidean n-space of the variables $x = (x_1, \ldots, x_n)$ and $T > 0$ is fixed. In this section, we consider the system

$$(1) \qquad u_t^i = \sum_{j,k=1}^{n} [a_{jk}^i(x, t) u^i]_{x_j x_k} - \sum_{j=1}^{n} [b_j^i(x, t) u^i]_{x_j} + f^i(x, t, u^1, \ldots, u^m), \qquad i = 1, \ldots, m.$$

We make the following assumptions:

(A_1) The coefficients a^i_{jk}, b^i_j and their derivatives $(a^i_{jk})_{x_j}$, $(a^i_{jk})_{x_jx_k}$, $(b^i_j)_{x_j}$ are defined and measurable in $\bar{S}$ and bounded in every cylinder $\bar{S}_R = (|x| \leqslant R)\times[0, T]$, where $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Also $a^i_{jk} = a^i_{kj}$.

(A_2) The quadratic forms

$$\sum_{j,k=1}^{n} a^i_{jk}(x, t)h_j h_k \qquad (i = 1, \ldots, m)$$

are positive semidefinite in $\bar{S}$.

(A_3) The functions f^i are defined for $(x, t)\in\bar{S}$ and arbitrary $u^1, \ldots, u^m$, and satisfy the Lipschitz condition

$$[f^i(x, t, u^1, \ldots, u^m)-f^i(x, t, \bar{u}^1, \ldots, \bar{u}^m)]\operatorname{sgn}(u^i-\bar{u}^i)$$
$$\leqslant \sum_{s=1}^{m} c^i_s(x, t)\,|u^s-\bar{u}^s| \qquad (i = 1, \ldots, m),$$

where $c^i_s(x, t) \geqslant 0$ for $s \neq i$, $(x, t)\in\bar{S}$ and $c^i_s = c^s_i$. Moreover, c^i_s are measurable in $\bar{S}$ and bounded in any finite cylinder $\bar{S}_R$. The symbol $\operatorname{sgn} x$ denotes $+1$ for $x \geqslant 0$ and -1 for $x < 0$.

By a *solution* of (1) we mean a system of functions $u(x, t) = \{u^1(x, t), \ldots, u^m(x, t)\}$ continuous in $\bar{S}$, having the derivatives appearing in (1) which are measurable in $\bar{S}$, bounded in every cylinder $\bar{S}_R$, and satisfy system (1) in S.

Now we prove the following uniqueness theorem for the solutions of the Cauchy problem for system (1):

THEOREM 1. *Let assumptions* (A_1)-(A_3) *be satisfied.*

Suppose there exists a vector-valued function $\Phi(x, t) = \{\Phi^1(x, t), \ldots, \Phi^m(x, t)\}$ *of class* $C^2(\bar{S})$ *such that* $\Phi^i(x, t) > 0$ *in* S *and*

$$(2)\qquad L^{*i}\Phi \equiv \sum_{j,k=1}^{n} a^i_{jk}\Phi^i_{x_jx_k}+\sum_{j=1}^{n} b^i_j\Phi^i_{kj}+\sum_{s=1}^{m} c^i_s\Phi^s+\Phi^i_t \leqslant 0 \qquad (i = 1, \ldots, m)$$

almost everywhere in S. *If* $u_1 = \{u^i_1)$ *and* $u_2 = \{u^i_2\}$, $i = 1, \ldots, m$, *are two solutions of system* (1) *satisfying the growth condition*

$$(3)\qquad \iint_S |u^i_l|\Big[\max_j \sum_k |a^i_{jk}\Phi^i_{x_k}|+\Phi^i(\max_{j,k}|a^i_{jk}|+\max_j|b^i_j|\Big]\,dx\,dt < \infty$$
$$(i = 1, \ldots, m;\ l = 1, 2)$$

and $u^i_1(x, 0) = u^i_2(x, 0)$, $i = 1, \ldots, m$, *for* $x\in E^n$, *then* $u^i_1(x, t) = u^i_2(x, t)$, $i = 1, \ldots, m$, *for* $(x, t)\in\bar{S}$.

Proof. We define $u^i = u_1^i - u_2^i$, $w^i = [(u^i)^2+\varepsilon]^{1/2}$, $\varepsilon > 0$, $u = \{u^1, \ldots, u^m\}$, $w = \{w^1, \ldots, w^m\}$. The functions u^i satisfy the relations

$$(4)\quad u_t^i = \sum_{j,k=1}^{n} (a_{jk}^i u^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i u^i)_{x_j} + f^i(x, t, u_1^1, \ldots, u_1^m) - f^i(x, t, u_2^1, \ldots, u_2^m) \qquad (i = 1, \ldots, m)$$

and $u^i(x, 0) = 0$ for $x \in E$. Now we multiply the i-th relation of (4) by u^i/w^i $(i = 1, \ldots, m)$ and make use of assumptions (A_2) and (A_3) to obtain

$$(5)\quad w_t^i \leqslant \sum_{j,k=1}^{n} (a_{jk}^i w^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i w^i)_{x_j} - \left[\sum_{j,k=1}^{n} (a_{jk}^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i)_{x_j}\right] \frac{\varepsilon}{w^i} + \frac{|u^i|}{w^i} \sum_{s=1}^{m} c_s^i |u^s|.$$

The assumption $c_s^i \geqslant 0$, $s \neq i$, implies the inequality

$$\frac{|u_i|}{w^i} \sum_{s=1}^{m} c_s^i |u^s| \leqslant \sum_{s=1}^{m} c_s^i w^s - \frac{\varepsilon}{w_i} c_i^i.$$

Hence by (5) we get

$$(6)\quad L^i w \equiv \sum_{j,k=1}^{n} (a_{jk}^i w^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i w^i)_{x_j} + \sum_{s=1}^{m} c_s^i w^s - w_t^i \geqslant A^i \frac{\varepsilon}{w^i},$$

where

$$A^i = \sum_{j,k=1}^{n} (a_{jk}^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i)_{x_j} + c_i^i.$$

Let $v(x, t) = \{v^i(x, t)\}$, $i = 1, \ldots, m$, where $v^i(x, t)$ are non-negative functions of class $C^2(\bar{S})$ with compact support as functions of x in E^n. We make use of the identities

$$(7)\quad v^i\Big(L^i w - \sum_{s=1}^{m} c_s^i w^s\Big) = w^i\Big(L^{*i} v - \sum_{s=1}^{m} c_s^i v^s\Big) + \sum_{j=1}^{n} \Big[v^i \sum_{k=1}^{n} (a_{jk}^i w^i)_{x_k} - w^i \sum_{k=1}^{n} a_{jk}^i v_{x_k}^i - b_j^i v^i w^i\Big]_{x_j} - (v^i w^i)_t \qquad (i = 1, \ldots, m).$$

Summing identities (7) over i and taking into account inequalities (6) and the relation

$$\sum_{i,s=1}^{m} c_s^i v^i w^s = \sum_{i,s=1}^{m} c_s^i v^s w^i$$

we get

$$\sum_{i=1}^{m} A^i v^i \frac{\varepsilon}{w^i} \leqslant \sum_{i=1}^{m} w^i L^{*i} v + \tag{8}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\Big[v^i \sum_{k=1}^{n}(a^i_{jk} w^i)_{x_k} - w^i \sum_{k=1}^{n} a^i_{jk} v^i_{x_k} - b^i_j v^i w^i\Big]_{x_j} - \Big(\sum_{i=1}^{m} v^i w^i\Big)_t.$$

Let (ξ, τ) be an arbitrary point of S. We show that $u^i(\xi, \tau) = 0$ $(i = 1, \ldots, m)$. Denote by S_τ the strip $E^n \times [0, \tau]$.

Integrating both sides of (8) over S_τ yields

$$\iint_{S_\tau} \sum_{i=1}^{m} A^i v^i \frac{\varepsilon}{w^i}\, dx\, dt \tag{9}$$

$$\leqslant \iint_{S_\tau} \sum_{i=1}^{m} w^i L^{*i} v\, dx\, dt - \int_{E^n} \sum_{i=1}^{m} (v^i w^i)|_{t=\tau} dx + \int_{E^n} \sum_{i=1}^{m} (v^i w^i)|_{t=0} dx.$$

Passing to the limit in (9) as $\varepsilon \to 0$ we obtain, in view of $u^i(x, 0) = 0$,

$$\int_{E^n} \sum_{i=1}^{m} |u^i(x, \tau)| v^i(x, \tau)\, dx \leqslant \iint_{S_\tau} \sum_{i=1}^{m} |u^i| L^{*i} v\, dx\, dt. \tag{10}$$

Now set $v^i = \gamma^R(x) \Phi^i$, where $\gamma^R(x)$ with $R > 1$ is a function of class $C^2(E^n)$ such that $\gamma^R(x) = 1$ for $|x| \leqslant R-1$, $\gamma^R(x) = 0$ for $|x| \geqslant R$, $0 \leqslant \gamma^R(x) \leqslant 1$, and $\sum_{j,k=1}^{n} |\gamma^R_{x_j x_k}| + \sum_{j=1}^{n} |\gamma^R_{x_j}|$ is bounded in E^n by a constant independent of R. We have

$$L^{*i} v = \gamma^R L^{*i} \Phi + 2 \sum_{j,k=1}^{n} a^i_{jk} \gamma^R_{x_j} \Phi^i_{x_k} + \Phi^i \Big(\sum_{j,k=1}^{n} a^i_{jk} \gamma^R_{x_j x_k} + \sum_{j=1}^{n} b^i_j \gamma^R_{x_j}\Big). \tag{11}$$

Combining (2), (10) and (11) we find

$$\int_{E^n} \sum_{i=1}^{m} |u^i| \gamma^R \Phi^i|_{t=\tau} dx \tag{12}$$

$$\leqslant \iint_{S_\tau} \sum_{i=1}^{m} |u^i| \Big| 2 \sum_{j,k=1}^{n} a^i_{jk} \gamma^R_{x_j} \Phi^i_{x_k} + \Phi^i \Big(\sum_{j,k=1}^{n} a^i_{jk} \gamma^R_{x_j x_k} + \sum_{j=1}^{n} b^i_j \gamma^R_{x_j}\Big)\Big| dx\, dt.$$

By (3), the right-hand side of (12) approaches zero as $R \to \infty$, whence

$$\int_{E^n} \sum_{i=1}^{m} |u^i| \Phi^i|_{t=\tau} dx \leqslant 0.$$

This implies that $u^i(x, \tau) = 0$ and, in particular, $u^i(\xi, \tau) = 0$ $(i = 1, \ldots, m)$, which was to be proved.

The following theorem is a consequence of theorem 1:

THEOREM 2. *We preserve assumptions* (A_1)-(A_3). *Suppose that the coefficients* a^i_{jk} *and* b^i_j *of* (1) *and the functions* c^i_s *satisfy the growth conditions*

$$
(13)\qquad
\begin{gathered}
|a^i_{jk}| \leqslant M_1(|x|^2+1)^{(2-\lambda)/2}[\ln(|x|^2+1)+1]^{-\mu},\ |b^i_j| \leqslant M_2(|x|^2+1)^{1/2},\\
\sum_{s=1}^{m} c^i_s \leqslant M_3(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^{\mu},\ j, k = 1, \ldots, n;\ i = 1, \ldots, m,
\end{gathered}
$$

almost everywhere in $\bar{S}$ *for some* $M_1, M_2, M_3 > 0$. *The constant* λ *is supposed to be non-negative, while* μ *may be any real number if* $\lambda > 0$, *and* $\mu \geqslant 1$ *if* $\lambda = 0$. *Let* $u_1 = \{u^i_1\}$ *and* $u_2 = \{u^i_2\}$, $i = 1, \ldots, m$, *be two solutions of* (1). *We assume that there exists a constant* $a_0 \geqslant 0$ *such that*

$$
(14)\qquad \iint_S |u^i_l| \exp\{-a_0(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^{\mu}\}\,dx\,dt < \infty
$$

for $i = 1, \ldots, m$ *and* $l = 1, 2$.

If, moreover, $u^i_1(x, 0) = u^i_2(x, 0)$, $i = 1, \ldots, m$, *for* $x \in E^n$, *then* $u^i_1(x, t) = u^i_2(x, t)$, $i = 1, \ldots, m$, *for* $(x, t) \in \bar{S}$.

Proof. At first the proof is carried out for a strip $S_\delta = E^n \times (0, \delta]$, where δ is a sufficiently small constant, and then it is extended step by step in a standard manner. To deduce the theorem from theorem 1 we set

$$
\Phi^i = \exp\{-(a+\beta t)(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^{\mu}\} \qquad (i = 1, \ldots, m).
$$

One can show that the constants $a > a_0$, $\beta = \beta(a) > 0$ and $\delta = \delta(a) > 0$ can be determined in such a way that assumption (2) of Theorem 1 is satisfied in S_δ. Then it can be verified that (14) implies (3). Thus Theorem 2 follows from Theorem 1. We omit the easy (but lengthy) computations. The details do not differ much from those in the proof of Theorem 1 of [1].

Another version of a corollary from Theorem 1 is the following

THEOREM 3. *Let* (A_1)-(A_3) *be satisfied. We assume that*

$$
(15)\qquad
\begin{gathered}
|a^i_{jk}| \leqslant M_1(|x|^2+1)[\ln(|x|^2+1)+1]^{2-\mu},\\
|b^i_j| \leqslant M_2(|x|^2+1)[\ln(|x|^2+1)+1],\\
\sum_{s=1}^{m} c^i_s \leqslant M_3[\ln(|x|^2+1)+1]^{\mu} \qquad (j, k = 1, \ldots, n;\ i = 1, \ldots, m)
\end{gathered}
$$

almost everywhere in $\bar{S}$ *for some* $M_1, M_2, M_3 > 0$ *and* $1 \leqslant \mu \leqslant 2$. *Consider two solutions* $u_1 = \{u_1^i\}$, $u_2 = \{u_2^i\}$ *of system* (1) *such that*

$$\iint_S |u_l^i| \exp\{-\alpha_0[\ln(|x|^2+1)+1]^\mu\} dx\,dt < \infty \tag{16}$$

$$(i = 1, \ldots, m;\ l = 1, 2)$$

for some $\alpha_0 \geqslant 0$.

Under these assumptions, if $u_1^i(x, 0) = u_2^i(x, 0)$ *in* E^n, *then* $u_1^i(x, t) = u_2^i(x, t)$ *in* $\bar{S}$ $(i = 1, \ldots, m)$.

For the proof, we set in Theorem 1

$$\Phi^i = \exp\{-(\alpha+\beta t)[\ln(|x|^2+1)+1]^\mu\} \qquad (i = 1, \ldots, m)$$

and select constants $\alpha > \alpha_0$, $\beta = \beta(\alpha) > 0$ and $\delta = \delta(\alpha) > 0$ so that assumption (2) is satisfied in S_δ. Then we show that (16) implies (3). We omit the computational details.

2. We shall say that a region of E^n is *normal with respect to a coordinate plane* $x_i = 0$ if it can be described by the inequalities

$$\varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \leqslant x_i \leqslant \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

where φ and ψ are continuous functions on a closed domain of the plane $x_i = 0$.

Now let D be an open region of the n-space having the following property:

(P) There exists an increasing sequence $\{D_\nu\}$, $\nu = 1, 2, \ldots$, of open bounded subregions of D such that the closure $\bar{D}_\nu$ of any D_ν can be represented as the union of a finite number of closed regions with disjoint interiors and normal with respect to every plane $x_i = 0$ $(i = 1, \ldots, n)$. Moreover, for any ball $K_R = \{x\colon |x| < R\}$ (of radius R large enough) there is a ν such that $D \cap K_R \subset D_\nu$.

Without loss of generality it may be assumed that the projections of every two members of the decomposition of $\bar{D}_\nu$ on any plane $x_i = 0$ $(1 \leqslant i \leqslant n)$ either coincide or their intersection contains at most some boundary points.

For the sake of simplicity of notations we shall assume hereafter that each of the regións $\bar{D}_\nu$ is normal with respect to every coordinate plane.

In this section we consider the system

$$\sum_{j,k=1}^{n} [a_{jk}^i(x) u^i]_{x_j x_k} - \sum_{j=1}^{n} [b_j^i(x) u^i]_{x_j} + f^i(x, u^1, \ldots, u^m) = 0, \tag{17}$$

$$i = 1, \ldots, m;\ x = (x_1, \ldots, x_n).$$

The following assumptions are introduced:

(A_1') The functions a^i_{jk} $(a^i_{jk} = a^i_{kj})$, b^i_j, $(a^i_{jk})_{x_j}$, $(a^i_{jk})_{x_j x_k}$, $(b^i_j)_{x_j}$ are defined and measurable in $\bar{D}$ and bounded in every bounded subset of $\bar{D}$.

(A_2') The forms

$$\sum_{j,k=1}^{n} a^i_{jk}(x) h_j h_k \qquad (i = 1, \ldots, m)$$

are positive semidefinite in $\bar{D}$.

(A_3') The functions f^i are defined for $x \in \bar{D}$ and arbitrary $u^1, \ldots, u^m$, and

$$[f^i(x, u^1, \ldots, u^m) - f^i(x, \bar{u}^1, \ldots, \bar{u}^m)]\operatorname{sgn}(u^i - \bar{u}^i) \leqslant \sum_{s=1}^{m} c^i_s(x)|u^s - \bar{u}^s| \qquad (i = 1, \ldots, m),$$

where $c^i_s(x) \geqslant 0$ for $s \neq i$, $x \in \bar{D}$, and $c^i_s = c^s_i$. Furthermore, c^i_s are measurable in $\bar{D}$ and bounded in every bounded subset of $\bar{D}$. The system of functions $u(x) = \{u^1(x), \ldots, u^m(x)\}$ will be said to be a *solution of* (17) if $u^i(x)$, $i = 1, \ldots, m$, are continuous in $\bar{D}$ and their derivatives occurring in (17) are measurable in $\bar{D}$, bounded in every bounded subset of $\bar{D}$, and satisfy (17) in D.

A counterpart of Theorem 1 for the elliptic problem is the following

THEOREM 4. *Let D be a domain having property* (P) *and let assumptions* (A_1')-(A_3') *be satisfied. We assume that there exists a vector-valued function $\Phi(x) = \{\Phi^1(x), \ldots, \Phi^m(x)\}$ of class $C^2(\bar{D})$ such that $\Phi^i > 0$ in D and*

$$(18) \qquad L^{*i}\Phi \equiv \sum_{j,k=1}^{n} a^i_{jk}(x)\Phi^i_{x_j x_k} + \sum_{j=1}^{n} b^i_j(x)\Phi^i_{x_j} + \sum_{s=1}^{m} c^i_s(x)\Phi^s < 0 \qquad (i = 1, \ldots, m)$$

almost everywhere in D. Consider solutions $u_1 = \{u^i_1(x)\}$ and $u_2 = \{u^i_2(x)\}$ of (17) *which satisfy the condition*

$$(19) \qquad \int_D |u^i_l| \Big[\max_j \sum_k |a^i_{jk}\Phi^i_{x_k}| + \Phi^i(\max_{j,k}|a^i_{jk}| + \max_j |b^i_j|)\Big] dx < \infty \qquad (i = 1, \ldots, m;\ l = 1, 2).$$

Under these assumptions, if $u^i_1(x) = u^i_2(x)$, $i = 1, \ldots, m$, on the boundary of D, then $u^i_1(x) = u^i_2(x)$, $i = 1, \ldots, m$, in $\bar{D}$.

Proof. Let us assume that $u^i = u^i_1 - u^i_2$, $w^i = [(u^i)^2 + \varepsilon]^{1/2}$, $\varepsilon > 0$, where $u = \{u^1, \ldots, u^m\}$ and $w = \{w^1, \ldots, w^m\}$. We have the relations

$$(20) \quad \sum_{j,k=1}^{n} (a^i_{jk}u^i)_{x_j x_k} - \sum_{j=1}^{n} (b^i_j u^i) + f^i(x, u^1_1, \ldots, u^m_1) - f^i(x, u^1_2, \ldots, u^m_2) = 0$$

and $u^i = 0$ on the boundary ∂D of D $(i = 1, \ldots, m)$.

We multiply the i-th relation of (20) by u^i/w^i $(i = 1, \ldots, m)$. In view of assumptions (A_2') and (A_3') we arrive at

$$(21)\qquad \sum_{j,k=1}^{n} (a_{jk}^i w^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i w^i)_{x_j} - \Big[\sum_{j,k=1}^{n} (a_{jk}^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i)_{x_j}\Big]\frac{\varepsilon}{w^i} + \\ + \frac{|u^i|}{w^i}\sum_{s=1}^{m} c_s^i |u^s| \geqslant 0 \qquad (i = 1, \ldots, m).$$

Hence

$$(22)\qquad L^i w \equiv \sum_{j,k=1}^{n} (a_{jk}^i w^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i w^i)_{x_j} + \sum_{s=1}^{m} c_s^i w^s \geqslant A^i \frac{\varepsilon}{w^i},$$

where

$$A^i = \sum_{j,k=1}^{n} (a_{jk}^i)_{x_j x_k} - \sum_{j=1}^{n} (b_j^i)_{x_j} + c_i^i.$$

Let $v = \{v^1, \ldots, v^m\}$, where $v^i(x)$ are non-negative functions of class $C^2(\bar{D})$, vanishing in the part of D lying outside a sphere K with centre at the origin and radius R.

Now we make use of the identities

$$(23)\qquad v^i\Big(L^i w - \sum_{s=1}^{m} c_s^i w^s\Big) \\ = w^i\Big(L^{*i} v - \sum_{s=1}^{m} c_s^i v^s\Big) + \sum_{j=1}^{m}\Big[v^i \sum_{k=1}^{n} (a_{jk}^i w^i)_{x_k} - w^i \sum_{k=1}^{n} a_{jk}^i v_{x_k}^i - b_j^i v^i w^i\Big]_{x_j}.$$

By summation over i, taking advantage of (22), and by integration over D we obtain

$$(24)\qquad \int_D \sum_{i=1}^{m} A^i v^i \frac{\varepsilon}{w_i}\, dx \leqslant \int_D \sum_{i=1}^{m} w^i L^{*i} v\, dx + \sum_{i=1}^{m}\sum_{j=1}^{n}\int_D B_{x_j}^{ij}(x, \varepsilon)\, dx,$$

where

$$(25)\qquad B^{ij}(x, \varepsilon) = v^i \sum_{k=1}^{n} (a_{jk}^i w^i)_{x_k} - w^i \sum_{k=1}^{n} a_{jk}^i v_{x_k}^i - b_j^i v^i w^i.$$

Consider the sequence $\{D_\nu\}$ occurring in the definition of property (P) of domain D. By property (P), for any fixed R there exists an index ν such that $D \cap K_R \subset D_\nu$. This relation implies the relation $\bar{D} - \bar{D}_\nu \subset \bar{D} - K_R$. Observe that

$$(26)\qquad \lim_{\varepsilon \to 0} B^{ij}(x, \varepsilon) = 0 \quad \text{for } x \in \partial\bar{D}_\nu,$$

because $u^i(x) = 0$ for $x \in \partial\bar{D}$ and $v^i(x) = v_{x_k}^i = 0$ for $x \in \partial\bar{D}_\nu - \partial\bar{D}$.

We have assumed for simplicity that each $\bar{D}_\nu$ is normal with respect to every plane $x_i = 0$ $(i = 1, \ldots, n)$. Thus, denoting by D_ν^j the projection of D_ν on the plane $x_j = 0$, we have, for $x \in D_\nu$,

$$\varphi_\nu^j(x_j') \leqslant x_j \leqslant \psi_\nu^j(x_j') \qquad (j = 1, \ldots, n),$$

where $x_j' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and $\varphi_\nu^j(x'), \psi_\nu^j(x_j')$ are continuous functions in $x_j' \in \bar{D}_\nu^j$.

The iterated integrals formula yields

$$\begin{aligned}(27) \qquad \int_D B_{x_j}^{ij}(x, \varepsilon)\,dx &= \int_{\bar{D}_\nu} B_{x_j}^{ij}(x, \varepsilon)\,dx \\ &= \int_{\bar{D}_\nu^j} B^{ij}(x, \varepsilon)\Big|_{x_j=\psi_\nu^j(x_j')} dx_j' - \int_{\bar{D}_\nu^j} B^{ij}(x, \varepsilon)\Big|_{x_j=\varphi_\nu^j(x_j')} dx_j'.\end{aligned}$$

By (26), (27) and the Lebesgue convergence theorem we obtain

$$(28) \qquad \lim_{\varepsilon\to 0} \int_D B_{x_j}^{ij}(x, \varepsilon)\,dx = 0.$$

By passage to the limit in (24) we obtain the inequality

$$(29) \qquad \int_D \sum_{i=1}^m |u^i| L^{*i} v\,dx \geqslant 0.$$

Let $\gamma^R = \gamma^R(x) \in C^2(E^n)$, $\gamma^R(x) = 1$ for $|x| \leqslant R-1$, $\gamma^R(x) = 0$ for $|x| \geqslant R$, $0 \leqslant \gamma^R(x) \leqslant 1$ in E^n and let the expression

$$\sum_{j=1}^n |\gamma_{x_j}^R| + \sum_{j,k=1}^n |\gamma_{x_j x_k}^R|$$

be bounded by a constant independent of R.

Put $v\,(x) = \gamma^R(x)\Phi^i(x)$. Then all the conditions required previously for $v^i(x)$ are satisfied. Moreover, we have

$$(30) \quad L^{*i} v = \gamma^R L^{*i}\Phi + 2\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j}^R \Phi_{x_k}^i + \Phi^i\Big(\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j x_k}^R + \sum_{j=1}^n b_j^i \gamma_{x_j}^R\Big).$$

By (29) and (30) we obtain

$$\begin{aligned}(31) \quad &\int_D \sum_{i=1}^m |u^i| \gamma^R L^{*i}\Phi\,dx + \\ &+ \int_D \sum_{i=1}^m |u^i| \Big\{2\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j}^R \Phi_{x_k}^i + \Phi^i\Big(\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j x_k}^R + \sum_{j=1}^n b_j^i \gamma_{x_j}^R\Big)\Big\}\,dx \geqslant 0.\end{aligned}$$

The definition of $\gamma^R(x)$ and condition (19) imply

$$(32) \quad \lim_{R\to\infty} \int_D \sum_{i=1}^m |u^i| \Big\{2\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j}^R \Phi_{x_k}^i + \Phi^i\Big(\sum_{j,k=1}^n a_{jk}^i \gamma_{x_j x_k}^R + \sum_{j=1}^n b_j^i \gamma_{x_j}^K\Big)\Big\}\,dx = 0.$$

Let $\bar{x}$ be an arbitrary point of D. We shall show that $u^i(\bar{x}) = 0$ for $i = 1, \ldots, m$. Suppose $u^i(\bar{x}) \neq 0$ for some i. Then by continuity of u^i and by (18) we have

$$\int\limits_{D\cap K_\varrho} \sum_{i=1}^{m} |u^i| L^{*i} \Phi \, dx = -\delta, \tag{33}$$

where $\varrho > |\bar{x}|$ and $\delta > 0$ are some constants.

By (32) we deduce that one can choose R in (31) so large that $R > \varrho + 1$ and

$$\int\limits_{D\cap K_\varrho} \sum_{i=1}^{m} |u^i| L^{*i} \Phi \, dx = \int\limits_{D\cap K_\varrho} \sum_{i=1}^{m} |u^i| \gamma^R L^{*i} \Phi \, dx > -\frac{\delta}{2}.$$

This contradicts (33) and thus the theorem is proved.

Remark. Clearly the theorem remains true if the domain D is replaced by the whole space E^n. In this case the boundary equality of the solutions in question does not occur in assumptions.

Now we state some theorems which can be derived from Theorem 4 as corollaries.

THEOREM 5. *Assume D to have property* (P) *and assumptions* (A_1')-(A_3') *to hold true. Suppose that the inequalities*

$$|a^i_{jk}| \leqslant M_1(|x|^2+1)^{1-\lambda}, \quad |b^i_j| \leqslant M_2(|x|^2+1)^{(1-\lambda)/2}, \quad \sum_{s=1}^{m} c^i_s(x) \leqslant -M_3 \tag{34}$$
$$(i = 1, \ldots, m;\ j, k = 1, \ldots, n)$$

are satisfied almost everywhere in $\bar{D}$, where $M_1, M_2, M_3 > 0$ and $0 < \lambda \leqslant 1$ are constants. Let $u_1 = \{u^i_1\}$ and $u_2 = \{u^i_2\}$ be solutions of (17) *such that*

$$\int\limits_D |u^i_l| \exp\{-a_0(|x|^2+1)^{\lambda/2}\} dx < \infty \quad (i = 1, \ldots, m;\ l = 1, 2), \tag{35}$$

a_0 being a positive constant less than the positive root of the quadratic equation (with respect to a)

$$Q(a) \equiv nM_1\lambda^2 a^2 + n\lambda[(2-\lambda)M_1 + M_2]a - M_3 = 0. \tag{36}$$

If, moreover, $u^i_1(x) = u^i_2(x)$, $i = 1, \ldots, m$, for $x \in \partial D$, then $u^i_1(x) \equiv u^i_2(x)$, $i = 1, \ldots, m$, in $\bar{D}$.

Proof. Let a_1 be the positive root of (36). Select $0 < a_0 < a_1$ and set $a_2 = (a_0 + a_1)/2$. It can readily be shown that the vector-function $\Phi = \{\Phi^1, \ldots, \Phi^m\}$, where

$$\Phi^i = \exp\{-a_2(|x|^2+1)^{\lambda/2}\} \quad (i = 1, \ldots, m),$$

satisfies the inequalities $L^{*i}\Phi \leqslant \Phi^i Q(a_2) < 0$. Thus the assumption (18) of Theorem 4 is fulfilled. Moreover, one can easily check that

$$\max_j \sum_k |a^i_{jk}\Phi^i_{x_k}| + \Phi^i(\max_{j,k}|a^i_{jk}| + \max_j |b^i_j|) \leqslant K\exp\{-a_0(|x|^2+1)^{\lambda/2}\},$$

where K is a constant which depends on M_1, M_2, λ, a_0, a_1 and n. This means that (35) implies (19). Therefore Theorem 5 is a consequence of Theorem 4.

We list below a few more theorems which can be derived from Theorem 4 by specializing the vector-valued function Φ.

THEOREM 6. *We assume that D has property* (P) *and hypotheses* (A_1')-(A_3') *hold true. Let inequalities*

$$|a^i_{jk}| \leqslant M_1(|x|^2+1), \qquad |b^i_j| \leqslant M_2(|x|^2+1)^{1/2}, \qquad \sum_{s=1}^m c^i_s(x) \leqslant -M_3$$

$$(i = 1, \ldots, m;\ j, k = 1, \ldots, n)$$

with $M_1 > 0$, $M_2 > 0$ and $M_3 > 2n(4M_1+M_2)$ be satisfied almost everywhere in $\bar{D}$. Suppose that the solutions $u_1 = \{u_1^i\}$ and $u_2 = \{u_2^i\}$ of (17) *satisfy the condition*

$$\int_D |u^i_l|(|x|^2+1)^{-a_0/2}dx < \infty \qquad (i = 1, \ldots, m;\ l = 1, 2),$$

where $0 < a_0 < a_1-2$, a_1 being the positive root of the quadratic equation

$$Q(a) \equiv nM_1a^2+n(2M_1+M_2)a-M_3 = 0.$$

If the solutions in question coincide on the boundary of D, then they are identical in $\bar{D}$.

For the proof we choose

$$\Phi^i = (|x|^2+1)^{-a_0/2-1} \qquad (i = 1, \ldots, m)$$

and show that $L^{*i}\Phi \leqslant \Phi^i Q(a_0+2) < 0$ almost everywhere in $\bar{D}$. Then we verify that

$$\max_j \sum_k |a^i_{jk}\Phi^i_{x_k}| + \Phi^i(\max_{j,k}|a^i_{jk}| + \max_j |b^i_j|) \leqslant K(|x|^2+1)^{-a_0/2},$$

where $K > 0$ is a constant. Thus Theorem 6 follows from Theorem 4.

THEOREM 7. *Assume that the domain D is the strip*

$$(37) \qquad D = \{x:\ -\infty < x_j < \infty\ (j = 1, \ldots, n-1),\ 0 < x_n < h\}.$$

We retain assumptions (A_1')-(A_3'). *For* $i = 1, \ldots, m$ *and* $j, k = 1, \ldots, n-1$ *let the coefficients of* (17) *satisfy, almost everywhere in* $\bar{D}$, *the inequalities*

$$|a_{jk}^i| \leqslant M_1(|x|^2+1)^{1-\lambda}, \quad |a_{jn}^i| \leqslant M_2(|x|^2+1)^{(1-\lambda)/2}, \quad M_3 \leqslant a_{nn}^i \leqslant M_4(|x|^2+1)^{\mu},$$

$$|b_j^i| \leqslant M_5(|x|^2+1)^{(1-\lambda)/2}, \quad 0 \leqslant b_n^i \leqslant M_6(|x|^2+1)^{\mu} \quad (or\ -M_6(|x|^2+1)^{\mu} \leqslant b_n^i \leqslant 0),$$

$$\sum_{s=1}^{m} c_s^i \leqslant 0,$$

$M_1, \ldots, M_6 > 0$, $0 < \lambda \leqslant 1$, $\mu \geqslant 0$ *being constants.*

Consider two solutions $u_1 = \{u_1^i\}$, $u_2 = \{u_2^i\}$, $i = 1, \ldots, m$, *of* (17) *such that*

$$\int_D |u_l^i| \exp\{-a_0(|x|^2+1)^{\lambda/2}\}\, dx < \infty \qquad (i = 1, \ldots, m;\ l = 1, 2),$$

where $0 < a_0 < a_1$, a_1 *being the positive root of the quadratic equation*

$$\lambda^2 M_1(n-1)(1+h^{2(1-\lambda)})a^2 + \\ +\lambda(n-1)\left[M_1(2-\lambda)(1+h^{2(1-\lambda)}) + \left(\frac{\pi M_2}{2h} + M_5\right)(1+h^{1-\lambda})\right]a - \frac{\pi^2 M_3}{16h^2} = 0.$$

If the solutions are identical on the boundary of D, *i.e. for* $x_n = 0$ *and* $x_n = h$, *then they are identical in the whole strip* $\bar{D}$.

Proof. This theorem follows from Theorem 4 by choosing

$$\Phi^i = \cos\frac{\pi x_n}{4h} \exp\{-a_2(|x'|^2+1)^{\lambda/2}\} \qquad \text{if } 0 \leqslant b_n^i \leqslant M_6(|x|^2+1)^{\mu}$$

and

$$\Phi^i = \sin\frac{\pi(x_n+h)}{4h} \exp\{-a_2(|x'|^2+1)^{\lambda/2}\} \qquad \text{if } -M_6(|x|^2+1)^{\mu} \leqslant b_n^i \leqslant 0$$

$(i = 1, \ldots, m)$, where $a_2 = (a_0+a_1)/2$, $x' = (x_1, \ldots, x_{n-1})$ and

$$|x'| = \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2}.$$

THEOREM 8. *Let* D *be the strip defined by* (37) *and let assumptions* (A_1')-(A_3') *hold true. Suppose the coefficients of* (17) *satisfy, almost everywhere in* $\bar{D}$, *the inequalities*

$$|a_{jk}^i| \leqslant M_1(|x|^2+1), \qquad |a_{jn}^i| \leqslant M_2(|x|^2+1)^{1/2}, \qquad M_3 \leqslant a_{nn}^i \leqslant M_4(|x|^2+1)^{\mu},$$

$$|b_j^i| \leqslant M_5(|x|^2+1)^{1/2}, \quad 0 \leqslant b_n^i \leqslant M_6(|x|^2+1)^{\mu} \quad (or\ -M_6(|x|^2+1)^{\mu} \leqslant b_n^i \leqslant 0),$$

$$\sum_{s=1}^{m} c_s^i \leqslant 0 \qquad (i = 1, \ldots, m;\ j, k = 1, \ldots, n-1),$$

where $M_1, M_2, M_5, M_6 > 0$, $\mu \geqslant 1$, $M_4 \geqslant M_3$ *and*

$$M_3 > \frac{32\mu h^2(n-1)}{\pi^2}\left[2M_1(1+h^2)(\mu+1)+\left(\frac{\pi M_2}{2h}+M_5\right)(1+h)\right].$$

Let $u_1 = \{u_1^i\}$, $u_2 = \{u_2^i\}$ *be solutions of* (17) *such that*

$$\int_D |u_l^i|(|x|^2+1)^{-a_0/2}dx < \infty \qquad (i = 1, \dots, m;\ l = 1, 2),$$

where $0 < a_0 < a_1 - 2\mu$, a_1 *being the positive root of the equation*

$$M_1(n-1)(1+h^2)a^2+ \\ +(n-1)\left[2M_1(1+h^2)+\frac{\pi M_2}{2h}(1+h)+M_5(1+h)\right]a+\frac{\pi^2 M_3}{16h^2} = 0.$$

Under these assumptions, if $u_1^i = u_2^i$ $(i = 1, \dots, m)$ *on the boundary of* D*, then the solutions coincide in* $\bar{D}$.

This theorem is obtained from Theorem 4 by selecting

$$\Phi^i = (|x'|^2+1)^{-a_0/2-\mu}\cos\frac{\pi x_n}{4h}$$

in the case $0 \leqslant b_n^i \leqslant M_6(|x|^2+1)^{\mu}$ and

$$\Phi^i = (|x'|^2+1)^{-a_0/2-\mu}\sin\frac{\pi(x_n+h)}{4h}$$

in the case $-M_6(|x|^2+1)^{\mu} \leqslant b_n^i \leqslant 0$.

THEOREM 9. *Consider system* (17) *in the half-space*

$$D\{-\infty < x_j < \infty\ (j = 1, \dots, n-1),\ x_n > 0\} \tag{38}$$

and retain assumptions (A_1')-(A_3'). *For* $i = 1, \dots, m$ *and* $j, k = 1, \dots, n-1$ *let the coefficients of* (17) *satisfy almost everywhere in* $\bar{D}$ *the conditions*

$$|a_{jk}^i| \leqslant M_1(|x'|^2+1)^{1-\lambda}, \quad |a_{jn}^i| \leqslant M_2(|x'|^2+1)^{(1-\lambda)/2}, \quad a_{nn}^i \leqslant M_3,$$

$$|b_j^i| \leqslant M_4(|x'|^2+1)^{(1-\lambda)/2}, \quad M_5 \leqslant b_n^i \leqslant M_6(|x'|^2+1)^{\mu}, \quad \sum_{s=1}^{m} c_s^i \leqslant M_7,$$

where $0 < \lambda \leqslant 1$, $\mu \geqslant 0$, $M_1, \dots, M_6 > 0$, $M_6 \geqslant M_5$ *and* $4M_3M_7 < M_5^2$. *Let* $u_1 = \{u_1^i\}$, $u_2 = \{u_2^i\}$ *be solutions of* (17) *such that*

$$\int_D |u_l^i|\exp\{-a_0(|x|^2+1)^{\lambda/2}\}dx < \infty \qquad (i = 1, \dots, m;\ l = 1, 2),$$

where $0 < a_0 < a_1$, a_1 *being the positive root of the equation*

$$[\lambda(n-1)(M_1\lambda+2M_2)+M_3]a^2+$$
$$\lambda(n-1)\left[M_1(2-\lambda)+\frac{M_2M_5}{M_3}+M_4\right]a+\frac{M_5^2-4M_3M_7}{4M_3}=0.$$

Under these assumptions, if $u_1^i = u_2^i$ $(i = 1, \ldots, m)$ *on the boundary of* D, *i.e. for* $x_n = 0$, *then* $u_1^i \equiv u_2^i$ $(i = 1, \ldots, m)$ *in* $\bar{D}$.

Proof. Theorem 9 is deduced from Theorem 4 by choosing the components of the vector-valued function Φ as

$$\Phi^i = e^{-(a_2+a_3)x_n}e^{-a_2(|x'|^2+1)^{\lambda/2}} \qquad (i = 1, \ldots, m)$$

with $a_2 = (a_0+a_1)/2$ and $a_3 = M_5/2M_3$.

THEOREM 10. *Let* D *be the half-space defined by* (38) *and assume* (A_1')-(A_3'). *Suppose that the coefficients of* (17) *and functions* c_s^i *satisfy the boundedness conditions*

$$|a_{jk}^i| \leqslant M_1(|x'|^2+1), \qquad |a_{jn}^i| \leqslant M_2(|x'|^2+1)^{1/2}, \qquad a_{nn}^i \leqslant M_3,$$
$$|b_j^i| \leqslant M_4(|x'|^2+1), \qquad M_5 \leqslant b_n^i \leqslant M_6(|x'|^2+1)^{\mu}, \qquad \sum_{s=1}^{m} c_s^i \leqslant M_7,$$

where $\mu \geqslant 1$, $M_1, \ldots, M_6 > 0$, $M_6 \geqslant M_5$ *and*

$$M_7 < -4[(n-1)(M_1+2M_2)+M_3]\mu^2-2[(n-1)(2M_1+M_4)-M_5]\mu.$$

Let $u_1 = \{u_1^i\}$, $u_2 = \{u_2^i\}$ *be solutions of* (17) *such that*

$$\int_D |u_l^i|(|x^2|+1)^{-a_0/2}dx < \infty \qquad (i = 1, \ldots, m;\ l = 1, 2),$$

where $0 < a_0 < a_1-2\mu$, a_1 $(> 2\mu)$ *being the greater positive root of the equation*

$$[(n-1)(M_1+2M_2)+M_3]a^2+[(n-1)(2M_1+M_4)-M_5]a+M_7 = 0.$$

The above assumptions imply that if $u_1^i = u_2^i$ $(i = 1, \ldots, m)$ *for* $x_n = 0$, *then* $u_1^i \equiv u_2^i$ *in* $\bar{D}$.

The proof consists in applying Theorem 4 with functions

$$\Phi^i = e^{-a_2x_n}(|x'|+1)^{-a_2/2} \qquad (i = 1, \ldots, m),$$

where $a_2 = a_0+2\mu$.

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Reçu par la Rédaction le 22. 8. 1967