COLLOQUIUM MATHEMATICUM

VOL. XXVIII 1973 FASC. 2

THE SEMIRING OF QUOTIENTS OF COMMUTATIVE SEMIRINGS

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1. Introduction. In [2] Lambek constructs the complete ring of quotients of a commutative ring. The purpose of this note is to indicate how one can carry this construction to the case of semirings. The embedding of certain semirings into semirings with identities was done by Grillet [1]. Our method is more general than given by Mosher [5] for the generalized quotients of semirings.

The concept of a semiring is in the literature for quite long (cf. Weinert [4]). To recall, a *semiring* R is a set together with two binary operations $+$, $\cdot$, respectively, such that

(i) R^{+} and $R^{\cdot}$ are semigroups.

(ii) The addition and multiplication are connected by the distributive laws:

$$a\cdot(b+c) = a\cdot b+a\cdot c, \quad (b+c)\cdot a = b\cdot a+c\cdot a.$$

We call R a *commutative semiring* if both the operations are commutative.

Definition 1.1. A *two-sided ideal* I of R is an additive subsemigroup of R^{+} such that for $w\in R$ and $i\in I$, $wi\in R$ and $iw\in R$. The concepts of left and right ideals are as usual.

Definition 2.1. By a *right R-semimodule* M, over a semiring R, we mean an additive abelian semigroup together with a multiplication $M\times R\to M$, $(m, r)\to mr$, such that

$$m(r+r') = mr+mr', \quad (m+m')r = mr+m'r, \quad (mr)r' = m(rr').$$

Definition 1.3. By a *right R-homomorphism* of right semimodules H and H' we mean a function $f: H\to H'$ such that

$$f(h+h') = fh+fh', \quad f(hr) = f(h)r.$$

PROPOSITION 1.4. *If D is any right ideal of a commutative semiring R, then the set of all right R-homomorphisms* $\mathrm{Hom}_R(D, R)$ *is a right R-semimodule.*

2. Total semiring of quotients. We assume that our semiring R is commutative and check how easily Lambek's construction goes over to our case.

Definition 2.1. An ideal J of R is called *element separating* if for any two distinct elements $r_1, r_2 \in R$ there exists a $j \in J$ such that $jr_1 \neq jr_2$. If R is a semiring with 1, then the unit ideal R is obviously element separating.

Henceforth we shall assume that R has at least one element separating ideal.

PROPOSITION 2.2. (a) *If D is element separating and $D \subset D'$, then D' is element separating.*

(b) *If D and D' are element separating, so is DD' and $D \cap D'$.*

Proof. Since (a) is trivial, we prove (b). Let $r_1 \neq r_2$ be two elements of R. Since D' is element separating, there exists a d' such that $d'r_1 \neq d'r_2$. Again there exists a $d \in D$ such that $d(d'r_1) \neq d(d'r_2)$, i.e., $(dd')r_1 \neq (dd')r_2$, where $dd' \in DD'$; thus DD' is element separating. Since $DD' \subseteq D \cap D'$, part (a) gives the result.

Definition 2.3. By a *fraction* we mean an element $f \in \mathrm{Hom}_R(D, R)$, where D is an element separating ideal of R.

Addition and multiplication of fractions $f_i : D_i \rightarrow R$ $(i = 1, 2)$ are defined by

$$f_1 + f_2 \in \mathrm{Hom}_R(D_1 \cap D_2, R), \qquad (f_1 + f_2)(d) = f_1(d) + f_2(d),$$

$$f_1 f_2 \in \mathrm{Hom}_R(f_2^{-1} D_1, R), \qquad f_1 f_2(d) = f_1[f_2(d)].$$

Here $f_2^{-1} D_1 = \{r \in R \mid f_2 r \in D_1\}$ is element separating, since it clearly contains $D_2 D_1$.

Clearly, then the fractions form an additive abelian semigroup $(F, +)$ and an abelian semigroup $(F, \cdot)$. To check that it is a semiring we must have the distributive laws like $f_1(f_2 + f_3) = f_1 f_2 + f_1 f_3$.

We note that $f_1(f_2 + f_3) \in \mathrm{Hom}_R[(f_2 + f_3)^{-1} D_1, R]$ whereas $f_1 f_2 + f_1 f_3 \in \mathrm{Hom}_R[f_2^{-1} D_1 \cap f_3^{-1} D_1, R]$.

We shall write $f_1 \theta f_2$ to mean that f_1 and f_2 agree on the intersection of their domains, i.e., $f_1(d) = f_2(d)$ for every $d \in D_1 \cap D_2$.

LEMMA 2.4. *$f_1 \theta f_2 \Leftrightarrow f_1$ and f_2 agree on some separating ideal.*

Proof. $(\Rightarrow)$ If $f_1 \theta f_2$, then by the definition f_1, f_2 agree on $D_1 \cap D_2$.

$(\Leftarrow)$ Suppose that f_1, f_2 agree on some separating ideal D'. We shall check that $f_1(d) = f_2(d)$ for every $d \in D_1 \cap D_2$.

Suppose $f_1(d) \neq f_2(d)$ for some d. Then there exists a $d' \in D'$ such that $f_1(dd') \neq f_2(dd')$, i.e., $f_1(d^*) \neq f_2(d^*)$, where $d^* = dd' \in D'$, i.e., f_1 and f_2 do not agree on D', a contradiction to our hypothesis.

As shown by Lambek [2]

LEMMA 2.5. *θ is a congruence on the system $(F, +, \cdot)$.*

THEOREM 2.6. *If R is a commutative semiring, having a non-empty set of separating ideals, then the system $(F, +, \cdot)/\theta = Q(R)$ is also a commutative semiring. Further, if R has an identity, so has $Q(R)$.*

We call $Q(R)$ the *total semiring of quotients of the commutative semiring R.*

Proof. Since θ is a congruence relation, F/θ satisfy all the identities that F satisfy. We need only to check the distributive laws, which is now obvious since for example $f_1(f_2+f_3)$ and $f_1f_2+f_1f_3$ both agree on the separating ideal $D_1D_2D_3$, whence $[f_1(f_2+f_3)]\theta[f_1f_2+f_1f_3]$ is in F. If R has an identity, we can take $1\in \mathrm{Hom}_R(R, R)$ as the fraction on R and $(F, \cdot, 1)$ becomes an abelian monoid; hence so does F/θ.

Finally, when the unit ideal R is itself separating, we can associate a fraction $\hat{r} : R\to R$ by defining $\hat{r}(s) = sr$.

The mapping $r\to\theta(\hat{r})$ is easily seen to be a homomorphism of semirings (i.e., to preserve sums and products). It is indeed a monomorphism for $\theta\hat{r} = \theta\hat{r}' \Rightarrow \hat{r}\theta\hat{r}'$, that is, $\hat{r}$ and $\hat{r}'$ coincides on R, i.e., $\hat{r}R = \hat{r}'R$ so $\hat{r} = \hat{r}'$.

We call the mapping $r\to\theta\hat{r}$ the *canonical monomorphism.*

PROPOSITION 2.7. *An element d of a semiring having at least one separating ideal is multiplicatively cancellable if and only if the principal ideal dR is separating in R.*

Proof. Suppose k is cancellable. If for two distinct elements t_1 and t_2 we have $t_1kl = t_2kl$ for each $l\in R$, then $t_1k = t_2k$. Since R itself is separating, $t_1 = t_2$ as k is cancellable.

Conversely, if dR is separating and $tk = t'k$, then, for each $r\in R$, $tkr = t'kr$, i.e., $t = t'$.

Thus, for $r\in R$ and for d cancellable, we have a *classical fraction* $r|d\in \mathrm{Hom}_R(dR, R)$, defined by $r|d(ds) = rs$ for any $s\in R$.

THEOREM 2.8. *The equivalence classes $\theta(r|d)$ for $r\in R$ and $d\in R$, cancellable, form a subsemiring of $Q(R)$, which is called the classical semiring of quotients of R and is denoted by $Q_{cl}(R)$.*

If $\theta(r_1|d_1) = \theta(r_2|d_2)$, i.e., $r_1|d_1 \equiv r_2|d_2$, for d_1 and d_2 cancellable, then, for each $d_1rd_2r'\in d_1R\cdot d_2R \subseteq d_1R\cap d_2R$, there is

$$r_1|d_1d_1rd_2r' = r_1rd_2r' = r_2d_1rr' = r_2|d_2d_1r\cdot d_2r',$$

i.e., $r_1d_1 = r_2d_2$.

The "only if" part of the above assertion also holds, i.e., if $r_1d_1 = r_2d_2$, then $\theta(r_1|d_1) = \theta(r_2|d_2)$.

The latter part of Lambek's construction on rational completion can be carried over and will be left for a subsequent paper.

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Reçu par la Rédaction le 17. 3. 1972