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# On a class of vector-valued analytic functions*

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**Abstract.** Given a complex Banach space $X$ and an open connected set $\mathscr{D}$ in the complex plane, denote by $\mathscr{A}(\mathscr{D}, X)$ the class of those analytic functions $f\colon \mathscr{D} \to X$ for which there exists a complex-valued function $\varphi$ defined and analytic on $\mathscr{D}$ and such that $\|f(\zeta)\| = |\varphi(\zeta)|$ $(\zeta \in \mathscr{D})$. In the paper a characterization of $\mathscr{A}(\mathscr{D}, X)$ is given. $\mathscr{D}$ being a neighbourhood of infinity, it is proved that every function in $\mathscr{A}(\mathscr{D}, X)$ is a complex-valued analytic function multiplied by a polynomial. For $f$ as above let us say that property $p$ holds if $f$ is entire and if $\|f(\zeta)\| = 1$ $(\operatorname{Re}\zeta < 0)$, $\|f(\zeta)\| = |\varphi(\zeta)|$ $(\operatorname{Re}\zeta > 0)$, where $\varphi$ is a complex-valued analytic function. This being the case, it is shown that $\varphi(\zeta) \equiv e^{i\beta}e^{\gamma\zeta}$ ($\beta$, $\gamma$ real constants). Next, $A$ being a bounded normal operator on a complex Hilbert space and $\psi$ a complex-valued entire analytic function, the necessary and sufficient conditions are obtained for $\zeta \to \psi(\zeta A)$ to belong to $\mathscr{A}(\mathscr{D}, X)$. Finally, $\psi$ being as above with $\psi'(0) \neq 0$ and $B$ being a bounded linear operator on a complex Hilbert space, let $\zeta \to \psi(\zeta B)$ have the property $p$. Then $\psi(\zeta) \equiv e^{i\beta}e^{\gamma\zeta}$ with $\beta$ real and $B = e^{i\alpha}P$, where $P$ is a positive operator and $\alpha$ is a real constant.

**0. Introduction.** In this paper we give a characterization of those analytic functions with values in a complex Banach space $X$ which have the norm equal to the absolute value of a complex-valued analytic function. Using this characterization, we investigate some special classes of such functions.

By the result of E. Thorp and R. Whitley [4] it is easy to see that every analytic function with the above property is trivial (i.e. equal to a fixed vector multiplied by a complex-valued analytic function) if and only if every point of the unit sphere of $X$ is a complex extreme point. On the other hand, for a large class of complex Banach spaces (which contains complex Banach spaces with sup norm of dimension greater than 1, Example 2.2, and algebras of linear operators over complex Hilbert spaces of dimension greater than 1, Example 4.2) there exist non-trivial analytic functions with the above property.

0.0. DEFINITION. Let $X$ be a complex Banach space and let $\mathscr{D}$ be a domain in the complex plane. We say that an analytic function $f\colon \mathscr{D} \to X$

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belongs to the class $\mathscr{A}(\mathscr{D}, X)$ if there exists a complex-valued function $\varphi$ defined and analytic on $\mathscr{D}$ and such that $\|f(\zeta)\| = |\varphi(\zeta)|$ $(\zeta \in \mathscr{D})$.

The paper has four sections. In Section 1 we give a characterization of the functions of class $\mathscr{A}(\mathscr{D}, X)$. Next we study two special cases of the functions of class $\mathscr{A}(\mathscr{D}, X)$: in Section 2 $\mathscr{D}$ is a neighbourhood of infinity and in Section 3 $\mathscr{D}$ is a half-plane. Section 4 deals with the functions of class $\mathscr{A}(\mathscr{D}, X)$, where $X$ is the algebra of bounded linear operators over a complex Hilbert space.

Throughout the paper we denote by $C$ the complex plane. An open connected subset of $C$ is called a *domain*. If $X$ is a complex Banach space, we denote by $S(X) = \{x \in X: \|x\| = 1\}$ the unit sphere of $X$, by $X'$ the dual space of $X$ and by $L(X)$ the algebra of all bounded linear operators with domain $X$ and with range in $X$; if $A \in L(X)$ we denote by $\varrho(A)$, $\sigma(A)$ and $R(\lambda, A)$ the resolvent set, the spectrum and the resolvent of $A$, respectively. By $\langle x, u\rangle$ we denote the image of $x \in X$ under $u \in X'$. If $\mathscr{B}$ is a commutative complex Banach algebra we denote by $\mathfrak{M}(\mathscr{B})$ the set of all multiplicative linear functionals over $\mathscr{B}$.

## 1. A characterization of the class $\mathscr{A}(\mathscr{D}, X)$.

1.0. THEOREM. *Let $X$ be a complex Banach space and $\mathscr{D}$ a domain in the complex plane and let $f: \mathscr{D} \to X$ be an analytic function. Then $f$ belongs to the class $\mathscr{A}(\mathscr{D}, X)$ if and only if an $u \in S(X')$ exists such that $|\langle f(\zeta), u\rangle| = \|f(\zeta)\|$ $(\zeta \in \mathscr{D})$. If the latter condition is satisfied, then every complex-valued function $\varphi$, defined and analytic on $\mathscr{D}$ and satisfying*

$$\|f(\zeta)\| = |\varphi(\zeta)| \quad (\zeta \in \mathscr{D}) \tag{1.0}$$

*has the form $\varphi(\zeta) = e^{ia}\langle f(\zeta), u\rangle$ $(\zeta \in \mathscr{D})$, where $a$ is a real constant.*

Proof. Let an $u \in S(X')$ exist such that $|\langle f(\zeta), u\rangle| = \|f(\zeta)\|$ $(\zeta \in \mathscr{D})$. By the analyticity of $f$ it follows that $\zeta \mapsto \varphi(\zeta) = \langle f(\zeta), u\rangle$ is a complex-valued function, defined and analytic on $\mathscr{D}$, which satisfies (1.0).

To prove the converse, let a complex-valued function $\varphi$, defined and analytic on $\mathscr{D}$, satisfy (1.0). If $\varphi = 0$, then we have nothing to prove, so assume that $\varphi \neq 0$. Then there exists a subdomain $\mathscr{D}_1 \subset \mathscr{D}$ such that $\varphi(\zeta) \neq 0$ $(\zeta \in \mathscr{D}_1)$. It follows that the function $\zeta \mapsto g(\zeta) = [1/\varphi(\zeta)]f(\zeta)$ is analytic on $\mathscr{D}_1$, and by (1.0) we have $\|g(\zeta)\| = 1$ $(\zeta \in \mathscr{D}_1)$. Now, choose $\zeta_0 \in \mathscr{D}_1$. By the Hahn–Banach theorem an $u \in S(X')$ exists such that $1 = \|g(\zeta_0)\| = \langle g(\zeta_0), u\rangle$. Since $|\langle g(\zeta), u\rangle| \leqslant \|g(\zeta)\| \cdot \|u\| = 1$ $(\zeta \in \mathscr{D}_1)$, it follows by the classical maximum modulus theorem (cf. [3]) that $\langle g(\zeta), u\rangle = 1$ $(\zeta \in \mathscr{D}_1)$. This means that $\varphi(\zeta) = \langle f(\zeta), u\rangle$ $(\zeta \in \mathscr{D}_1)$. Since the functions $f$ and $\varphi$ are both analytic on $\mathscr{D}$, we have $\varphi(\zeta) = \langle f(\zeta), u\rangle$ $(\zeta \in \mathscr{D})$. By (1.0) it follows that $|\langle f(\zeta), u\rangle| = \|f(\zeta)\|$ $(\zeta \in \mathscr{D})$.

The last statement of the theorem is trivial since any two complex-valued functions $\varphi_1, \varphi_2$, defined and analytic on $\mathscr{D}$ and satisfying $|\varphi_1(\zeta)| = |\varphi_2(\zeta)|$ $(\zeta \in \mathscr{D})$, can differ only in a factor $e^{i\alpha}$, $\alpha$ real. Q. E. D.

1.1. COROLLARY. *Let $X$ be a complex Banach space and $\mathscr{D}$ a domain in the complex plane and $f\colon \mathscr{D} \to X$ an analytic function. Let $\mathscr{D}_1$ be a subdomain of $\mathscr{D}$. If $\|f(\zeta)\| = |\varphi(\zeta)|$ $(\zeta \in \mathscr{D}_1)$, where $\varphi$ is a complex-valued function, defined and analytic on $\mathscr{D}_1$, then the function $\varphi$ can be continued analytically to all $\mathscr{D}$.*

Proof. By Theorem 1.0 an $u \in S(X')$ exists such that $\varphi(\zeta) = \langle f(\zeta), u\rangle$ $(\zeta \in \mathscr{D}_1)$, which proves the assertion. Q. E. D.

**2. The entire functions of class $\mathscr{A}(\mathscr{D}, X)$, where $\mathscr{D}$ is a neighbourhood of infinity.**

2.0. THEOREM. *Let $X$ be a complex Banach space and let $f\colon C \to X$ be an entire analytic function. Let there exist a constant $R < \infty$ such that*

$$\|f(\zeta)\| = |\varphi(\zeta)| \quad (|\zeta| > R), \tag{2.0}$$

*where $\varphi$ is a complex-valued function, defined and analytic for $|\zeta| > R$.*

*Then there exist an entire complex-valued analytic function $\psi$ and vectors $a_0, a_1, \ldots, a_{n-1} \in X$ such that*

$$f(\zeta) = \psi(\zeta)(a_0 + a_1\zeta + a_2\zeta^2 + \ldots + a_{n-1}\zeta^{n-1}) \quad (\zeta \in C). \tag{2.1}$$

*Further, $\varphi$ is an entire function and the degree $n-1$ is equal to the number of zeros of $\varphi$* (counted with multiplicity) *on the disc $|\zeta| \leqslant R$. If $f$ has $m$ zeros on the disc $|\zeta| \leqslant R$, then among the vectors $a_0, a_1, \ldots, a_{n-1}$ at most $n-m$ are linearly independent.*

Proof. By Corollary 1.1 $\varphi$ is an entire function. If $\varphi = 0$, then we have nothing to prove, so assume that $\varphi \neq 0$. Then $\varphi$ has a finite number of zeros $\zeta_0, \zeta_1, \ldots, \zeta_{n-2}$ on the disc $|\zeta| \leqslant R$. The function $\zeta \mapsto [1/\varphi(\zeta)]f(\zeta)$ is a meromorphic function, analytic for $|\zeta| > R$ since by (2.0) $\|[1/\varphi(\zeta)]f(\zeta)\| = 1$ $(|\zeta| > R)$. Its poles on the disc $|\zeta| \leqslant R$ coincide with the zeros $\zeta_0, \zeta_1, \ldots, \zeta_{n-2}$ of $\varphi$ so that $\zeta \mapsto h(\zeta) = (\zeta-\zeta_0)(\zeta-\zeta_1)\ldots(\zeta-\zeta_{n-2})[1/\varphi(\zeta)]f(\zeta)$ is an entire function. Since $\|h(\zeta)\| = |(\zeta-\zeta_0)(\zeta-\zeta_1)\ldots(\zeta-\zeta_{n-2})|$ $(|\zeta| > R)$, it follows by the generalized theorem of Liouville that for every $u \in S(X')$, $\langle h(\zeta), u\rangle$ is a polynomial of degree less than $n$. Further, by Theorem 1.0 there exists an $u_0 \in S(X')$ such that $|(\zeta-\zeta_0)(\zeta-\zeta_1)\ldots(\zeta-\zeta_{n-2}) = |\langle h(\zeta), u_0\rangle|$ $(|\zeta| > R)$, which means that $\langle h(\zeta), u_0\rangle$ is a polynomial of degree $n-1$. This implies that $h(\zeta)$ is a polynomial of degree $n-1$:

$$h(\zeta) = a_0 + a_1\zeta + \ldots + a_{n-1}\zeta^{n-1},$$

where $a_0, a_1, \ldots, a_{n-1} \in X$, $a_{n-1} \neq 0$. So we have $f(\zeta) = \psi(\zeta)h(\zeta)$ $(\zeta \in C)$, where $\zeta \mapsto \psi(\zeta) = \varphi(\zeta)/[(\zeta-\zeta_0)\cdot(\zeta-\zeta_1)\ldots(\zeta-\zeta_{n-2})]$ is an entire complex-valued analytic function which has no zeros on the disc $|\zeta| \leqslant R$. Further, it is evident that $\varphi$ has $n-1$ zeros on the disc $|\zeta| \leqslant R$.

To prove the last statement of the theorem, assume that $f$ has $m$ zeros on the disc $|\zeta| \leqslant R$. Since $\psi(\zeta) \neq 0$ ($|\zeta| \leqslant R$) it follows that the polynomial $a_0 + a_1\zeta + \ldots + a_{n-1}\zeta^{n-1}$ has at least $m$ zeros. Now it is easy to see that among its coefficients at most $n-m$ are linearly independent. Q. E. D.

2.1. COROLLARY. *Let $X$ be a complex Banach space and let $f\colon C \to X$ be an entire analytic function. Let there exist a constant $R < \infty$ such that*

$$\|f(\zeta)\| = |\varphi(\zeta)| \quad (|\zeta| > R),$$

*where $\varphi$ is a complex-valued function, defined and analytic for $|\zeta| > R$. If the (entire) function $\varphi$ has no zeros on the disc $|\zeta| \leqslant R$, then there exists an $x \in X$ such that $f(\zeta) = \varphi(\zeta)x$ ($\zeta \in C$).*

2.2. EXAMPLE. The converse of Theorem 2.0 does not hold in general. To see this, we construct a polynomial $f$ which does not belong to $\mathscr{A}(\mathscr{D}, X)$ for any neighbourhood $\mathscr{D}$ of infinity. Let $X$ be the complex Banach space of complex number pairs $z = \{z_1, z_2\}$, where $\|z\| = \max\{|z_1|, |z_2|\}$. Let $f(\zeta) = \{\zeta+1, \zeta-1\} = \zeta\{1, 1\} + \{1, -1\}$. Then

$$\|f(\zeta)\| = \begin{cases} |\zeta+1| & (\mathrm{Re}\,\zeta > 0), \\ |\zeta-1| & (\mathrm{Re}\,\zeta < 0), \end{cases}$$

and it is clear that there exists no analytic function $\varphi$ such that $\|f(\zeta)\| \equiv |\varphi(\zeta)|$ in a neighbourhood of infinity. If we write $\mathscr{D}_1 = \{\zeta\colon \mathrm{Re}\,\zeta > 0\}$, the same example shows that $\mathscr{A}(\mathscr{D}_1, X)$ contains non-trivial functions.

**3. Some entire functions of class $\mathscr{A}(\mathscr{D}, X)$, where $\mathscr{D}$ is a half-plane.** We shall need two lemmas. The first one is an easy consequence of a theorem in [1], p. 220.

3.0. LEMMA. *Let $\varphi$ be an entire complex-valued analytic function satisfying*

$$\mathrm{Re}\,\varphi(\zeta) \geqslant 0 \quad (\mathrm{Re}\,\zeta > 0),$$

$$\mathrm{Re}\,\varphi(\zeta) = 0 \quad (\mathrm{Re}\,\zeta = 0).$$

*Then*

$$\varphi(\zeta) = i\beta + \gamma\zeta \quad (\zeta \in C),$$

*where $\beta$ and $\gamma$ are real constants with $\gamma \geqslant 0$.*

3.1. LEMMA. *Let $X$ be a complex Banach space and let $f\colon C \to X$ be an entire analytic function. Let there exist a domain $\mathscr{D}$ such that $\|f(\zeta)\| = 1$ ($\zeta \in \mathscr{D}$). Then $\|f(\zeta)\| \geqslant 1$ ($\zeta \in C$).*

Proof. By Theorem 1.0 there exists a $u \in S(X')$ such that $\langle f(\zeta), u\rangle = 1$ ($\zeta \in \mathscr{D}$). By the analytic continuation principle it follows that $\langle f(\zeta), u\rangle = 1$ ($\zeta \in C$), which proves the assertion. Q. E. D.

3.2. THEOREM. *Let $f$ be an entire analytic function with values in a complex Banach space. Let*

$$\|f(\zeta)\| = 1 \quad (\mathrm{Re}\,\zeta < 0) \tag{3.0}$$

*and let there exist a complex-valued function $\varphi$ defined and analytic for* $\mathrm{Re}\,\zeta > 0$ *and such that*

$$\|f(\zeta)\| = |\varphi(\zeta)| \quad (\mathrm{Re}\,\zeta > 0). \tag{3.1}$$

*Then*

$$\varphi(\zeta) = e^{i\beta}e^{\gamma\zeta} \quad (\mathrm{Re}\,\zeta > 0),$$

*where $\beta$ and $\gamma$ are real constants with $\gamma \geqslant 0$.*

Proof. By Lemma 3.1, (3.0) implies

$$\|f(\zeta)\| \geqslant 1 \quad (\mathrm{Re}\,\zeta > 0). \tag{3.2}$$

By Corollary 1.1 $\varphi$ is an entire function and by (3.1) and (3.2) we have $|\varphi(\zeta)| \geqslant 1$ $(\mathrm{Re}\,\zeta > 0)$ and $|\varphi(\zeta)| = 1$ $(\mathrm{Re}\,\zeta = 0)$. So $\varphi$ has no zeros for $\mathrm{Re}\,\zeta \geqslant 0$. Consequently, the function $\zeta \mapsto \psi(\zeta) = \log\varphi(\zeta)$ is analytic for $\mathrm{Re}\,\zeta \geqslant 0$ and satisfies

$$\begin{aligned} &\mathrm{Re}\,\psi(\zeta) \geqslant 0 \quad (\mathrm{Re}\,\zeta > 0),\\ &\mathrm{Re}\,\psi(\zeta) = 0 \quad (\mathrm{Re}\,\zeta = 0). \end{aligned} \tag{3.3}$$

By the Riemann–Schwarz symmetry principle $\psi$ is an entire function. So by (3.3) Lemma 3.0 implies that $\psi(\zeta) = i\beta + \gamma\zeta$ $(\zeta \in C)$, where $\beta$ and $\gamma$ are real constants, $\gamma \geqslant 0$. Now the assertion follows immediately. Q. E. D

**4. Some examples of the operator-valued functions of class $\mathscr{A}(\mathscr{D}, X)$.** Let $f$ be an entire, complex-valued analytic function. So

$$f(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \ldots, \tag{4.0}$$

where the series converges absolutely in the entire complex plane. If $X$ is a complex Hilbert space and $A \in L(X)$, then also the series

$$f(\zeta A) = c_0 I + c_1(\zeta A) + c_2(\zeta A)^2 + \ldots \tag{4.1}$$

(here $I$ is the identity operator) converges absolutely in the entire complex plane and so it defines an entire operator-valued analytic function. In the special case where the operator $A$ is normal, the following theorem asserts that the function $\zeta \mapsto f(\zeta A)$ belongs to the class $\mathscr{A}(\mathscr{D}, L(X))$.

4.0. THEOREM. *Let $X$ be a complex Hilbert space, $A \in L(X)$ a normal operator and $f$ a non-constant, complex-valued entire analytic function. Let $\mathscr{D}$ be a domain in the complex plane.*

*Then a complex-valued function $g$, defined and analytic on $\mathscr{D}$ and satisfying $\|f(\zeta A)\| = |g(\zeta)|$ $(\zeta \in \mathscr{D})$, exists if and only if the spectrum $\sigma(A)$ contains* *a point* $\eta_0$ *such that*

$$|f(\zeta\eta)| \leqslant |f(\zeta\eta_0)| \qquad (\zeta \in \mathscr{D};\ \eta \in \sigma(A)).$$

*Let either condition be satisfied. Then*

$$|g(\zeta)| = |f(\zeta\eta_0)| \qquad (\zeta \in \mathscr{D}).$$

*Further,* $\eta_0$ *lies in the boundary of* $\sigma(A)$ *and* $\eta_0 = 0$ *if and only if* $g$ *is a constant.*

Proof. Since $A$ is normal, it follows by (4.1) that $f(\zeta A)$ is normal for every $\zeta$.

Let $\eta_0 \in \sigma(A)$ exist such that

$$|f(\zeta\eta)| \leqslant |f(\zeta\eta_0)| \qquad (\zeta \in \mathscr{D};\ \eta \in \sigma(A)).$$

Define $g(\zeta) = f(\zeta\eta_0)$ $(\zeta \in \mathscr{D})$. Since $f$ is an entire function, the same holds for $g$. Further, the operators $f(\zeta A)$ being normal, it follows by the spectral mapping theorem (cf. [2]) that

$$\|f(\zeta A)\| = \operatorname{spr} f(\zeta A) = \sup_{\eta \in \sigma(A)} |f(\zeta\eta)| = |f(\zeta\eta_0)| = |g(\zeta)| \qquad (\zeta \in \mathscr{D})$$

(here spr means spectral radius).

To prove the converse, let

$$\|f(\zeta A)\| = |g(\zeta)| \qquad (\zeta \in \mathscr{D}), \tag{4.2}$$

where $g$ is a complex-valued function, defined and analytic on $\mathscr{D}$. Let $\mathscr{B} \subset L(X)$ be the second commutant of the set $\{R(\lambda, A):\ \lambda \in \varrho(A)\}$. It is known (cf. [2]) that $\mathscr{B}$ is a commutative complex Banach algebra with identity, which contains $A$, $f(\zeta A)$ $(\zeta \in C)$ and which has the property that the spectrum of an element of $\mathscr{B}$ with respect to $\mathscr{B}$ is equal to its spectrum with respect to $L(X)$. By Theorem 1.0, (4.2) holds if and only if a $u_0 \in S(\mathscr{B}')$ exists such that

$$|\langle f(\zeta A), u_0\rangle| = \|f(\zeta A)\| \qquad (\zeta \in \mathscr{D}). \tag{4.3}$$

A glance into the proof of Theorem 1.0, where such a $u_0$ was constructed, and the fact that by the normality of operators $f(\zeta A)$ we have $\|f(\zeta A)\| = \operatorname{spr} f(\zeta A)$ $(\zeta \in C)$, tell us by the theory of Gelfand (cf. [2]) that we may take $u_0 \in \mathfrak{M}(\mathscr{B})$. Now, by (4.3) we have

$$|\langle f(\zeta A), u\rangle| \leqslant |\langle f(\zeta A), u_0\rangle| \qquad (\zeta \in \mathscr{D};\ u \in \mathfrak{M}(\mathscr{B}));$$

hence it follows that

$$|f(\zeta\eta)| \leqslant |f(\zeta\eta_0)| \qquad (\zeta \in \mathscr{D};\ \eta \in \sigma(A)), \tag{4.4}$$

where $\eta_0 = \langle A, u_0\rangle \in \sigma(A)$.

Now we prove the last statement of the theorem. Let the above conditions be satisfied. By the first part of the theorem we have

$$|g(\zeta)| = |f(\zeta\eta_0)| \qquad (\zeta \in \mathscr{D}). \tag{4.5}$$

To prove that $\eta_0$ lies in the boundary of $\sigma(A)$, assume that $\sigma(A)$ contains a neighbourhood $\mathscr{U}(\eta_0)$ of the point $\eta_0$. Let $\zeta_0 \in \mathscr{D}$, $\zeta_0 \neq 0$. Then $\zeta_0\mathscr{U}(\eta_0) = \{z = \zeta_0\eta;\ \eta \in \mathscr{U}(\eta_0)\}$ is a neighbourhood of the point $\tau_0 = \zeta_0\eta_0$. For each $\tau$ in this neighbourhood we have $|f(\tau)| \leqslant |f(\tau_0)|$ by (4.4). By the classical maximum modulus theorem (cf. [3]) this is not possible since $f$ is not a constant. So $\eta_0$ lies in the boundary of $\sigma(A)$. Finally, since $f$ is a non-constant function, it follows by (4.5) that $g$ is constant if and only if $\eta_0 = 0$. Q. E. D.

4.1. Remark. The proof of Theorem 4.0 shows that the assumption that $f$ is an entire function is not essential — if we drop it, some restrictions on the domain of definition of $f$ are necessary.

4.2. Example. Let $X$ be a complex Hilbert space of dimension greater than 1, and $A \in L(X)$ a selfadjoint operator whose spectrum is contained in the interval $[0, 1]$ and contains the points 0 and 1. By Theorem 4.0 we have

$$\|e^{\zeta A}\| = 1 \quad (\operatorname{Re}\zeta < 0)\rangle, \qquad \|e^{\zeta A}\| = |e^{\zeta}| \quad (\operatorname{Re}\zeta > 0).$$

If we write $\mathscr{D} = \{\zeta\colon \operatorname{Re}\zeta > 0\}$, the above example shows that $\mathscr{A}(\mathscr{D}, L(X))$ contains non-trivial functions if the dimension of $X$ is greater than 1 (since in this case operators $A \in L(X)$ with the above properties always exist).

On the other hand, if $g$ is an operator-valued function, satisfying

$$\|g(\zeta)\| = 1 \quad (\operatorname{Re}\zeta < 0), \qquad \|g(\zeta)\| = |e^{\zeta}| \quad (\operatorname{Re}\zeta > 0), \tag{4.6}$$

then $g$ does not necessarily have the form $g(\zeta) = e^{\zeta A}$ with $A \in L(X)$. To see this, let $f$ be a function whose values are linear operators over a three-dimensional complex Hilbert space, given by

$$f(\zeta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\zeta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in some orthonormal basis. Clearly $f$ is an entire analytic function which satisfies (4.6). Since $f(\zeta)^{-1}$ never exists, $f$ cannot have the form $f(\zeta) = e^{\zeta A}$. However, we have the following theorem.

4.3. Theorem. *Let $X$ be a complex Hilbert space and $A \in L(X)$, $A \neq 0$. Let $f$ be an entire complex-valued analytic function satisfying $f'(0) \neq 0$.*

*Let*

$$\|f(\zeta A)\| = 1 \qquad (\mathrm{Re}\,\zeta \leqslant 0) \tag{4.7}$$

*and let, on the half-plane* $\mathrm{Re}\,\zeta > 0$, *the norm* $\|f(\zeta A)\|$ *be equal to the absolute value of a complex-valued analytic function.*

*Then there exist a constant* $\gamma$ *and a real constant* $\beta$ *such that* $f(\zeta) = e^{i\beta}e^{\gamma\zeta}$ $(\zeta \in C)$. *Further, the operator* $A$ *has the form* $A = e^{i\alpha}P$, *where* $P$ *is a positive operator and* $\alpha$ *is a real constant.*

For the proof we need the following lemma.

4.4. LEMMA (cf. [5]). *Let* $X$ *be a complex Hilbert space and* $A \in L(X)$. *Denote by* $I \in L(X)$ *the identity operator. If* $\|I+i\tau A\| = 1+o(\tau)$ $(\tau$ *real,* $\tau \to 0)$, *then* $A$ *is a selfadjoint operator.*

Proof of Theorem 4.3. Expand $f$ into the Taylor series

$$f(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \ldots$$

Then

$$f(\zeta A) = c_0 I + c_1(\zeta A) + c_2(\zeta A)^2 + \ldots$$

By the assumption we have $c_1 \neq 0$ and by (4.7) we have $|c_0| = 1$. Now, (4.7) implies

$$\|f(\zeta A)\| = \|I + (c_1/c_0)(\zeta A) + (c_2/c_0)(\zeta A)^2 + \ldots\| = 1 \qquad (\mathrm{Re}\,\zeta \leqslant 0).$$

In particular, if $\zeta = i\tau$ with $\tau$ real, then we have

$$\|I + i\tau[(c_1/c_0)A] + (i\tau)^2(c_2/c_0)A^2 + \ldots\| = 1 \qquad (\tau \text{ real})$$

so

$$\|I + i\tau[(c_1/c_0)A]\| = 1 + o(\tau) \qquad (\tau \text{ real},\ \tau \to 0).$$

By Lemma 4.4 it follows that the operator $(c_1/c_0)A$ is selfadjoint.

Further, since in the half plane $\mathrm{Re}\,\zeta > 0$ $\|f(\zeta A)\|$ is equal to the absolute value of a complex-valued analytic function and since we have (4.7), it follows by Theorem 3.2 that

$$\|f(\zeta A)\| = |e^{\delta\zeta}| \qquad (\mathrm{Re}\,\zeta > 0), \tag{4.8}$$

where $\delta \geqslant 0$. In our case we have $\delta > 0$, since $\delta = 0$ would imply by (4.7) and (4.8) that $\|f(\zeta A)\| = 1$ $(\zeta \in C)$ and by the theorem of Liouville it would follow that $f(\zeta A)$ is a constant, which is not possible since by the assumptions $A \neq 0$ and $c_1 \neq 0$.

Now, since $(c_1/c_0)A$ is a selfadjoint operator and since $c_1/c_0 \neq 0$, $A$ is a normal operator. By (4.8) and by Theorem 4.0 a point $\eta_0 \in \sigma(A)$ exists such that $|f(\zeta\eta_0)| = |e^{\delta\zeta}|$ $(\mathrm{Re}\,\zeta > 0)$. Since $\delta \neq 0$, by Theorem 4.0 it follows that $\eta_0 \neq 0$ so that

$$|f(\zeta)| = |e^{(\delta/\eta_0)\zeta}| \qquad (\mathrm{Re}\,\zeta > 0),$$

which proves that $f(\zeta) = e^{i\beta}e^{\gamma\zeta}$ ($\zeta \in C$) with $\beta$ real. Further, by Theorem 4.0, (4.7) implies $|f(\zeta\eta)| \leqslant |f(0)|$ ($\mathrm{Re}\,\zeta < 0$; $\eta \in \sigma(A)$), so that

$$|e^{(\delta/\eta_0)\zeta\eta}| \leqslant 1 \quad (\mathrm{Re}\,\zeta < 0,\ \eta \in \sigma(A)). \tag{4.9}$$

Assume that $\eta \neq 0$, $\pm\eta \in \sigma(A)$. Now (4.9) gives $|e^{(\delta/\eta_0)\zeta\eta}| = 1$ ($\mathrm{Re}\,\zeta < 0$), which is not possible since $\delta > 0$ and since $\eta \neq 0$. This shows (since $(c_1/c_0)A$ is a selfadjoint operator with $c_1/c_0 \neq 0$) that $\sigma(A)$ lies on a ray with the beginning at the point 0, which proves the last statement of the theorem. Q. E. D.

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