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On the behavior of solutions of parabolic equations with unbounded coefficients

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1. Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space R^n and let t be a non-negative number. The distance of the point $x \in R^n$ from the origin of R^n is denoted by $|x|$. Denote by Ω_T a strip $R^n \times (0, T)$ in the $(n+1)$-dimensional half space $R^n \times (0, +\infty)$, where $T < +\infty$. A point in Ω_T is represented by its coordinate (x, t).

Consider a parabolic differential equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + k^2(|x|^2+1)u = \frac{\partial u}{\partial t}, \qquad k > 0 \tag{1}$$

in $R^n \times (0, +\infty)$. Krzyżański and Szybiak [4] proved the existence of the fundamental solution of this equation. By using this fundamental solution, we can see that the solution $u(x, t)$ of the above equation with the Cauchy data $u(x, 0) = e^{-\mu|x|^2}$ $(\mu > 0)$ is uniquely determined in $\Omega_{\frac{\pi}{4k}}$ and is given by

$$u(x, t) = \left(\frac{k}{2\mu \sin 2kt + k\cos 2kt}\right)^{n/2} \exp\left\{-\frac{k(2\mu\cos 2kt - k\sin 2kt)}{2(2\mu \sin 2kt + k\cos 2kt)}\,|x|^2 + k^2 t\right\}.$$

So, if $0 \leqslant t < t_0 = \dfrac{1}{2k}\tan^{-1}\dfrac{2\mu}{k}$, then $u(x, t)$ decays exponentially as $|x|$ tends to infinity, and $u(x, t_0)$ is equal to a positive constant and further, if $t_0 < t < \dfrac{\pi}{4k}$, then $u(x, t)$ grows exponentially as $|x|$ tends to infinity (cf. Kusano [5]). This fact leads us to a question whether the similar situation to the above holds or not for solutions of general parabolic equations of unbounded coefficients with a suitable Cauchy data.

2. The following result of Chen [2] gives us an answer to the question in part:

Let

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu - \frac{\partial u}{\partial t} = 0 \tag{2}$$

be a parabolic equation in Ω_T, where coefficients a_{ij} $(= a_{ji})$, b_i and c are functions defined in Ω_T such that

$$\begin{cases} 0 < \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leqslant K_1(|x|^2+1)^{1-\lambda} |\xi|^2 \\ \qquad\qquad \text{for any real vector } \xi = (\xi_1, \ldots, \xi_n) \neq 0, \\ |b_i| \leqslant K_2(|x|^2+1)^{1/2} \quad (1 \leqslant i \leqslant n), \\ c \leqslant K_3(|x|^2+1) \end{cases} \tag{3}$$

in Ω_T for some positive constants K_1, K_2, K_3 and $\lambda \in (0, 1]$. Assume that $K = 4K_1K_3 - [K_2 n - 2(\lambda-1)K_1]^2 > 0$. If the solution $u(x, t)$ of equation (2) in Ω_T satisfies $|u(x, t)| \leqslant K_4 \exp\{\mu_0(|x|^2+1)^\lambda\}$ in Ω_T and $|u(x, 0)| \leqslant K_5 \exp\{-\mu(|x|^2+1)^\lambda\}$ for some positive constants K_4, K_5, μ_0 and μ and if

$$T_0 = \mathrm{Min}\left(T, \frac{1}{\lambda\sqrt{K}} \tan^{-1} \frac{\lambda\sqrt{K}}{-4\lambda(\lambda-1)K_1 + 2\lambda K_2 n + 2K_3\mu^{-1}}\right),$$

then there exists a positive constant $\tilde{\mu}$ such that

$$|u(x, t)| \leqslant K_5 \exp\{-\tilde{\mu}(|x|^2+1)^\lambda\}$$

in $\overline{\Omega}_{T'}$ for any fixed T' $(< T_0)$.

In this article we shall deal with the question stated in Section 1 under a somewhat stronger condition for coefficients and give an affirmative answer.

3. The following minimum principle due to Bodanko [1] plays an essential role in the later treatment.

LEMMA 1. *Suppose that coefficients of L in* (2) *satisfy condition* (3) *in Ω_T and that $u = u(x, t)$ continuous in $\overline{\Omega}_T = R^n \times [0, T]$ satisfies $Lu \leqslant 0$ and $u(x, t) \geqslant -K_4 \exp\{\mu_0(|x|^2+1)^\lambda\}$ in Ω_T for some positive constants K_4 and μ_0. If $u(x, 0) \geqslant 0$, then $u(x, t) \geqslant 0$ throughout $\overline{\Omega}_T$.*

Using this minimum principle, we can prove the following which is a general form of Krzyżański's theorem [3].

LEMMA 2. *Assume that coefficients of L in (2) satisfy*

$$0 < \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant K_1(|x|^2+1)^{1-\lambda}\ |\xi|^2 \quad \textit{for any real vector } \xi \neq 0,$$

$$(4) \quad |b_i| \leqslant K_2(|x|^2+1)^{1/2} \quad (1 \leqslant i \leqslant n),$$

$$k_3(|x|^2+1)^{\lambda} \leqslant c$$

for some positive constants K_1, K_2, k_3 and $\lambda \in (0, 1]$. Let $u = u(x, t)$ coninuous in $\overline{\Omega_T}$ satisfy $Lu \leqslant 0$ and $u(x, t) \geqslant -K_4\exp\{\mu_0(|x|^2+1)^{\lambda}\}$ in Ω_T tor positive constants K_4 and μ_0. If there exists a positive constant K_5 such that $u(x, 0) \geqslant K_5$, then it holds that

$$u(x, t) \geqslant K_5\exp\{\mu^*(|x|^2+1)^{\lambda}t\}$$

in $\overline{\Omega_T}$ for a positive constant μ^*.

Proof. Take μ^* as such as

$$0 < \mu^* \leqslant \frac{k_3}{[4\lambda(1-\lambda)K_1+2\lambda nK_2]T+1}$$

and put

$$v(x, t) = K_5\exp\{\mu^*(|x|^2+1)^{\lambda}t\}.$$

Then, from (4) we see easily that

$$\begin{aligned}
\frac{Lv}{v} &= [4\mu^{*2}\lambda^2(|x|^2+1)^{2\lambda-2}t^2+4\mu^*\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}t]\sum_{i,j=1}^{n} a_{ij}x_ix_j+ \\
&\quad +2\mu^*\lambda(|x|^2+1)^{\lambda-1}t\sum_{i=1}^{n}(a_{ii}+b_ix_i)+c-\mu^*(|x|^2+1)^{\lambda} \\
&\geqslant 4\mu^*\lambda(\lambda-1)TK_1-2\mu^*\lambda(|x|^2+1)^{\lambda}TK_2n+(k_3-\mu^*)(|x|^2+1)^{\lambda} \\
&\geqslant (|x|^2+1)^{\lambda}\big[\mu^*\big(4\lambda(\lambda-1)TK_1-2\lambda TK_2n-1\big)+k_3\big] \\
&\geqslant 0
\end{aligned}$$

in Ω_T. Putting $w(x, t) = u(x, t)-v(x, t)$ and applying Lemma 1 to this function $w(x, t)$, we have $w(x, t) \geqslant 0$ in $\overline{\Omega_T}$, that is, $u(x, t) \geqslant v(x, t)$ in $\overline{\Omega_T}$, which proves the lemma.

4. Now we assume that the coefficients of L in (2) satisfy the condition

$$(5) \quad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqslant \displaystyle\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any real vector } \xi, \\ |b_i| \leqslant K_2(|x|^2+1)^{1/2} \quad (1 \leqslant i \leqslant n), \\ k_3(|x|^2+1)^{\lambda} \leqslant c \leqslant K_3(|x|^2+1)^{\lambda} \end{cases}$$

in Ω_T for positive constants k_1, k_3, K_1, K_2, K_3 and $\lambda\in(0,1]$.

Let $u=u(x,t)$ continuous in $\bar{\Omega}_T$ satisfy $Lu\leqslant 0$ and $u(x,t)\geqslant -K_4\exp\{\mu_0(|x|^2+1)^\lambda\}$ in Ω_T and $u(x,0)\geqslant K_5\exp\{-\mu(|x|^2+1)^\lambda\}$ for positive constants K_4, K_5, μ_0 and μ. Suppose that these constants fulfil the inequality

$$-2\lambda K_2 n+k_3\mu^{-1}>0. \tag{6}$$

We introduce a parameter ϱ (>1) and put

$$\begin{aligned}&v(x,t)\\&=K_5\exp\left\{-\mu(|x|^2+1)^\lambda\varrho^{-\gamma_0 t}-\frac{2\lambda K_1 n}{\lambda_0\log\varrho}\mu(1-\varrho^{-\gamma_0 t})-\frac{2\lambda^2 k_1}{\gamma_0\log\varrho}\mu^2(1-\varrho^{-2\gamma_0 t})\right\},\end{aligned}$$

where

$$\gamma_0=(4\lambda^2 k_1\mu\varrho^{-1}-2\lambda K_2 n+k_3\mu^{-1})(\log\varrho)^{-1}.$$

From (6) we see $\gamma_0>0$. Since $\lambda\in(0,1]$, it is easy to see that

$$\begin{aligned}\frac{Lv}{v}&=4\mu^2\lambda^2\varrho^{-2\gamma_0 t}(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j-\\&\quad-4\mu\lambda(\lambda-1)\varrho^{-\gamma_0 t}(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j-\\&\quad-2\mu\lambda\varrho^{-\gamma_0 t}(|x|^2+1)^{\lambda-1}\sum_{i=1}^n(a_{ii}+b_i x_i)+c-\\&\quad-\left[\mu(|x|^2+1)^\lambda\gamma_0\varrho^{-\gamma_0 t}\log\varrho-\left(4\lambda(\lambda-1)k_1+2\lambda K_1 n\right)\mu\varrho^{-\gamma_0 t}\right]\\&\geqslant(|x|^2+1)^\lambda\mu\varrho^{-\gamma_0 t}[4\lambda^2 k_1\mu\varrho^{-\gamma_0 t}-2\lambda nK_2+k_3(\mu\varrho^{-\gamma_0 t})^{-1}-\lambda_0\log\varrho].\end{aligned}$$

If $0\leqslant t<\gamma_0^{-1}$, then

$$4\lambda^2 k_1\mu\varrho^{-\gamma_0 t}-2\lambda nK_2+k_3(\mu\varrho^{-\gamma_0 t})^{-1}-\gamma_0\log\varrho\geqslant 0.$$

Hence it follows that $Lv\geqslant 0$ provided that $0<t<\gamma_0^{-1}$. In the following we assume $\gamma_0^{-1}\leqslant T$. By putting $w(x,t)=u(x,t)-v(x,t)$, we see easily $w(x,0)\geqslant 0$, $Lw\leqslant 0$ in $\Omega_{\gamma_0^{-1}}$ and $w(x,t)\geqslant -K_4'\exp\{\mu_0(|x|^2+1)^\lambda\}$ in $\Omega_{\gamma_0^{-1}}$ for a suitable positive constant K_4'. Therefore Lemma 1 implies $w(x,t)\geqslant 0$ in $\bar{\Omega}_{\gamma_0^{-1}}$, so $u(x,t)\geqslant v(x,t)$ in $\bar{\Omega}_{\gamma_0^{-1}}$. Hence we have

$$\begin{aligned}&u(x,\gamma_0^{-1})\geqslant v(x,\gamma_0^{-1})\\&=K_5\exp\left\{-\mu\varrho^{-1}(|x|^2+1)^\lambda-\frac{2\lambda K_1 n}{\gamma_0\log\varrho}\mu(1-\varrho^{-1})-\frac{2\lambda^2 k_1}{\gamma_0\log\varrho}\mu^2(1-\varrho^{-2})\right\}.\end{aligned} \tag{7}$$

We consider $t=\gamma_0^{-1}$ as to be the initial time and (7) as to be the initial condition for u. Repeating the above procedure, we obtain

$$u(x,t) \geqslant K_5' \exp\Big\{-\mu\varrho^{-1}(|x|^2+1)^\lambda \varrho^{-\gamma_1(t-\gamma_0^{-1})} -$$
$$- \frac{2\lambda K_1 n}{\gamma_1 \log\varrho}\mu\varrho^{-1}(1-\varrho^{-\gamma_1(t-\gamma_0^{-1})}) - \frac{2\lambda^2 k_1}{\gamma_1\log\varrho}\mu^2\varrho^{-2}(1-\varrho^{-2\gamma_1(t-\gamma_0^{-1})})\Big\}$$

in $R^n \times [\gamma_0^{-1}, \gamma_0^{-1}+\gamma_1^{-1}]$, where

$$\gamma_1 = (4\lambda^2 k_1\mu\varrho^{-2} - 2\lambda K_2 n + k_3\mu^{-1}\varrho)(\log\varrho)^{-1}$$

and

$$K_5' = K_5\exp\Big\{-\frac{2\lambda K_1 n}{\gamma_0\log\varrho}\mu(1-\varrho^{-1}) - \frac{2\lambda^2 k_1}{\gamma_0\log\varrho}\mu^2(1-\varrho^{-2})\Big\},$$

provided that $\gamma_0^{-1}+\gamma_1^{-1} < T$. Hence

$$u(x,\gamma_0^{-1}+\gamma_1^{-1}) \geqslant K_5\exp\Big\{-\frac{2\lambda K_1 n}{\log\varrho}\mu(1-\varrho^{-1})(\gamma_0^{-1}+\varrho^{-1}\gamma_1^{-1}) -$$
$$- \frac{2\lambda^2 k_1}{\log\varrho}\mu^2(1-\varrho^{-2})(\gamma_0^{-1}+\varrho^{-2}\gamma_1^{-1})\Big\}\exp\{-\mu\varrho^2(|x|^2+1)^\lambda\}.$$

In general, if $\gamma_0^{-1}+\ldots+\gamma_j^{-1} < T$, then it holds that

(8) $$u(x,\gamma_0^{-1}+\ldots+\gamma_j^{-1})$$
$$\geqslant K_5\exp\Big\{-\frac{2\lambda K_1 n}{\log\varrho}\mu(1-\varrho^{-1})(\gamma_0^{-1}+\varrho^{-1}\gamma_1^{-1}+\ldots+\varrho^{-j}\gamma_j^{-1}) -$$
$$- \frac{2\lambda^2 k_1}{\log\varrho}\mu^2(1-\varrho^{-2})(\gamma_0^{-1}+\varrho^{-2}\gamma_1^{-1}+\ldots+\varrho^{-2j}\gamma_j^{-1})\Big\}\exp\{-\mu\varrho^{-j-1}(|x|^2+1)^\lambda\},$$

where

$$\gamma_j = (4\lambda^2 k_1\mu\varrho^{-j-1} - 2\lambda K_2 n + k_3\mu^{-1}\varrho^j)(\log\varrho)^{-1}.$$

Now we suppose

$$\sigma(\varrho) = \sum_{j=0}^{\infty}\gamma_j^{-1} < T.$$

First we estimate the sum $\sigma(\varrho)$ from above and below. For the brevity we put $4\lambda^2 k_1\mu = f$, $-2\lambda K_2 n = g$ and $k_3\mu^{-1} = h$. Then

$$\sigma(\varrho) = \log\varrho\sum_{j=0}^{\infty}\frac{1}{f\varrho^{-j-1}+g+h\varrho^j}.$$

The function $(f\varrho^{-\tau-1}+g+h\varrho^\tau)^{-1}$ of $\tau\in(-\infty,\infty)$ has its maximum at $\tau = \tau_0 = \frac{1}{2}\log_\varrho\frac{f}{h\varrho}$. Assume that

(9) $$4fh - g^2 = 4\lambda^2[4k_1k_3 - K_2^2 n^2] > 0.$$

There are two cases: (i) $f > h$ and (ii) $f \leqslant h$.

In the case (i), we can find a number ϱ_0 (>1) such that $\varrho_0 > \varrho > 1$ implies $f > h\varrho$ and such that $4fh\varrho^{-1} - g^2 > 0$. For such a number ϱ it is evident that $\tau_0 > 0$ and there exists an integer p ($\geqslant 0$) satisfying $p < \tau_0 \leqslant p+1$. So, if $\varrho_0 > \varrho > 1$, then

$$\sigma(\varrho) \geqslant \log\varrho\left[\int_0^p \frac{d\tau}{f\varrho^{-\tau-1}+g+h\varrho^\tau} + \int_{p+1}^\infty \frac{d\tau}{f\varrho^{-\tau-1}+g+h\varrho^\tau}\right]$$

$$= \frac{2}{\sqrt{4fh\varrho^{-1}-g^2}} \times$$

$$\times \tan^{-1}\left[\frac{\sqrt{4fh\varrho^{-1}-g^2}\,[4fh\varrho^{-1}-g^2+(2h\varrho^p+g)(2h+g)+2h(\varrho^p-1)(2h\varrho^{p+1}+g)]}{(2h\varrho^{p+1}+g)[4fh\varrho^{-1}-g^2+(2h\varrho^p+g)(2h+g)]-4fh\varrho^{-1}-g^2)\,2h(\varrho^p-1)}\right].$$

We denote by $T_1(\varrho)$ the right-hand side of the above. It is easy to see that

$$\sigma(\varrho) \leqslant \log\varrho\left[\int_0^p \frac{d\tau}{f\varrho^{-\tau-1}+g+h\varrho^\tau} + \int_{p+1}^\infty \frac{d\tau}{f\varrho^{-\tau-1}+g+h\varrho^\tau}\right] + \gamma_p^{-1} + \gamma_{p+1}^{-1}$$

$$\leqslant T_1(\varrho) + \log\varrho\left(\frac{1}{f\varrho^{-p-1}+g+h\varrho^p} + \frac{1}{f\varrho^{-p-2}+g+h\varrho^{p+1}}\right) \quad (1 < \varrho < \varrho_0).$$

In the case (ii), it is obvious that $\tau_0 \leqslant 0$ for any $\varrho > 1$. As in the case (i), there is a ϱ_0 (>1) such that $4fh\varrho^{-1} - g^2 > 0$ for any ϱ satisfying $\varrho_0 > \varrho > 1$. So for such a ϱ we get

$$\sigma(\varrho) \geqslant \log\varrho \int_0^\infty \frac{d\tau}{f\varrho^{-\tau-1}+g+h\varrho^\tau} = \frac{2}{\sqrt{4fh\varrho^{-1}-g^2}} \tan^{-1}\frac{\sqrt{4fh\varrho^{-1}-g^2}}{2h+g}.$$

Denoting the right-hand side of the above by $T_2(\varrho)$, we see easily

$$\sigma(\varrho) \leqslant T_2(\varrho) + \frac{\log\varrho}{f\varrho^{-1}+g+h} \quad (1 < \varrho < \varrho_0).$$

Therefore, in both cases (i) and (ii), we have

$$\lim_{\varrho\to 1}\sigma(\varrho) = \frac{2}{\sqrt{4fh-g^2}}\tan^{-1}\frac{\sqrt{4fh-g^2}}{2h+g}$$

from the supposition (6).

Next we estimate the sum of the series

$$\sum_{j=0}^\infty \varrho^{-j}\gamma_j^{-1}.$$

It is easy to see from (6) that

$$(10)\qquad \sum_{j=0}^{\infty} \varrho^{-j}\gamma_j^{-1} = \log\varrho \sum_{j=0}^{\infty} \frac{\varrho^{-j}}{4\lambda^2 k_1 \mu \varrho^{-j-1} - 2\lambda K_2 n + k_3\mu^{-1}\varrho^j}$$
$$\leqslant \log\varrho \sum_{j=0}^{\infty} \frac{1}{\varrho^j}\,\frac{1}{-2\lambda K_2 n + k_3\mu^{-1}} = \frac{1}{-2\lambda K_2 n + k_3\mu^{-1}}\,\frac{\log\varrho}{1-\varrho^{-1}}.$$

By the same reasoning as the above, it follows that

$$(11)\qquad \sum_{j=0}^{\infty} \varrho^{-2j}\gamma_j^{-1} \leqslant \frac{1}{-2\lambda K_2 n + k_3\mu^{-1}}\,\frac{\log\varrho}{1-\varrho^{-2}}.$$

5. Now we can prove the following

THEOREM. *Let*

$$L \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t}$$

be a parabolic differential operator in Ω_T, where coefficients a_{ij} $(= a_{ji})$, b_i and c satisfy condition (5) *in Ω_T for positive constants k_1, k_3, K_1, K_2, K_3 and $\lambda \in (0, 1]$. Let $u(x, t) \geqslant -K_4 \exp\{\mu_0(|x|^2+1)^\lambda\}$ in Ω_T for some positive constants K_4 and μ_0 and $u(x, 0) \geqslant K_5 \exp\{-\mu(|x|^2+1)^\lambda\}$ for positive constants K_5 and μ. Assume that conditions* (6) *and* (9) *are valid.*

If

$$T_0^* = \frac{1}{\lambda\sqrt{4k_1k_3 - K_2^2 n^2}} \tan^{-1} \frac{\sqrt{4k_1k_3 - K_2^2 n^2}}{-\lambda K_2 n + k_3\mu^{-1}} < T,$$

then there exists a positive constant K_6 such that $u(x, T_0^*) \geqslant K_6$. Further if $T_0^* < t < T$, then there exists a positive constant μ^* such that

$$u(x, t) \geqslant K_6 \exp\{\mu^*(t - T_0^*)(|x|^2+1)^\lambda\}.$$

Proof. By Lemma 2 it suffices to show the existence of a constant K_6 in our Theorem. As was shown already, the function $\sigma(\varrho)$ in Section 4 satisfies

$$\lim_{\varrho\to 1} \sigma(\varrho) = T_0^*.$$

So, given any positive number ε, we can find ϱ_0 (>1) such that if $\varrho_0 > \varrho > 1$, then $u(x, T_0^*) > u(x, \sigma(\varrho)) - \varepsilon/2$. On the other hand, there exists an integer N_0 (>0) such that $N \geqslant N_0$ implies $u(x, \sigma(\varrho)) > u(x, \sum_{j=0}^{N} \gamma_j^{-1}) - \varepsilon/2$. Therefore it holds that $u(x, T_0^*) > u(x, \sum_{j=0}^{N} \gamma_j^{-1}) - \varepsilon$. Hence (8), (10) and (11) yield

$$u(x, T_0^*) > K_5 \exp\left\{-\frac{2\lambda K_1 n\mu + 2\lambda^2 k_1 \mu^2}{-2\lambda K_2 n + k_3 \mu^{-1}}\right\} \exp\{-\mu\varrho^{-N-1}(|x|^2+1)^\lambda\} - \varepsilon.$$

We fix $x \in R^n$ arbitrary. Letting N tend to infinity and ε to zero, we get

$$u(x, T_0^*) \geqslant K_5 \exp\left\{-\frac{2\lambda K_1 n\mu + 2\lambda^2 k_1 \mu^2}{-2\lambda K_2 n + k_3 \mu^{-1}}\right\}.$$

Taking K_6 equal to the right-hand term of the above, we have $u(x, T_0^*) \geqslant K_6$ at every point $x \in R^n$.

Thus we have the theorem.

6. In our Theorem we assume $Lu = 0$, $|u(x, t)| \leqslant K_4 \exp\{\mu_0(|x|^2+1)^\lambda\}$ in Ω_T and $u(x, 0) = K_5 \exp\{-\mu(|x|^2+1)^\lambda\}$. Then the assertions of our theorem and of Chen's result stated in Section 2 are both valid. Thus, if $t < T_0$, then $u(x, t)$ decays exponentially as $|x|$ tends to infinity and, if $T_0^* < t < T$, then $u(x, t)$ grows exponentially as $|x|$ tends to infinity. It would be a hard problem to determine the behavior of $u(x, t)$ for $t \in (T_0, T_0^*)$ as $|x|$ tends to infinity.

For equation (1), we may take $k_1 = K_1 = 1$, $k_3 = K_3 = k^2$ and $\lambda = 1$ and further K_2 can be taken as small as we want. So, in this case, it is clear that $T_0 = T_0^* = \frac{1}{2k} \tan^{-1} \frac{2\mu}{k}$ and we can conclude the property of the solution $u(x, t)$ stated in Section 1 without use of the fundamental solution of (1).

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