COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 2

DIRECT PRODUCT DECOMPOSITION OF ZERO-PRODUCT-ASSOCIATIVE RINGS WITHOUT NILPOTENT ELEMENTS

BY

ALEXANDER ABIAN (AMES, IOWA)

In what follows* A stands for a (not necessarily associative or commutative) ring satisfying the following property:

(a) *A has no nilpotent element and a product of elements of A which is equal to zero remains equal to zero no matter how its factors are associated.*

In this paper, as expected, "nilpotent" means "non-zero nilpotent". Also, for the sake of brevity, we refer to the second property in (a) as: *zero-product-associative.*

In [1] it is shown that (a) is equivalent to:

(a') *A has no nilpotent element and a product of elements of A which is equal to zero remains equal to zero no matter how its factors are associated or permuted.*

Remark 1. In [4] it is shown that an alternative ring without nilpotent elements satisfies (a'). Therefore, whatever is implied by (a) or (a') holds true for the case of alternative rings without nilpotent elements. In view of [5], the same applies for the case of right alternative rings without nilpotent elements and of characteristic not equal to 2.

In [1] it is shown that A is partially ordered by $\leqslant$, where, for any elements x and y of A,

$$x \leqslant y \quad \text{if and only if} \quad xy = x^2. \tag{1}$$

In what follows any reference to order is made in connection with the partial order given by (1).

As shown in [1], if $(x_i)_{i\in I}$ is a subset of A such that $\sup_i x_i$ exists, then, for every element r of A, $\sup_i rx_i$ as well as $\sup_i x_i r$ exists and

$$r\sup_i x_i = \sup_i rx_i \quad \text{with } i \in I, \tag{2}$$

* This research was partially supported by Iowa State University SHRI.

whereas

(3) $$(\sup_i x_i)r = \sup_i x_i r \quad \text{with } i \in I.$$

A subset M of A is called a *maximal multiplicative system* (*of* A) *not containing* 0 if M is closed under multiplication and $0 \notin M$ and, for every subset N of A, whenever N is closed under multiplication, $M \subseteq N$ and $M \neq N$ imply $0 \in N$.

Definition. An element a of the ring A is called an *atom of* A if a belongs to one and only one maximal multiplicative system (of A) not containing 0. Moreover, A is called *atomic* if for every non-zero element r of A there exists an atom a such that $a \leqslant r$.

Following [2] and [3], a subset S of A is called *orthogonal* if $xy = 0$ for every two distinct elements x and y of A. Furthermore, A is called *orthogonally complete* if $\sup S$ of every orthogonal subset S of A exists.

Remark 2. In [1] it is shown that the ring A is isomorphic to a subdirect product of rings without zero divisors. As shown in [4], the same applies to alternative rings without nilpotent elements. We shall prove the following result:

THEOREM. *The ring A is isomorphic to a direct product of rings without zero divisors if and only if A is atomic and orthogonally complete.*

Also, in view of Remark 1, we have

COROLLARY. *An alternative ring A without nilpotent elements is isomorphic to a direct product of alternative rings without zero divisors if and only if A is atomic and orthogonally complete.*

The Theorem is an immediate consequence of two lemmas which we are going to prove.

LEMMA 1. *Let $(D_i)_{i \in K}$ be a set of rings D_i without zero divisors. Then the direct product*

$$D = \prod_{i \in K} D_i$$

is a not necessarily associative or commutative ring which has property (a). *Moreover, D is atomic and orthogonally complete.*

Proof. For the sake of simplicity, we assume that K is an ordinal number and, as usual, we identify every element of D with a sequence (of type K)

$$(\ldots d_i \ldots) \quad \text{with } d_i \in D_i.$$

Also, (instead of 0_i) we often write 0 for the zero of every ring D_i as well as for the zero of D, and we denote by $(0\ d_i\ 0)$ an element of D

all coordinates of which, except perhaps the i-th coordinate, are equal to 0.

The fact that D satisfies property (a) is easily verified.

For every $i \in K$, it is readily seen that the set

$$M(i) = \{(\dots d_i \dots) \mid d_i \neq 0 \text{ and } d_i \in D\} \tag{4}$$

is a maximal multiplicative system (of D) not containing 0. However, not every maximal multiplicative system (of D) not containing 0 is necessarily equal to $M(i)$ for some $i \in K$. For instance, such is the case if D is an infinite direct product of 2-element fields.

Next, we observe that an element a of D is an atom if and only if

$$a = (0 \ d_i \ 0) \quad \text{for some } d_i \neq 0. \tag{5}$$

Clearly, no atom of D can possibly have two non-zero coordinates, since then it would belong to two distinct maximal multiplicative systems not containing 0 and of type (4). On the other hand, a as given by (5) is an element of $M(i)$, as given by (4). Moreover, a cannot belong to a maximal multiplicative system M (of D) not containing 0 and such that $M \neq M(i)$. Indeed, otherwise,

$$(\dots 0_i \dots) \in M \quad \text{and} \quad (\dots 0_i \dots)\cdot(0 \ d_i \ 0) = 0 \in M$$

which is a contradiction.

Obviously, if $(\dots d_i \dots)$ is a non-zero element of D with $d_i \neq 0$, then, in view of (1), we see that

$$(0 \ d_i \ 0) \leqslant (\dots d_i \ \dots).$$

Thus D is atomic.

Finally, let T be an orthogonal set of atoms of D. Since D_i has no zero divisors, in view of (1), the least upper bound of T is an obvious element of D. But then it follows that every orthogonal subset of D has a least upper bound. Thus A is orthogonally complete.

LEMMA 2. *Let the ring A be atomic and orthogonally complete. Then A is isomorphic to a direct product of rings without zero divisors.*

Proof. As shown in [1], if M_h is a maximal multiplicative system (of A) not containing 0, then $A - M_h$ is a completely prime ideal of A and the quotient ring $A/(A - M_h)$ has no zero divisors. Moreover, in [1] it is shown that, in view of property (a), every non-zero element of A belongs to a maximal multiplicative system (of A) not containing 0. Thus, if $(M_h)_{h \in H}$ is the set of all maximal multiplicative systems (of A) not containing 0, then

$$\bigcap_{h \in H} (A - M_h) = \{0\},$$

which implies that A is isomorphic to a subdirect product of the quotient rings $A/(A-M_h)$ with $h \in H$. From this (in view of the definition of an atom) it follows that the set of all atoms a_{hi} of A such that $a_{hi} \in M_h$ together with 0 forms a ring A_h without zero divisors, i.e., for every $h \in H$,

$$(6) \qquad A_h = \{a_{hi} | \, a_{hi} \text{ is an atom and } a_{hi} \in M_h\} \cup \{0\}$$

is a ring without zero divisors.

Let $(A_h)_{h \in N}$ be the set of all A_h's such that $A_h - \{0\}$ is not empty (a case where some of $A_h - \{0\}$ are empty is provided by any infinite Boolean algebra). To prove the lemma, it is enough to show that A is isomorphic to the direct product of A_h's with $h \in N$, i.e.,

$$(7) \qquad A = \prod_{h \in N} A_h.$$

Obviously, $\left(\bigcup_{h \in N} A_h \right) - \{0\}$ is the set of all atoms of A. Thus, an atom of A is an a_{hi} with $h \in N$ and $i \in I(h)$ for some index set $I(h)$. From the subdirect product decomposition of A and the definition of an atom it follows that

$$(8) \qquad a_{hi} a_{kj} = 0 \quad \text{if and only if} \quad h \neq k.$$

Let P as well as Q be an orthogonal set of atoms. We show that

$$(9) \qquad P = Q \quad \text{if and only if} \quad \sup P = \sup Q.$$

Clearly, it is sufficient to prove that $\sup P = \sup Q$ implies $P = Q$ Assume the contrary and let, say, $a_{hi} \in P$ and $a_{hi} \neq Q$. But then from (2) and (8) it follows that

$$a_{hi}(\sup Q) = a_{hi}(\sup P) = a_{hi}^2,$$

which, again by (2) and (8), implies $a_{hj} = a_{hi}$ for some $a_{hj} \in Q$, since A_h has no zero divisors. Consequently, $a_{hi} \in Q$ which is a contradiction. Hence (9) is established.

Next, for every element x of A, let $A(x)$, given by

$$(10) \qquad A(x) = \{a_{hi}, a_{kj}, a_{uv}, \ldots\},$$

denote the set of all the atoms of A which are less than or equal to x. From the subdirect decomposition of A and (1), and the fact that for every $h \in N$ the subring A_h has no zero divisors it follows that no two distinct atoms appearing in (10) are elements of the same A_h. Thus, $A(x)$ as given by (10) is an orthogonal subset of A. Based on the orthogonal completeness of A we show that, for every element x of A,

$$(11) \qquad x = \sup A(x),$$

where $0 = \sup \emptyset$. Since every atom appearing in (10) is less than or equal to x, we see that $\sup A(x) \leqslant x$. Assume on the contrary that $x \neq \sup A(x)$. But then

$$\sup A(x) < x \quad \text{and} \quad x - \sup A(x) \neq 0.$$

Since A is atomic, there exists an atom a such that

$$a \leqslant x - \sup A(x).$$

Since $\sup A(x) \leqslant x$, from (1) and the subdirect product decomposition of A it follows that $a \in A(x)$. Thus, $a \leqslant x$ and $a \leqslant \sup A(x)$. But then, in view of (1) and (2), we have

$$0 \neq a^2 = a\big(x - \sup A(x)\big) = ax - a\sup A(x) = a^2 - a^2 = 0,$$

which is a contradiction. Thus, (11) is established.

Finally, let f be a function from A into $\prod_{h\in N} A_h$ defined by

$$f(x) = A(x) \cup \{0_t \mid 0_t = 0 \text{ and } t \in (N - H)\},$$

where $A(x)$ is given by (10), and the subset H of N is defined by $t \in H$ if and only if $a_{tu} \in A(x)$ for some $u \in I(t)$. From (9) and (11) it follows that f is one-to-one and onto. From the subdirect product decomposition of A it is readily seen that f is an isomorphism. Hence (7) is proved.

REFERENCES

[1] A. Abian, *Order in a special class of rings and a structure theorem*, Proceedings of the American Mathematical Society 52 (1975), p. 45-49.

[2] — *Direct product decomposition of commutative semisimple rings*, ibidem 24 (1970), p. 502-507.

[3] M. Chacron, *Direct product of division rings and a paper of Abian*, ibidem 29 (1971), p. 259-262.

[4] I. R. Hentzel, *Alternative rings without nilpotent elements*, ibidem 42 (1974), p. 373-376.

[5] E. Kleinfeld, *Right alternative rings*, ibidem 4 (1953), p. 939-944.

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA

Reçu par la Rédaction le 23. 5. 1976