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The Ritt order of the derivative of an entire function

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Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \sqrt[n]{|c_n|} \to 0,$$

be the Maclaurin series of an entire function and let $M(r, f) = \max_{|z|=r} |f(z)|$. Since the sequence $\{a_n\}$ determines the function completely, it should in principle be possible to discover all the properties of the function by examining the coefficients. Thus the order ϱ is given by (for references see [2], pp. 9-12)

$$(1) \qquad \varrho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)},$$

and if $0 < \varrho < \infty$ then the type by

$$T = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\varrho}} = \limsup_{n\to\infty} \frac{n}{e\varrho} |a_n|^{\varrho/n}.$$

From the above it follows that the order of a function is the same as the order of its derivative. The same remark holds about the type of a function of finite non-zero order.

If the entire function $f(z)$ $(z = x+iy)$ is defined by an everywhere absolutely convergent Dirichlet series

$$(3) \qquad \sum_{n=1}^{\infty} a_n \exp(\lambda_n z) \qquad (0 < \lambda_n < \lambda_{n+1} \to \infty)$$

and $m(x, f) = \operatorname*{l.u.b.}_{-\infty<y<\infty} |f(x+iy)|$, then

$$(4) \qquad \varrho_{R,f} = \limsup_{x\to\infty} \frac{\log\log m(x, f)}{x}$$

is called the *Ritt order* of $f(z)$ and for $0 < \varrho_{R,f} < \infty$ the *type* $T_{R,f}$ has been defined as

$$\limsup_{x\to\infty} \frac{\log m(x, f)}{\exp(x\varrho_{R,f})}.$$

Improving upon a theorem of Ritt, it has recently been proved by Azpeitia ([1]) that if

(5) $$\lim_{n\to\infty}\frac{\lambda_n\log\lambda_n}{\log n}=\infty$$

then

(6) $$\varrho_{R,f}=\limsup_{n\to\infty}\frac{\lambda_n\log\lambda_n}{\log(1/|a_n|)}.$$

On the other hand, it is also known ([4], p. 71) that if

(7) $$\lim_{n\to\infty}\frac{\lambda_n}{\log n}=\infty$$

then

(8) $$T_{R,f}=\limsup_{n\to\infty}\frac{\lambda_n}{e\varrho_{R,f}}\,|a_n|^{\varrho_{Rf}/\lambda_n}.$$

Thus, subject to the restrictions (5) and (7) respectively we have

(9) $$\varrho_{R,f'}=\varrho_{R,f}$$

and

(10) $$T_{R,f'}=T_{R,f}.$$

The purpose of this note is to show that (9) and (10) are true in general. We shall in fact prove that for a function of finite Ritt order

(11) $$\log m(x,f')\sim\log m(x,f)\qquad\text{as}\qquad x\to\infty.$$

It is known ([2], p. 13) that

$$\frac{M(r,f)-|f(0)|}{r}\leqslant M(r,f')\leqslant\frac{M(2r,f)}{r},$$

and our proof of (11) depends on obtaining an analogous relationship between $m(x,f)$ and $m(x,f')$.

Now let $f(x)$ be analytic in the half plane $x\leqslant X$ and defined there by an absolutely convergent Dirichlet series (3). Then $\log m(x,f)$ is ([3]) an increasing downward convex function of x for $x\leqslant X$. For $\operatorname{Re} z_0=x_0\leqslant x\leqslant X$ we have

(12) $$f(z)=\int_{z_0}^{z}f'(z)\,dz+f(z_0).$$

Let $\operatorname{Im} z_0=y$ and consider the integral along the straight line joining z_0 and z. If $m(x,f)$ is attained at a finite point of the vertical line of abscissa x then we can choose z such that $|f(z)|=m(x,f)$. If, however, $m(x,f)$ is not attained at a finite point of the vertical line $\operatorname{Re} z=x$, we can still find a point z of the line such that $|f(z)| > m(x, f) - \varepsilon$ for a given $\varepsilon > 0$. Hence it follows from (12) that

$$m(x, f) - \varepsilon < (x - x_0)\, m(x, f') + |f(z_0)| . \tag{13}$$

On the other hand, let z^* be a finite point of $\operatorname{Re} z = x < X$ for which $|f'(z^*)| > m(x, f') - \varepsilon'$, $\varepsilon' > 0$. By Cauchy's theorem on the derivative applied to the point z^* and the circle Γ: $|z - z^*| < \delta \leqslant X - \operatorname{Re} z^*$, we get

$$m(x, f') - \varepsilon' < \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z^*)^2}\, dz \right| \leqslant \frac{1}{\delta} (\text{maximum of } |f(z)| \text{ on } \Gamma) \tag{14}$$

$$\leqslant \frac{1}{\delta}\, m(x + \delta, f) .$$

Hereafter, let $f(z)$ be entire and of finite Ritt order $\varrho_{R,f}$. Since $\log m(x, f)$ is an increasing convex function of x, we have

$$\log m(x, f) = \log m(x_0, f) + \int_{x_0}^{x} w(t)\, dt , \tag{15}$$

where $x_0 < x$ and $w(t)$ is a non-decreasing function of t. If ε is a fixed positive number then for sufficiently large x

$$\int_{x}^{x+2} w(t)\, dt < \log m(x + 2, f) < \exp\{(x + 2)(\varrho_{R,f} + \varepsilon)\}$$

and since $w(t)$ is non-decreasing, we get

$$2w(x) < \exp\{(x + 2)(\varrho_{R,f} + \varepsilon)\} .$$

As ε is arbitrary, we can even write

$$w(x) < \exp\{x(\varrho_{R,f} + \varepsilon)\} \tag{16}$$

for sufficiently large x.

Equality (15) then gives

$$\log m(x + \delta, f) = \log m(x, f) + \int_{x}^{x+\delta} w(t)\, dt ,$$

where the integral is smaller than

$$\exp\{(x + \delta)(\varrho_{R,f} + \varepsilon)\}\delta ,$$

and if we take

$$\delta = \exp\{-x(\varrho_{R,f} + \varepsilon)\}$$

which tends to zero with $1/x$, we obtain

$$\log m(x + \delta, f) < \log m(x, f) + e ,$$

provided x is sufficiently large. Putting this value of δ and the corresponding value of $m(x+\delta, f)$ in (14) we see that for every $\varepsilon > 0$, and sufficiently large x

$$m(x, f') \leqslant m(x, f) \exp\{x(\varrho_{R,f} + \varepsilon)\}. \tag{17}$$

The inequalities (13) and (17) show that, for every entire function of finite Ritt order,

$$\log m(x, f') \sim \log m(x, f).$$

References

[1] A. G. Azpeitia, *A remark on the Ritt order of entire functions defined by Dirichlet series*, Proc. Amer. Math. Soc. 12 (1961), pp. 722-723.

[2] R. P. Boas, Jr., *Entire functions*, New York 1954.

[3] G. Doetsch, *Über die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden*, Math. Zeitschr. 8 (1920), pp. 237-240.

[4] Yu Chia-Yung, *Sur les droites de Borel de certaines fonctions entières*, Ann. École Norm. 68 (1951), pp. 65-104.

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