ANNALES
POLONICI MATHEMATICI
XIII (1963)

Integration of infinite systems of differential inequalities

by W. MLAK and C. OLECH (Kraków)

In this paper we investigate infinite systems of ordinary differential inequalities of the form

$$D\varphi_i(t) \leqslant f_i\big(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots\big), \qquad i = 1, 2, \ldots, n, \ldots$$

$D\psi(t)$ stands here for any of the Dini derivatives of the function $\psi(t)$ at the point t. The first section of the paper deals with some simple linear differential inequalities. The second section concerns the non-linear case. We there introduce the right-hand maximum solution of a countable system of differential equations.

1. For the sake of clarity we will consider, in the following, inequalities with right-hand upper derivatives. Remember that the right-hand upper derivative denoted by $\overline{D}_+\psi(t)$ is defined by

$$\overline{D}_+\psi(t) = \limsup_{h\to 0+} \frac{\psi(t+h)-\psi(t)}{h}.$$

It can be remarked that our theorems remain true if $\overline{D}_+$ is replaced by any other Dini derivative.

We start with the following fundamental lemma:

LEMMA 1 (see [2]). *Let Ω be an open subset of the n-dimensional space of points $(y_1, \ldots, y_n)$. Suppose that the functions $g_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ are continuous on $\langle 0, a) \times \Omega$, and satisfy there the condition:*

(W) *If $\overline{y}_k \leqslant \overline{\overline{y}}_k$ for $k \neq i$ then* $g_i(t, \overline{y}_1, \ldots, \overline{y}_{i-1}, y_i, \overline{y}_{i+1}, \ldots, \overline{y}_n) \leqslant g_i(t, \overline{\overline{y}}_1, \ldots, \overline{\overline{y}}_{i-1}, y_i, \overline{\overline{y}}_{i+1}, \ldots, \overline{\overline{y}}_n)$.

Let the continuous functions $\varphi_1(t), \ldots, \varphi_n(t)$ satisfy in $(0, a)$ the inequalities

$$\overline{D}_+\varphi_i(t) \leqslant g_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big), \qquad i = 1, \ldots, n.$$

Suppose that the right-hand maximum solution $\omega_1(t), \ldots, \omega_n(t)$ (1) of the

(1) For the definition and construction of the extremal solutions of finite systems of differential equations see [2].

system $y_i' = g_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ *exists in* $\langle 0, a)$ *and* $\varphi_i(0) \leqslant \omega_i(0)$. *Then* $\varphi_i(t) \leqslant \omega_i(t)$ *for* $i = 1, \ldots, n$ *and* $t \in \langle 0, a)$.

COROLLARY. *If* $\varphi_i(t)$ $(i = 1, \ldots, n)$ *satisfy in* $\langle 0, a)$ *the inequalities* $\overline{D}_+\varphi_i(t) \geqslant g_i(t, \varphi_1(t), \ldots, \varphi_n(t))$ $(i = 1, \ldots, n)$ *then* $\varphi_i(t) \geqslant \tau_i(t)$ *where* $\tau_i(t)$ *are the components of the right-hand minimum solution of the system* $y_i' = g_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ *such that* $\varphi_i(0) \geqslant \tau_i(0)$.

Consider an infinite matrix $\{a_{ik}\}$, $i, k = 1, 2, 3, \ldots$, with real constants a_{ik}. Denote by $\big(\overset{n}{u}_{is}(t)\big)_n$ the fundamental matrix of the system of differential equations

$$z_i' = \sum_{k=1}^{n} a_{ik} z_k , \quad i = 1, 2, \ldots, n .$$

Notice that $\sum_{s=1}^{n} \overset{n}{u}_{is}(0) = 1$. We will prove the following lemma:

LEMMA 2. *Suppose the elements of* $\{a_{ik}\}$ *satisfy* $a_{ik} \geqslant 0$ *for* $i \neq k$. *Assume that we are given a sequence of non-negative functions* $\{\psi_k(t)\}$ *which are continuous on* $\langle 0, a)$. *Let the series* $\sum_{k=1}^{\infty} a_{ik}\psi_k(t)$ *be convergent for every* $i = 1, 2, \ldots$ *and every* $t \in (0, a)$. *Suppose that*

$$\sum_{k=1}^{\infty} a_{ik}\psi_k(t) \leqslant \overline{D}_+\psi_i(t) , \quad i = 1, 2, \ldots, \quad t \in (0, a) \tag{1}$$

and $1 \leqslant \psi_i(0)$ *for* $i = 1, 2, \ldots$ *Under our assumptions we have*

$$\sum_{s=1}^{n} \overset{n}{u}_{is}(t) \leqslant \psi_i(t)$$

for arbitrary n *and* i, $t \in \langle 0, a)$.

Proof. It is a simple matter to verify that the functions $\overset{n}{u}_i(t) = \sum_{s=1}^{n} \overset{n}{u}_{is}(t)$, $i = 1, 2, \ldots, n$, satisfy the system

$$\overset{n}{u}{}_i'(t) = \sum_{k=1}^{n} a_{ik}\overset{n}{u}_k(t) , \quad i = 1, 2, \ldots, n$$

and $\overset{n}{u}_i(0) = 1$. On the other hand $a_{ik} \geqslant 0$ for $i \neq k$ and $\psi_k(t) \geqslant 0$ for $i, k = 1, 2, \ldots$ It follows then from (1) that

$$\sum_{k=1}^{n} a_{ik}\psi_k(t) \leqslant \overline{D}_+\psi_i(t) , \quad i = 1, 2, \ldots, n$$

and consequently

$$\sum_{k=1}^{n} a_{ik}[\psi_k(t)-\overset{n}{u}_i(t)] \leqslant \bar{D}_+[\psi_i(t)-\overset{n}{u}_i(t)]\,, \qquad i=1,\dots,n\,.$$

But $0 \leqslant \psi_k(0)-\overset{n}{u}_k(0)$, $k=1,\dots,n$. The assertion of the lemma follows from the last differential inequalities and from the above corollary.

Lemma 2 generalizes certain result of [1] (lemma 3, p. 249) where appeared the assumptions: $a_{ik} \geqslant 0$ for $i \neq k$ and $\sum_{k=1}^{\infty} a_{ik} = 0$ for each i. However, the above equality shows that we can take in our lemma $\psi_k(t) \equiv 1$.

THEOREM 1. *Let $\{a_{ik}\}$ be an infinite matrix of real constants such that*

$$a_{ik} \geqslant 0 \quad \textit{for} \quad i \neq k\,. \tag{2}$$

Suppose that there exists a sequence $\{\psi_k(t)\}$ of non-negative functions which are continuous in $\langle 0, a)$ and satisfy

$$\begin{gathered}\sum_{k=1}^{\infty} a_{ik}\psi_k(t) \leqslant \bar{D}_+\psi_i(t) \quad \textit{for} \quad i=1,2,\dots, \quad 0<t<a\,,\\ 1 \leqslant \psi_i(0) \quad \textit{for} \quad i=1,2,\dots\end{gathered} \tag{3}$$

Suppose we are given a sequence $\{\varphi_i(t)\}$ of functions, which are continuous on $\langle 0, a)$. Assume that for every i the series $\sum_{k=1}^{\infty} a_{ik}\varphi_k(t)$ is almost uniformly convergent on $\langle 0, a)$. Suppose that the functions $\varphi_i(t)$ satisfy the following inequalities

$$\begin{gathered}\bar{D}_+\varphi_i(t) \leqslant \sum_{k=1}^{\infty} a_{ik}\varphi_k(t)\,, \quad i=1,2,\dots, \quad t \in \langle 0, a)\,,\\ \varphi_i(0) \leqslant 0 \quad \textit{for} \quad i=1,2,\dots\end{gathered} \tag{4}$$

Let us assume that

$$\sigma_n(t) = \max_{k=1,\dots,n} \Big|\sum_{s=n+1}^{\infty} a_{ks}\varphi_s(t)\Big| \to 0 \tag{5}$$

almost uniformly on $\langle 0, a)$.

Under our assumptions the inequalities

$$\varphi_i(t) \leqslant 0$$

hold for $i=1,2,\dots$ and $t \in \langle 0, a)$.

Proof. Given a fixed n denote by $\lambda_i^n(t) = \sum_{k=n+1}^{\infty} a_{ik}\varphi_k(t)$. Take now the system

$$y_i' = \sum_{k=1}^{n} a_{ik}y_k + \lambda_i^n(t)\,, \qquad i=1,\dots,n$$

and denote by $\gamma_1(t), \ldots, \gamma_n(t)$ its solution such that $\gamma_i(0) = 0$ for $i = 1, \ldots, n$. Observe that (4) implies

$$\overline{D}_+\varphi_i(t) \leqslant \sum_{k=1}^{n} a_{ik}\varphi_k(t) + \lambda_i^n(t) , \qquad \varphi_i(0) \leqslant \gamma_i(0) , \qquad i = 1, \ldots, n .$$

Applying (2) and lemma 1 we thus get that

$$\varphi_i(t) \leqslant \gamma_i(t) \quad \text{for} \quad i = 1, \ldots, n , \quad 0 \leqslant t < a .$$

On the other hand

$$\gamma_i(t) = \int_0^t \sum_{k=1}^{n} \overset{n}{u}_{ik}(t-\tau)\lambda_k^n(\tau)\,d\tau .$$

Hence

$$\varphi_i(t) \leqslant \int_0^t \sum_{k=1}^{n} \overset{n}{u}_{ik}(t-\tau)\lambda_k^n(\tau)\,d\tau , \qquad i = 1, \ldots, n .$$

Notice now, that by the corollary $\overset{n}{u}_{ik}(t-\tau) \geqslant 0$ and obviously $\lambda_k^n(\tau) \leqslant \sigma^n(\tau)$. Therefore

$$\varphi_i(t) \leqslant \int_0^t \sum_{k=1}^{n} \overset{n}{u}_{ik}(t-\tau)\sigma_n(\tau)\,d\tau .$$

But (3) and lemma 2 imply that

$$\sum_{k=1}^{n} \overset{n}{u}_{ik}(t-\tau) \leqslant \psi_i(t-\tau) , \qquad i = 1, \ldots, n .$$

We have then

$$\varphi_i(t) \leqslant \int_0^t \psi_i(t-\tau)\sigma_n(\tau)\,d\tau , \quad \text{for} \quad i = 1, \ldots, n .$$

The limit passage in the above inequalities together with (5) proves the assertion.

2. This section deals with-non-linear inequalities. We assume in the following that the functions $f_i(t, y_1, y_2, \ldots, y_n, \ldots)$ $(i = 1, 2, \ldots)$ are defined for $t \in \langle 0, a)$ and for arbitrary real-valued sequences $y = \{y_k\}$. They are continuous in the following sense: for each i, if for every k, $y_k^\nu \underset{\nu\to\infty}{\to} y_k$ and $t^\nu \underset{\nu\to\infty}{\to} t$ then $f_i(t^\nu, y_1^\nu, y_2^\nu, \ldots, y_n^\nu, \ldots) \to f_i(t, y_1, y_2, \ldots, y_n, \ldots)$. The following condition generalizes the condition (W) of lemma 2:

(C) for every i, if $\overline{y}_k \leqslant \overline{\overline{y}}_k$ for $k \neq i$ then
$f_i(t, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_{i-1}, y_i, \overline{y}_{i+1}, \ldots) \leqslant f_i(t, \overline{\overline{y}}_1, \overline{\overline{y}}_2, \ldots, \overline{\overline{y}}_{i-1}, y_i, \overline{\overline{y}}_{i+1}, \ldots)$.

We begin with the following

THEOREM 2. *Suppose that the functions* $f_i(t, y_1, y_2, \ldots, y_n, \ldots)$ $(i = 1, 2, \ldots)$ *satisfy* (C). *We assume that there exist finite constants* $M_i > 0$ *such that*

$$|f_i(t, y_1, y_2, \ldots, y_n, \ldots)| \leqslant M_i, \quad i = 1, 2, \ldots \tag{6}$$

for $t \in \langle 0, a)$ *and* $y = \{y_k\}$ *arbitrary. Suppose we are given a sequence of functions* $\{\varphi_i(t)\}$ *which are continuous on* $\langle 0, a)$ *and satisfy on* $(0, a)$ *the following inequalities*

$$\overline{D}_+\varphi_i(t) \leqslant f_i\big(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots\big) . \tag{7}$$

Then there exists the solution $\{\omega_i(t)\}$ *in* $\langle 0, a)$ *of the infinite system*

$$\omega_i'(t) = f_i\big(t, \omega_1(t), \omega_2(t), \ldots, \omega_n(t), \ldots\big) , \quad i = 1, 2, \ldots \tag{8}$$

such that $\omega_i(0) = \varphi_i(0)$ *and*

$$\varphi_i(t) \leqslant \omega_i(t) \quad \text{for} \quad t \in \langle 0, a) \quad \text{and} \quad i = 1, 2, \ldots \tag{9}$$

Proof. Suppose that the continuous functions $\psi_i(t)$ satisfy on $(0, a)$ the inequalities

$$\overline{D}_+\psi_i(t) \leqslant f_i\big(t, \psi_1(t), \psi_2(t), \ldots, \psi_n(t), \ldots\big) , \quad i = 1, 2, \ldots \tag{10}$$

Let us consider the following differential equation

$$y' = F_i(t, y) \equiv f_i\big(t, \psi_1(t), \psi_2(t), \ldots, \psi_{i-1}(t), y, \psi_{i+1}(t), \ldots\big) . \tag{11}$$

Obviously F_i is continuous in (t, y). On the other hand (10) implies $\overline{D}_+\psi_i(t) \leqslant F_i\big(t, \psi_i(t)\big)$. Hence by lemma 1

$$\psi_i(t) \leqslant \psi_i^1(t) \quad \text{for} \quad t \in \langle 0, a) \tag{12}$$

where $\psi_i^1(t)$ is the right-hand maximum solution of (11) such that $\psi_i(0) = \psi_i^1(0)$. This maximum solution exists in the whole interval $\langle 0, a)$. This is an immediate consequence of the boundedness of f_i. We have also

$$\frac{d}{dt}\psi_i^1(t) = F_i\big(t, \psi_i^1(t)\big) . \tag{13}$$

By (12), (13) and condition (C) we get

$$\frac{d}{dt}\psi_i^1(t) \leqslant f_i\big(t, \psi_1^1(t), \psi_2^1(t), \ldots, \psi_n^1(t), \ldots\big) , \quad i = 1, 2, \ldots \tag{14}$$

We see now that to every sequence $\{\psi_k(t)\}$ of functions which satisfy (10) there corresponds a sequence $\{\psi_k^1(t)\}$ such that the conditions (12), (13) and (14) hold. We have just to do with a transformation law which maps $\psi = \{\psi_k(t)\}$ on the sequence $\psi^1 = \{\psi_k^1(t)\}$. Denote this transformation by F.

Hence $F\psi = \psi^1$. It follows from (14) that we can apply F to ψ^1, or more generally, that the sequence $\psi^{n+1} = F\psi^n$ is well defined. It is easy to prove that

$$\psi_i^n(t) \leqslant \psi_i^{n+1}(t) \tag{15}$$

and

$$\frac{d}{dt}\psi_i^n(t) = f_i\big(t, \psi_1^{n-1}(t), \ldots, \psi_{i-1}^{n-1}(t), \psi_i^n(t), \psi_{i+1}^{n-1}(t), \ldots\big) . \tag{16}$$

The inequalities $|\psi_i^n(t)| \leqslant M_i t + |\psi_i(0)|$, $\left|\frac{d}{dt}\psi_i^n(t)\right| \leqslant M_i$ show that for a fixed i the sequence $\{\psi_i^n(t)\}$ is equibounded and equicontinuous on every compact contained in $\langle 0, a)$. Hence, the limits $\lim_{n\to\infty} \psi_i^n(t) \overset{\text{df}}{=} \omega_i(t)$ exist and the convergence is almost uniform on $\langle 0, a)$. It follows from (16) that $\omega_i'(t) = f_i\big(t, \omega_1(t), \omega_2(t), \ldots, \omega_n(t), \ldots\big)$. By (15) we conclude that $\psi_i(t) \leqslant \omega_i(t)$ for $t \in \langle 0, a)$ and $i = 1, 2, \ldots$ Then assertion of our theorem follows if we put $\psi_i(t) = \varphi_i(t)$.

The supposed inequalities $|f_i| \leqslant M_i$ may be replaced by the following assumption: $|f_i(t, y_1, \ldots, y_i, \ldots)| \leqslant g_i(t, |y_i|)$, $g_i(t, z)$ are continuous and for an arbitrary initial value $\overset{0}{y}_i \geqslant 0$ the right-hand maximum solution of equation $y' = g_i(t, y)$ passing through the point $(0, \overset{0}{y}_i)$ exists in the whole interval $\langle 0, a)$.

It is easy to see that the method used in the proof of theorem 2 does not need the assumption that the considered systems are countable. Hence, in theorem 2 the countable systems may be replaced by an arbitrary infinite systems. However, in the case of countable systems theorem 2 can be proved by using following arguments: suppose that the sequence $\{\varphi_i(t)\}$ satisfies (7) and consider the finite system

$$y_i' = F_i^n(t, y_1, \ldots, y_n) \equiv f_i\big(t, y_1, y_2, \ldots, y_n, \varphi_{n+1}(t), \varphi_{n+2}(t), \ldots\big) , \quad i = 1, 2, \ldots, n . \tag{17}$$

Condition (C) implies that F_i^n satisfy contition (W) of lemma 1. Denote by $\overset{n}{\omega}_1(t), \ldots, \overset{n}{\omega}_n(t)$ the right-hand maximum solution of (17) such that $\overset{n}{\omega}_i(0) = \varphi_i(0)$. The inequalities (7) imply the following inequalities

$$\overline{D}_+\varphi_i(t) \leqslant F_i^n\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) , \quad i = 1, \ldots, n .$$

Hence, by lemma 1

$$\varphi_i(t) \leqslant \overset{n}{\omega}_i(t) \quad \text{for} \quad i = 1, 2, \ldots, n , \quad t \in \langle 0, a) . \tag{18}$$

Observe that by (C) and (18)

$$\overline{D}_+\varphi_{n+1}(t) \leqslant f_{n+1}\big(t, \overset{n}{\omega}_1(t), \ldots, \overset{n}{\omega}_n(t), \varphi_{n+1}(t), \varphi_{n+2}(t), \ldots\big) = F_{n+1}^{n+1}\big(t, \overset{n}{\omega}_1(t), \ldots, \overset{n}{\omega}_n(t), \varphi_{n+1}(t)\big) .$$

This inequality and the definition of $\overset{n}{\omega}_i(t)$ imply by lemma 1

$$\overset{n}{\omega}_i(t) \leqslant \overset{n+1}{\omega}_i(t)\,, \quad i = 1, \ldots, n\,, \quad \varphi_{n+1}(t) \leqslant \overset{n+1}{\omega}_{n+1}(t)\,.$$

Arguments similar to those used in the proof of theorem 2 show that the limits $\lim_{n\to\infty} \overset{n}{\omega}_i(t)$ are the components of a solution of system $y_i' = f_i(t, y_1, \ldots, y_n, \ldots)$ and obviously $\varphi_i(t) \leqslant \lim_{n\to\infty} \overset{n}{\omega}_i(t)$.

THEOREM 3. *Suppose that f_i satisfy* (C) *and* $|f_i| \leqslant M_i < +\infty$. *Then for every sequence $\overset{0}{y} = \{\overset{0}{y}_i\}$ there exists in $\langle 0, a)$ the right-hand maximum solution $\{\omega_i(t : \overset{0}{y})\}$ of the system*

$$y_i' = f_i(t, y_1, \ldots, y_n, \ldots)\,, \quad i = 1, 2, \ldots, n \tag{19}$$

such that $\omega_i(0 : \overset{0}{y}) = \overset{0}{y}_i$. If the functions $\varphi_i(t)$ are continuous on $\langle 0, a)$ and satisfy on $(0, a)$ the inequalities

$$\bar{D}_+\varphi_i(t) \leqslant f_i\big(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots\big)\,, \quad i = 1, 2, \ldots$$

then $\varphi_i(t) \leqslant \omega_i\big(t : \varphi(0)\big)$ $(\varphi(0) = \{\varphi_i(0)\})$ for $i = 1, 2, \ldots$ and $t \in \langle 0, a)$.

Proof. The functions $\psi_i(t) = -M_i t + \overset{0}{y}_i$ satisfy (10). It follows from theorem 2 that there exists on $\langle 0, a)$ at least one solution $\{\omega_i(t)\}$ of (19) such that $\omega_i(0) = \overset{0}{y}_i$. Denote by Ω_i the set of i-th components of solutions of (19) passing through $(0, \overset{0}{y})$. We define now

$$\omega_i(t : \overset{0}{y}) = \sup_{\omega \in \Omega_i} \omega(t)\,. \tag{20}$$

The functions $\omega \in \Omega_i$ are equibounded and equicontinuous in every compact subinterval of $\langle 0, a)$. We get therefore that $\omega_i(t : \overset{0}{y})$ is continuous in t on $\langle 0, a)$. Let $\{\omega_i(t)\}$ be an arbitrary solution of (19) such that $\omega_i(0) = \overset{0}{y}_i$. We have

$$\omega_i(t) = f_i\big(t, \omega_1(t), \ldots, \omega_n(t), \ldots\big)\,, \quad i = 1, 2, \ldots$$

and by (20)

$$\omega_i'(t) \leqslant f_i\big(t, \omega_1(t : \overset{0}{y}), \ldots, \omega_{i-1}(t : \overset{0}{y}), \omega_i(t), \omega_{i+1}(t : \overset{0}{y}), \ldots\big)\,. \tag{21}$$

It follows from (21) that

$$\omega_i(t) \leqslant \sigma_i(t)\,, \quad i = 1, 2, \ldots, \quad t \in \langle 0, a) \tag{22}$$

where $\sigma_i(t)$ is the right-hand maximum solution of the equation

$$y' = f_i\big(t, \omega_1(t:\overset{0}{y}), \ldots, \omega_{i-1}(t:\overset{0}{y}), y, \omega_{i+1}(t:\overset{0}{y}), \ldots\big) \tag{23}$$

such that $\sigma_i(0) = \overset{0}{y}_i$. The inequality (22) holds for an arbitrary solution. We get therefore

$$\omega_i(t:\overset{0}{y}) \leqslant \sigma_i(t) \tag{24}$$

and consequently by (20) and (C)

$$\sigma_i'(t) \leqslant f_i\big(t, \sigma_1(t), \ldots, \sigma_n(t), \ldots\big) .$$

By theorem 2 there exists a solution $\{\tau_i(t)\}$ of (19) such that $\tau_i(0) = \overset{0}{y}_i$ and

$$\sigma_i(t) \leqslant \tau_i(t) . \tag{25}$$

But $\tau_i(t) \leqslant \omega_i(t:\overset{0}{y})$ and by (24) and (25) we derive $\tau_i(t) = \omega_i(t:\overset{0}{y})$. We have just proved that $\{\omega_i(t:\overset{0}{y})\}$ is a solution of (19). It follows from (20) that this solution is the right-hand maximum one. The second part of the assertion follows easily from theorem 2 and from (20).

References

[1] G. E. H. Reuter and W. Ledermann, *On the differential equations for the transition probabilities of Markov processes with enumerably many states*, Proc. Cambridge Phil. Soc. 49 (2) (1953), p. 247-262.

[2] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Ann. Soc. Pol. Math. 23 (1950), p. 112-156.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 5. 10. 1960