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Fractional iteration of differentiable functions

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§ 1. In the present paper we shall be concerned with differentiable solutions φ of the functional equation

$$\varphi^2(x) = g(x)\,, \tag{1}$$

where g is a given function and φ^2 denotes the second functional iterate of φ. (More generally, for a given function f we denote by f^n its n-th iterate: $f^0(x) = x$, $f^{n+1}(x) = f[f^n(x)]$, $n = 0, 1, 2, \ldots$) These investigations have been originated by the following problem of Moszner [4]:

Let D_k^r, $1 \leqslant r \leqslant +\infty$, denote the class of all mappings

$$f\colon R^k \to R^k$$

that are of class C^r in the whole R^k and have a positive Jacobian in the whole R^k. Does there exist, for every $g \in D_k^r$, a solution φ of equation (1) belonging to the class D_k^r?

As we shall see, the answer to this question is negative.

§ 2. In the present section we exhibit an example of a function $g \in D_2^\infty$ for which equation (1) has no solution at all. We start with a lemma [1].

LEMMA 1. *Let E be an arbitrary set and g arbitrary function on E taking values in E. Further suppose that there exist in E points $a \neq b$ such that $g(a) = b$, $g(b) = a$, and $g^2(x) = x$ implies that either $x = a$, or $x = b$, or $g(x) = x$. Then equation (1) has no solution in E.*

Proof. Suppose that a function φ, defined in E, satisfies equation (1) in E, and put $c = \varphi(a)$, $d = \varphi(b)$. Then $g^2(c) = c$ and $g^2(d) = d$. If $c = a$, i.e. $\varphi(a) = a$, then $b = g(a) = \varphi^2(a) = a$, contrary to the assumption. If $c = b$, i.e. $\varphi(a) = b$, then $\varphi(b) = \varphi^2(a) = g(a) = b$ and $a = g(b) = \varphi^2(b) = b$, contrary to the assumption. Lastly, $g(c) = c$ implies $\varphi(b) = \varphi[g(a)] = \varphi^3(a) = g[\varphi(a)] = g(c) = c$, i.e. $\varphi(a) = \varphi(b)$, whence $g(a) = g(b)$ and $b = a$, again a contradiction.

[1] This lemma is implied by the general considerations of Isaacs [1]. (Cf. also [3], theorem 15.6.) But we sketch here a proof because of its simplicity.

Now we proceed to give the example announced.

EXAMPLE I. We consider the transform $g\colon R^2 \to R^2$ given by

$$g\colon \begin{cases} x' = -\tfrac{1}{2}(x^2+y^2+1)(x+y)\,, \\ y' = -\tfrac{1}{2}(x^2+y^2+1)y\,. \end{cases} \tag{2}$$

Let us write shortly $u = u(x, y) = \frac{1}{2}(x^2+y^2+1)$. Then $u_x = x$, $u_y = y$ and

$$\begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} -x(x+y)-u & -y(x+y)-u \\ -xy & -y^2-u \end{vmatrix}$$

$$= (x^2+xy+u)(y^2+u)-xy(xy+y^2+u) = u(x^2+y^2+u) \geqslant \tfrac{1}{4}\,.$$

Consequently $g \in D_2^\infty$. For $a = (1, 0)$ and $b = (-1, 0)$ we have $g(a) = b$ and $g(b) = a$. The equation $g^2(x, y) = (x, y)$ has the form

$$-u'(x'+y') = x\,, \qquad -u'y' = y\,, \tag{3}$$

where

$$u' = u(x', y') = \tfrac{1}{2}[(x')^2+(y')^2+1] = \tfrac{1}{2}+\tfrac{1}{2}u^2(x^2+2xy+2y^2)\,.$$

Thus (3) may be written as

$$u'u(x+2y) = x\,, \qquad u'uy = y\,. \tag{4}$$

If $u'u \neq 1$, then from the second equation of (4) we obtain $y = 0$, and if $u'u = 1$, then $y = 0$ results from the first equation of (4). Consequently y must be zero, $u'u$ reduces to

$$u'u = \left(\frac{1}{2}+\frac{x^2}{2}\left(\frac{x^2+1}{2}\right)^2\right)\frac{x^2+1}{2} = \frac{1}{16}\left(4+x^2(x^2+1)^2\right)(x^2+1)\,,$$

and system (4) to

$$u'ux = x\,.$$

If $u'u \neq 1$, then $x = 0$, and consequently we get the solution $(x, y) = (0, 0)$, which is a fixed point of g: $g(0, 0) = (0, 0)$. To solve the equation $u'u = 1$ we put $t = x^2+1$ and thus the equation becomes

$$t^4-t^3+4t-16 = 0\,,$$

i.e.

$$(t-2)(t^3+t^2+2t+8) = 0\,. \tag{5}$$

But, since $t = x^2+1 \geqslant 1$, we have $t^3+t^2+2t+8 \geqslant 12$ and consequently $t = 2$ is the only admissible solution of (5). It leads us to $x = 1$

and $x = -1$, i.e. $a = (1, 0)$ and $b = (-1, 0)$ are the only fixed points of g^2 that are not fixed points of g.

As we see, the function g given by (2) fulfils the assumptions of Lemma 1, and thus equation (1) has no solution in R^2.

§ 3. A similar situation cannot happen if $k = 1$. Equation (1) with a continuous and strictly increasing function g on $(-\infty, +\infty)$ always has a continuous and strictly increasing solution φ (cf. [3], corollary to theorem 15.7). In particular, we have the following

LEMMA 2 ([2], [3]). *Let $g(x)$ be continuous and strictly increasing in an interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, moreover, let $a < g(x) < x$ in (a, b).*

Further let x_0, y_0 be arbitrary two points of (a, b) such that $g(x_0) < y_0 < x_0$, and let $\varphi_0(x)$ be an arbitrary continuous and strictly increasing function on $\langle y_0, x_0\rangle$ such that

$$\varphi_0(x_0) = y_0, \quad \varphi_0(y_0) = g(x_0). \tag{6}$$

Then there exists a unique continuous and strictly increasing function $\varphi(x)$ on (a, b) satisfying equation (1) in (a, b) and such that

$$\varphi(x) = \varphi_0(x) \quad \textit{for } x \in \langle y_0, x_0\rangle. \tag{7}$$

As an immediate consequence of Lemma 2 we obtain the following

LEMMA 3. *If the function $g(x)$ fulfils the conditions of Lemma 2, then every continuous and strictly increasing solution $\varphi(x)$ of equation (1) in (a, b) is completely determined by its values in an arbitrary right neighbourhood $(a, a+\delta)$ of the point a.*

Now we shall prove an analogue of Lemma 2 for differentiable functions.

THEOREM 1. *Under conditions of Lemma 2, if, moreover, $g(x)$ is of class C^1 in (a, b) with $g'(x) > 0$ in (a, b), and $\varphi_0(x)$ is of class C^1 in $\langle y_0, x_0\rangle$ with $\varphi_0'(x) > 0$ in $\langle y_0, x_0\rangle$, and if* ([2])

$$\varphi_0'(y_0) = \frac{g'(x_0)}{\varphi_0'(x_0)}, \tag{8}$$

then the continuous and strictly increasing solution $\varphi(x)$ of equation (1) fulfilling (7) is of class C^1 in (a, b).

([2]) $\varphi_0'(x_0)$ and $\varphi_0'(y_0)$ denote here the left derivative and the right derivative of φ_0 at the points x_0 and y_0, respectively.

Proof. Let the sequence x_n be defined for integral n (³) by

$$x_n = \varphi^n(x_0) . \tag{9}$$

It is readily seen that the sequence x_n is strictly decreasing and

$$(a, b) = \bigcup_{n=n_0}^{\infty} \langle x_{n+1}, x_n) , \tag{10}$$

where $n_0 = -\infty$ if $\lim_{x \to b-0} g(x) = b$, and if $\lim_{x \to b-0} g(x) = b' < b$, then n_0 is determined by the condition $x_{n_0+1} \in \langle b', b)$ and we put $x_{n_0} \overset{\mathrm{df}}{=} b$. At any case $n_0 < 0$.

By (6) and (9)

$$\varphi_0^{-1}(x) \in (y_0, x_0) \quad \text{for } x \in (x_2, x_1) = \big(g(x_0), y_0\big) .$$

In virtue of (1) and (7) we get hence for $x \in \big(g(x_0), y_0\big)$

$$\varphi(x) = \varphi\Big(\varphi_0\big(\varphi_0^{-1}(x)\big)\Big) = \varphi\Big(\varphi\big(\varphi_0^{-1}(x)\big)\Big) = g[\varphi_0^{-1}(x)] ,$$

which shows that φ is of class C^1 in $\big(g(x_0), y_0\big) = (x_2, x_1)$. Now

$$\lim_{x \to y_0+0} \varphi'(x) = \lim_{x \to y_0+0} \varphi_0'(x) = \varphi_0'(y_0) ,$$

since φ_0 is of class C^1 in $\langle y_0, x_0 \rangle$, and

$$\lim_{x \to y_0-0} \varphi'(x) = \lim_{x \to x_0-0} \varphi'[\varphi(x)] = \lim_{x \to x_0-0} \frac{g'(x)}{\varphi'(x)} = \lim_{x \to x_0-0} \frac{g'(x)}{\varphi_0'(x)} = \frac{g'(x_0)}{\varphi_0'(x_0)} = \varphi_0'(y_0) ,$$

since φ satisfies equation (1) and φ_0 is of class C^1 in $\langle y_0, x_0 \rangle$ and fulfils (8). This proves, in view of the continuity of φ, that φ' exists and is continuous at $x = x_0$. Consequently φ is of class C^1 in (x_2, x_0).

Now suppose that φ is of class C^1 in (x_n, x_0) for an $n \geqslant 2$. According to (9) $\varphi(x_i) = x_{i+1}$ for all $i > n_0$, and consequently

$$\varphi^{-1}(x) \in (x_n, x_{n-2}) \quad \text{for } x \in (x_{n+1}, x_{n-1}) .$$

Furthermore, φ^{-1} is of class C^1 in (x_{n+1}, x_{n-1}), since differentiating (1) we get $\varphi'(x) \neq 0$ for $x \in (x_n, x_{n-2})$. Thus, for $x \in (x_{n+1}, x_{n-1})$,

$$\varphi(x) = \varphi\Big(\varphi\big(\varphi^{-1}(x)\big)\Big) = g[\varphi^{-1}(x)]$$

is of class C^1, and consequently φ is of class C^1 in (x_{n+1}, x_0). Induction now yields that φ is of class C^1 in (x_{n+1}, x_0) for every $n \geqslant 0$, and quite similarly one can prove that φ is of class C^1 in (x_1, x_{-n}) for every $n \leqslant -n_0$. By (10) φ is of class C^1 in (a, b), which was to be proved.

(³) Positive and negative. For negative n, φ^n denotes the inverse function to φ^{-n}.

The following lemma is easily established by induction.

LEMMA 4. *If a function φ is of class C^r in (a, b) and takes values in (a, b), then*

$$\frac{d^j}{dx^j}\varphi^2(x) = \sum_{i=1}^{j} P_{ij}\big(\varphi'(x), \ldots, \varphi^{(j)}(x)\big)\varphi^{(i)}[\varphi(x)]\,, \quad j = 1, \ldots, r\,,$$

where $P_{ij}(t_1, \ldots, t_j)$ are polynomials.

By almost the same argument as in the proof of Theorem 1 one can prove the following

THEOREM 2. *Under conditions of Lemma 2, if, moreover, $g(x)$ is of class C^r in (a, b), $1 \leqslant r \leqslant \infty$, with $g'(x) > 0$ in (a, b), and $\varphi_0(x)$ is of class C^r in $\langle y_0, x_0\rangle$ with $\varphi_0'(x) > 0$ in $\langle y_0, x_0\rangle$, and if* (4)

$$\sum_{i=1}^{j} P_{ij}\big(\varphi_0'(x_0), \ldots, \varphi_0^{(j)}(x_0)\big)\varphi_0^{(i)}(y_0) = g^{(j)}(x_0)\,, \quad j = 1, \ldots, r\,,$$

where P_{ij} are the polynomials occurring in Lemma 4, then the continuous and strictly increasing solution $\varphi(x)$ of equation (1) *fulfilling* (7) *is of class C^r in (a, b).*

The following result is now an easy consequence of Theorem 2.

THEOREM 3. *If $g \in D_1^r$, $1 \leqslant r \leqslant \infty$, and $g(x) \neq x$ in $(-\infty, +\infty)$, then equation* (1) *has a solution $\varphi \in D_1^r$.*

Proof. The condition $g(x) \neq x$ implies that either $g(x) < x$ in $(-\infty, +\infty)$, or $g(x) > x$ in $(-\infty, +\infty)$. In the former case Theorem 2 with $(a, b) = (-\infty, +\infty)$ shows that equation (1) has a strictly increasing solution φ of class C^r in $(-\infty, +\infty)$. Since $\varphi'[\varphi(x)]\varphi'(x) = g'(x) \neq 0$, φ must belong to D_1^r.

If $g(x) > x$ in $(-\infty, +\infty)$, then $f(x) = -g(-x) < x$ in $(-\infty, +\infty)$ and f belongs to D_1^r. Thus there exists a $\psi \in D_1^r$ such that $\psi^2(x) = f(x)$ in $(-\infty, +\infty)$. The function $\varphi(x) = -\psi(-x)$ belongs to D_1^r and satisfies equation (1) in $(-\infty, +\infty)$.

§ 4. The conclusion of Theorem 3 fails to hold if g has fixed points in $(-\infty, +\infty)$, as may be seen from examples IV and V in § 5. In the present section we shall consider equation (1) in an interval $\langle a, b)$, $-\infty < a < b \leqslant +\infty$, under the condition that $a < g(x) < x$ in (a, b), $g'(x)$ exists, is continuous and positive in $\langle a, b)$, in particular (5)

$$g'(a) = s > 0\,. \tag{11}$$

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(4) The derivatives of φ_0 at x_0 and y_0 denote the left and right derivatives, respectively.

(5) $g'(a)$ denotes the right derivative of g at a.

Suppose that equation (1) has a strictly increasing solution φ that is of class C^1 in $\langle a, b)$. Differentiating (1) we obtain

$$\varphi'[\varphi(x)]\varphi'(x) = g'(x) . \tag{12}$$

The function $\varphi(x)$ fulfils the condition (cf. [5])

$$g(x) < \varphi(x) < x \quad \text{in } (a, b) ,$$

whence it follows in view of the continuity of φ and g at a that

$$\varphi(a) = a . \tag{13}$$

Thus we obtain by (12) and (11)

$$\varphi'(a) = \sqrt{s} . \tag{14}$$

Replacing in (12) x by $\varphi(x)$ we get in view of (1)

$$\varphi'[g(x)]\varphi'[\varphi(x)] = g'[\varphi(x)] ,$$

whence by (12)

$$\frac{\varphi'(x)}{\varphi'[g(x)]} = \frac{g'(x)}{g'[\varphi(x)]} . \tag{15}$$

By (1) $\varphi[g^i(x)] = \varphi^{2i+1}(x) = g^i[\varphi(x)]$. Consequently we obtain by (15)

$$\frac{\varphi'[g^i(x)]}{\varphi'[g^{i+1}(x)]} = \frac{g'[g^i(x)]}{g'[g^i(\varphi(x))]} , \qquad i = 0, 1, 2, \ldots ,$$

whence

$$\frac{\varphi'(x)}{\varphi'[g^n(x)]} = \prod_{i=0}^{n-1} \frac{g'[g^i(x)]}{g'[g^i(\varphi(x))]} , \qquad n = 1, 2, 3, \ldots \tag{16}$$

As $n \to \infty$, we obtain from (16) and (14) for $x \in (a, b)$

$$\varphi'(x) = \sqrt{s} \prod_{i=0}^{\infty} \frac{g'[g^i(x)]}{g'[g^i(\varphi(x))]} . \tag{17}$$

(17) shows that for $y = \varphi(x)$ the infinite product

$$G(x, y) = \prod_{i=0}^{\infty} \frac{g'[g^i(x)]}{g'[g^i(y)]} \tag{18}$$

must converge and the function φ satisfies in (a, b) the differential equation

$$\varphi' = \sqrt{s}\, G(x, \varphi) . \tag{19}$$

Thus we have proved the following

THEOREM 4. *If $g(x)$ is of class C^1 in $\langle a, b)$, $a < g(x) < x$ in (a, b) and $g'(x) > 0$ in $\langle a, b)$, and if $\varphi(x)$ is a strictly increasing solution of equation* (1) *in $\langle a, b)$, of class C^1 in $\langle a, b)$, then φ satisfies in (a, b) the differential equation* (19), *where s and G are given by* (11) *and* (18), *respectively.*

A more detailed discussion of equation (19) as well as of problems of the uniqueness and existence connected therewith is deferred to a future publication.

§ 5. We conclude the paper with a few examples.

EXAMPLE II. Let $g(x) = s(x-a)+a$, $0 < s < 1$, $x \in \langle a, b)$. Then $g'(x) \equiv s$ and $G(x, y) \equiv 1$. Equation (19) becomes $\varphi' = \sqrt{s}$ which together with (13) implies that $\varphi(x) = \sqrt{s}(x-a)+a$.

If we have $g(x) = s(x-a)+a$ in an interval $(c, a\rangle$, $-\infty \leqslant c < a < +\infty$, and φ is a strictly increasing solution of equation (1), of class C^1 in $(c, a\rangle$, then the function $\psi(x) = 2a-\varphi(2a-x)$ is a strictly increasing solution of the equation

$$\psi^2(x) = 2a-g(2a-x) ,$$

of class C^1 in the interval $\langle a, 2a-c)$. But, since $2a-g(2a-x) = s(x-a)+a$, we must have $\psi(x) = \sqrt{s}(x-a)+a$ in $\langle a, 2a-c)$, whence also $\varphi(x) = \sqrt{s}(x-a)+a$ in $(c, a\rangle$.

If $s > 1$ and $\varphi(x)$ is a solution of equation (1) with $g(x) = s(x-a)+a$, of class C^1 in an interval I containing a, then $\psi(x) = \varphi^{-1}(x)$ is a solution of the equation

$$\psi^2(x) = s^{-1}(x-a)+a ,$$

of class C^1 in $\varphi(I)$. Consequently $\psi(x) = \left(\sqrt{s}\right)^{-1}(x-a)+a$ in $\varphi(I)$ and $\varphi(x) = \sqrt{s}(x-a)+a$ in I.

If $s = 1$, then $g(x) = x$ and $\varphi(x) = x$ is the only increasing solution of equation (1) in any interval I ([3], theorem 15.2).

Thus we arrive at the following conclusion:

THEOREM 5. *If*

$$g(x) = s(x-a)+a , \quad s > 0 ,$$

in an interval $I \subset (-\infty, +\infty)$, where $a \in I$, then

$$\varphi(x) = \sqrt{s}(x-a)+a$$

is the only strictly increasing solution of equation (1) *of class C^1 in I.*

EXAMPLE III. Let $\langle a, b) = \langle 0, \infty)$, $g(x) = x/(x+1)$. Then $g'(x) = 1/(x+1)^2$, $g^n(x) = x/(nx+1)$, $s = 1$, and

$$G(x, y) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \frac{\left(1+\dfrac{y}{iy+1}\right)^2}{\left(1+\dfrac{x}{ix+1}\right)^2} = \lim_{n\to\infty} \frac{(ny+1)^2}{(nx+1)^2} = \frac{y^2}{x^2}.$$

Equation (19) becomes

$$\varphi' = [\varphi]^2/x^2$$

and has the solutions $\varphi(x) \equiv 0$ and $\varphi(x) = x/(cx+1)$. The former evidently does not satisfy (1) and inserting the latter into (1) we see that c must be equal to $\frac{1}{2}$. Thus finally we get

$$\varphi(x) = \frac{x}{\frac{1}{2}x+1}.$$

Note that in this case not every solution of equation (19) satisfies equation (1).

EXAMPLE IV. Let us fix arbitrary s, s' such that

$$0 < s < 1 < s',$$

and put

$$u = \frac{1}{2}\,\frac{s'-1}{s'-\sqrt{s}}, \qquad v = 1 - \frac{1-\sqrt{s}\,u}{s'}.$$

Then

$$v - u = \frac{1}{s'}\left[(s'-1) - \left(s'-\sqrt{s}\right)u\right] = \frac{s'-1}{2s'} > 0,$$

and thus

$$0 < u < v < 1.$$

Let $h(x)$ be an arbitrary (but fixed) convex function, of class C^1 in $\langle u, v\rangle$, and fulfilling the following conditions:

$$h(u) = su, \quad h(v) = \sqrt{s}\,u, \quad h'(u) = s, \quad h'(v) = s'.$$

Such a function surely exists, since

$$h(v) - h(u) = \left(\sqrt{s} - s\right)u = \frac{1}{2}(s'-1)\frac{\sqrt{s}-s}{s'-\sqrt{s}}$$

$$< \frac{1}{2}(s'-1)\frac{\sqrt{s}-s}{1-\sqrt{s}} < \frac{s'-1}{2} = s'(v-u).$$

We define the function $g(x)$ in $(-\infty, +\infty)$ as follows (cf. Fig. 1):

$$(20) \qquad g(x) = \begin{cases} sx & \text{for } x \in (-\infty, u), \\ h(x) & \text{for } x \in \langle u, v\rangle, \\ s'(x-1)+1 & \text{for } x \in (v, +\infty). \end{cases}$$

The function g thus defined belongs to the class D_1^1, and if we choose $h(x)$ more thoroughly we may make g to belong to D_1^r with any $r \geqslant 1$. Moreover, let us note that, since h is convex on $\langle u, v\rangle$, we have

(21) $$s \leqslant g'(x) \leqslant s' .$$

We shall show that equation (1) with g defined by (20) has not a solution $\varphi \in D_1^1$. Supposing the contrary, let φ be such a solution. By Theorem 5

(22) $$\varphi(x) = \sqrt{s}\,x \quad \text{in } (-\infty, u) .$$

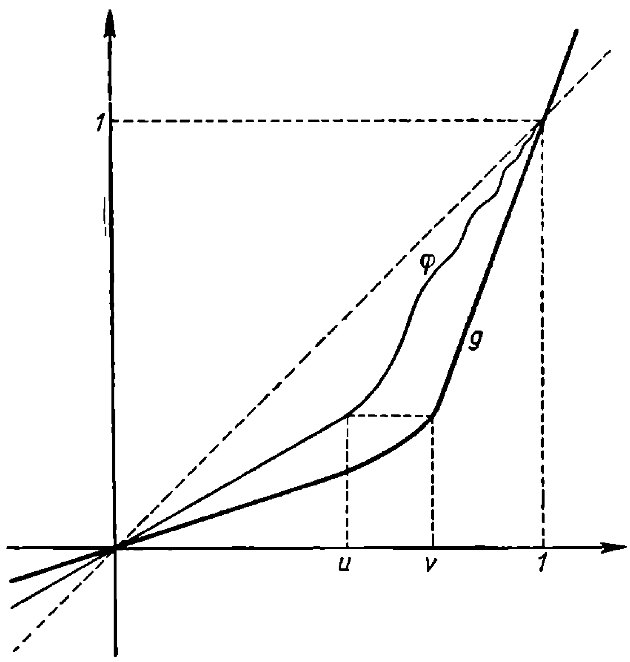

Fig. 1

In virtue of Lemma 3 the function $\varphi(x)$ is completely determined by (22) in $(0, 1)$. But it has been proved in [5] that $\lim_{x\to 1-0} \varphi'(x)$ does not exist. Consequently φ cannot be of class C^1 in $(-\infty, +\infty)$.

In the above example, though equation (1) has no solution of class C^1 in the whole $(-\infty, +\infty)$ (⁶), nevertheless every point of $(-\infty, +\infty)$ has a neighbourhood in which equation (1) has a local solution of class C^1. (The function indicated on Fig. 1 is of class C^1 in $(-\infty, 1)$, and a similar construction can be carried out in the interval $(0, +\infty)$.) But we can

(⁶) The argument presented shows that equation (1) has no strictly increasing solution of class C^1 in $(-\infty, +\infty)$; but every solution of (1) must be invertible ([3], lemma 15.1) and (1) cannot have strictly decreasing solutions, since function (20) has an even number of fixed points ([3], theorem 15.10).

modify this example as to obtain an equation that has not even a local solution of class C^1 in a neighbourhood of zero.

EXAMPLE V. Function (20) occurring in example IV depends on s and s'. (For given s and s' we consider the function $h(x)$ as fixed.) To make this dependence more explicit we shall denote function (20) restricted to the interval $\langle 0, 1\rangle$ by $g^*(s, s'; x)$.

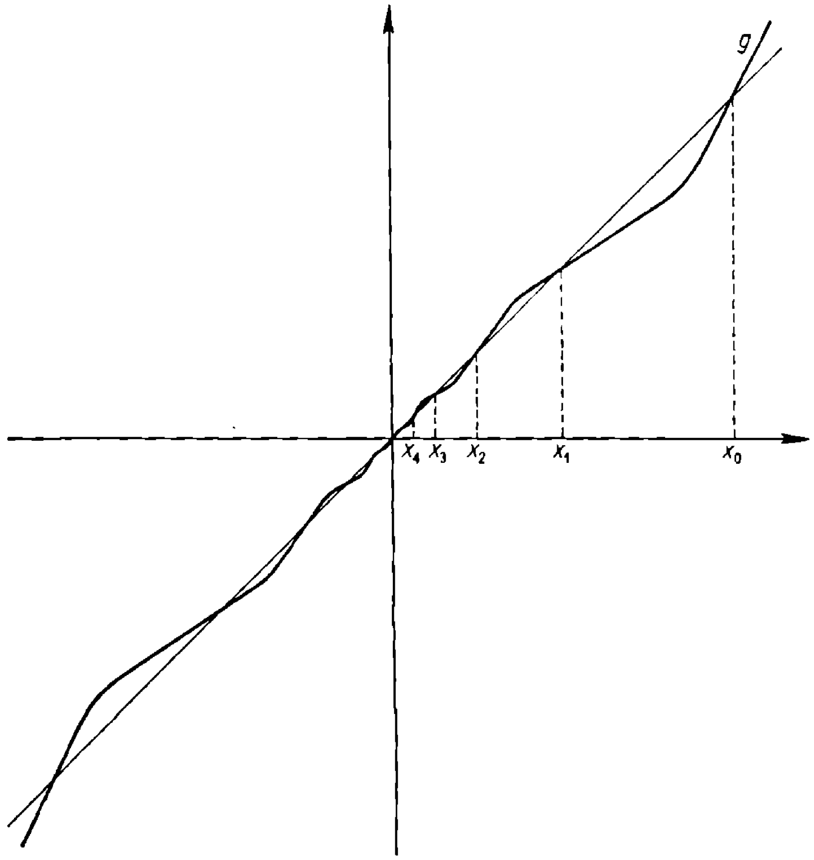

Fig. 2

We put $x_n = 2^{-n}$, $s_n = 2^{-(2^{-n})}$, $n = 0, \pm 1, \pm 2, \ldots$ These sequences are strictly monotonic and

$$\lim_{n\to+\infty} x_n = 0\,, \quad \lim_{n\to-\infty} x_n = +\infty\,,$$

$$\lim_{n\to+\infty} s_n = 1\,, \quad \lim_{n\to-\infty} s_n = 0\,.$$

For every n we define the function $g_n(x)$ on $\langle x_{n+1}, x_n\rangle$ by

$$g_n(x) = x_{n+1} + (x_n - x_{n+1})\, g^*\left(s_{n+1}, s_n^{-1}; \frac{x - x_{n+1}}{x_n - x_{n+1}}\right).$$

In view of (21) we have

$$s_{n+1} \leqslant g_n'(x) \leqslant s_n^{-1} \quad \text{for } x \in \langle x_{n+1}, x_n\rangle\,. \tag{23}$$

Now we define the function $g(x)$ on $\langle 0, +\infty)$ as follows:

$$(24)\qquad g(x) = \begin{cases} g_n(x) & \text{for } x \in (x_{n+1}, x_n\rangle\,,\ n = 0, \pm 2, \pm 4, \ldots\,, \\ g_n^{-1}(x) & \text{for } x \in (x_{n+1}, x_n\rangle\,,\ n = \pm 1, \pm 3, \ldots\,, \\ 0 & \text{for } x = 0\,, \end{cases}$$

and we extend $g(x)$ onto $(-\infty, +\infty)$ by the condition that it is odd:

$$(25)\qquad g(x) = -g(-x)$$

(cf. Fig. 2). Function (24) clearly is of class C^1 in $(-\infty, 0) \cup (0, +\infty)$. But condition (23) together with (25) show that

$$(26)\qquad \lim_{n\to\infty} g'(x) = 1\,.$$

Since $g(x)$ is continuous, condition (26) implies that $g'(0)$ exists and equals 1. Thus g is of class C^1 in $(-\infty, +\infty)$ and in fact $g \in D_1^1$ (cf. in particular (23) and (25)).

It follows from what has been shown in example IV that equation (1) with the function g given by (24) cannot have a strictly increasing solution φ of class C^1 in $\langle x_{n+1}, x_n\rangle$ (for any n). Consequently equation (1) cannot have a strictly increasing solution of class C^1 in any neighbourhood of the origin.

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