ON COVERING OF BOUNDED SETS
BY SETS WITH THE TWICE LESS DIAMETER

BY

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It is proved that every bounded plane set $A$ is the union of 7 sets with diameters less than or equal to $\frac{1}{4} \delta(A)$ and that there exist bounded plane sets $A$ which are not unions of 6 sets with diameters less than or equal to $\frac{1}{4} \delta(A)$.

1. For every $n = 1, 2, \ldots$ and for $0 < a \leq 1$ denote by $D_n(a)$ the smallest natural number $q$ such that every bounded set $A$ lying in the $n$-dimensional Euclidean space $E^n$ is covered by $q$ sets $A_1, \ldots, A_q$ with diameters $\delta(A_i) \leq a \delta(A)$ for $i = 1, \ldots, q$ (cf. [2]).

Since the geometric sphere $S^{n-1} \subset E^n$ with radius $r$ cannot be covered by less than $n$ sets with diameters less than $2r$ ([3], p. 178; see also [4]), we infer that

\[(1) \quad D_n(a) > n \quad \text{for every } 0 < a < 1 \text{ and } n = 1, 2, \ldots\]

On the other hand, one shows easily ([2], p. 249) that

\[(2) \quad D_n(a) \geq m_1 m_2 \ldots m_n,\]

where $m_1, \ldots, m_n$ are natural numbers such that

\[m_1^{-2} + m_2^{-2} + \ldots + m_n^{-2} \leq a^2.\]

Formulas (1) and (2) give an evaluation of $D_n(a)$. However, this evaluation is far from being satisfactory and the problem to compute the exact value of $D_n(a)$ remains open and seems to be hard.

The aim of the present note is to establish the following

Theorem. The number $D_2(\frac{1}{4})$ is equal to 7.

2. First let us show that

\[(3) \quad D_2\left(\frac{1}{2}\right) \leq 7.\]
It is evident that inequality (3) will be established if we show that every compact set $A \subset E^2$ with diameter 1 can be covered by 7 compact sets $A_1, \ldots, A_7$ such that $\delta(A_i) \leq \frac{1}{3}$ for $i = 1, \ldots, 7$.

It is known (see [1], p. 9) that in $E^2$ there exists a regular hexagon $P_6$ with diameter $\delta(P_6) = \frac{2}{3}\sqrt{3}$ containing $A$. Let $a_1, \ldots, a_6$ be vertices of $P_6$ (in a cyclic order) (see Fig. 1) and let $c$ denote the center of $P_6$, and $c_i$ — the center of the segment $a_ia_{i+1}$ for $i = 1, \ldots, 6$ (where $a_7 = a_1$). Moreover, let $c_i'$ denote the center of the segment $cc_i$. Let $T_i$ denote the regular triangle with vertices $c_i, c_i', c_{i+1}$, and $T_i'$ — the triangle with vertices $c_i, a_{i+1}, a_{i+1}$ (where $a_7 = a_1$). Denote by $A_7$ the circular disk with center $c$ and radius $g(c, c_i') = \frac{1}{3}$ and let

$$A_i = (T_i \cup T_i') \setminus A_7 \quad \text{for } i = 1, \ldots, 6.$$  

Then $A \subset P_6 = A_1 \cup \ldots \cup A_7$ and one sees easily that $\delta(A_i) = \frac{1}{3}$ for $i = 1, \ldots, 7$. Thus inequality (3) is proved.

![Fig. 1](image)

3. In order to complete the proof of the Theorem, it remains to show that

$$D_6\left(\frac{1}{2}\right) \geq 7,$$

that is to show that there exists a set $A \subset E^2$ with diameter 1, for which every covering consisting of 6 sets contains at least one set with diameter greater than $\frac{1}{3}$.

Consider a regular heptagon $P_7 \subset E^2$ with diameter 1 and let $c$ denote its barycenter, and $B$ — its boundary. Let $a_1, \ldots, a_7$ be vertices of $P_7$ in a cyclic order (see Fig. 2) and let us set, for every integer $k$,  

$$a_{i+7k} = a_i \quad \text{for } i = 1, \ldots, 7.$$
Then
\[ 1 = \delta(P_i) = \varrho(a_i, a_{i+1}) < 2\varrho(a_i, c) \quad \text{for every } i. \]

If \( c' \) denotes the center of the segment \( a_1a_4 \), then
\[ \varrho(a_1, c) > \varrho(a_1, c') = \frac{1}{2}. \]

The segments \( a_ia_{i+1} \) are called sides of \( P_i \). Let \( c_i \) denote the center of \( a_ia_{i+1} \). Two distinct sides of \( P_i \) are said to be adjacent one to the other if they have a common vertex. They are said to be opposite one to the other if there exists no side adjacent to each of them.

Let \( a'_i \) denote the point of \( a_ia_{i+1} \) such that \( \varrho(a'_i, a_{i-1}) = \frac{1}{2} \), and \( a''_i \) — the point of \( a_ia_{i-1} \) such that \( \varrho(a''_i, a_{i+1}) = \frac{1}{2} \).

If \( x, y \) are two distinct points of \( B \), then \( B \) is the union of two arcs with endpoints \( x, y \). If the lengths of those arcs are not equal, then we denote by \( (x, y) \) the shorter of those arcs.

Suppose now, contrary to (4), that there exists a covering of \( P_i \) consisting of 6 sets \( A_1, \ldots, A_6 \) with diameters less than or equal to \( \frac{1}{3} \). We may assume that the sets \( A_i \) (\( i = 1, \ldots, 6 \)) are closed and convex and that the center \( c \) of \( P_i \) belongs to \( A_4 \). Then (5) implies that \( A_4 \) does not contain any vertex \( a_i \) of \( P_i \). Now, let us distinguish two cases:
Case I. Each of the sets $A_1, \ldots, A_5$ contains at least one vertex of $P_7$.

Case II. One of the sets $A_1, \ldots, A_5$ (say $A_i$) does not contain any vertex of $P_7$.

It remains to show that in both cases our hypotheses lead to a contradiction.

Consider first the case I. Since $\varrho(a_i, a_{i+2}) > \frac{1}{2}$, none of the sets $A_1, \ldots, A_5$ contains three vertices. It follows that there exist two non-adjacent sides of $P_7$, each contained in one of the sets $A_1, \ldots, A_5$. We may assume that one of those sides lies in $A_4$ and the other in $A_5$. If those sides are adjacent to one side of $P_7$ (say to $a_2a_5$), then the center $c_3$ of $a_2a_5$ and the points $a_3, a_4, a_5$ belong to $A_1 \cup A_3 \cup A_4$. We may assume that $c_3 \in A_3$. Since $\varrho(c_3, a_i) > \frac{1}{2}$ for $i = 5, 6, 7$, we infer that three vertices $a_5, a_6, a_7$ belong to $A_1 \cup A_3$, whence one of the sets $A_1, A_2$ contains one of the sides $a_3a_5, a_5a_7$. But the sides $a_1a_2$ and $a_4a_5$ are opposite one to the other, as well the sides $a_3a_4, a_5a_7$ are opposite one to the other. Consequently, in the case I there exist two opposite sides of $P_7$ such that each of them is contained in one of the sets $A_1, \ldots, A_5$.

Thus we may assume that

\begin{equation}
(a_1a_2 \subset A_5, \quad a_4a_5 \subset A_4 \quad \text{and} \quad a_5 \in A_3)
\end{equation}

Now let us consider the points $a'_3 \in a_3a_5$ and $a''_3 \in a_5a_4$ and let us distinguish the following subcases:

1. **Subcase I.** $A_5 \cap (a'_3, a''_3) = \emptyset$.
2. **Subcase I.** $A_5 \cap (a'_3, a''_3) \neq \emptyset$.

In the subcase I, we infer by the inequality $\varrho(a'_3, a''_3) > \frac{1}{2}$ and by (6) that at least one of the sets $A_1, A_2$ (say $A_2$) intersects $(a'_3, a''_3)$. Then the sets $A_1, A_3, A_4$, and $A_5$ do not intersect the interior of the arc $(a'_3, a''_3)$ and we infer that this last arc is a subset of $A_1 \cup A_5$. But $a_6, a_7 \in (a'_3, a''_3) \setminus A_5$, whence $a_6, a_7 \in A_1$. It follows that $a_5, c_7 \in A_5$, which contradicts the inequality $\varrho(a_5, c_5) > \frac{1}{2}$. Thus the subcase I is impossible.

In the subcase II, the sets $A_4$ and $(a'_3, a''_3)$ are disjoint. Then (6) implies that the arc $(a'_3, a''_3)$ must be covered by $A_1, A_2, A_3$ and, since $a_5 \in A_5$, the arc $(a'_3, a''_3)$ lies in $A_1 \cup A_2$. We may assume that $a'_3 \in A_1$ and $a''_3 \in A_2$. Since $\varrho(c_3, a''_3) > \frac{1}{2}$ and $\varrho(c_3, a'_3) > \frac{1}{2}$, we infer that the point $c_6 \in (a'_3, a''_3)$ does not belong to $A_1 \cup A_2$, which contradicts the inclusion $(a'_3, a''_3) \subset A_1 \cup A_2$. Thus the subcase II is also impossible.

In the case II, all vertices $a_1, \ldots, a_7$ belong to the set $A_1 \cup A_3 \cup A_4 \cup A_5$. Since none of the sets $A_i$ contains three vertices, we infer that there
exist three sides of $P_7$ such that each of them lies in one of the sets $A_1, \ldots, A_4$ and no two of those sides are adjacent one to the other. We may assume that

$$a_1a_2 \subset A_4, \quad a_4a_5 \subset A_3 \quad \text{and} \quad a_4a_7 \subset A_4.$$ 

Consider the points $a'_4$ and $a''_4$ (see Fig. 2) and the arc $(a'_4, a''_4)$. Then the vertex $a_3$ does not belong to $A_3 \cup A_4 \cup A_4$, whence $a_3 \in A_1$. Observe that both points $c_3, c_3$ do not belong to $A_3 \cup A_3 \cup A_4$ and that at least one of them does not belong to $A_1$. Denote this last point by $d$. Consequently, $d$ belongs to one of the sets $A_3, A_6$, say to $A_6$. Then $A_6$ and the set $a_3a_6 \cup a_1a_7$ are disjoint and we infer that

$$a_3a_6 \cup a_1a_7 \subset A_3 \cup A_3 \cup A_4 \cup A_4.$$ 

But it is clear that the centers $c_5$ of $a_3a_6$ and $c_7$ of $a_1a_7$ do not belong to $A_3 \cup A_3 \cup A_4$. Consequently, $c_5, c_7 \in A_1$, which contradicts the inequality $\varrho(a_5, a_7) > \frac{1}{2}$. Thus the case II is also impossible and the proof of the Theorem is complete.

REFERENCES


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