A JACOBSON-SEMI-SIMPLE BANACH ALGEBRA
WITH A DENSE NIL SUBALGEBRA

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In this note we give an example of an algebra as described in the title. This answers a question of Grabiner [1].

Let \( A \) be the free Banach algebra on generators \( \mathcal{X}_n \ (n \in \mathbb{Z}^+) \) of norm 1, with the relations: \( M = 0 \) for every monomial \( M = X_{i_1} \ldots X_{i_r} \) containing more than \( n \) occurrences of \( X_n \), where \( n = \max \{ i_1, \ldots, i_r \} \). (The Banach algebra \( A \) is obtained by taking the algebra \( A_0 \) generated algebraically by \( \{ X_n : n \in \mathbb{Z}^+ \} \) with these relations, giving it the norm
\[
\left\| \sum_{i=0}^{n} \lambda_i M_i \right\| = \sum_{i=0}^{n} |\lambda_i|
\]
(\( \lambda_i \) scalars, \( M_i \) monomials), and completing.)

The dense subalgebra \( A_0 \) (the algebra of polynomials) is clearly nil, since if \( P \in A_0 \), we can find
\[
N = \max \{ n : X_n \text{ occurs in } P \}
\]
and then \( P^{(N+1)!} = 0 \).

We show that \( A \) is Jacobson-semisimple by proving that it has no non-zero ideals of topologically nilpotent elements (see [2], Chapter II, Section 3). We show that, given a non-zero \( T \in A \), there is a \( Y \in A \) with \( TY \) not topologically nilpotent. Now \( T \) is a linear combination of monomials \( M_i \),
\[
T = \sum_{i=0}^{\infty} \lambda_i M_i \quad \text{with} \quad \|T\| = \sum_{i=0}^{\infty} |\lambda_i| < \infty \quad \text{and} \quad \lambda_0 \neq 0.
\]

Let \( N > \max \{ n : X_n \text{ occurs in } M_0 \} \), and write
\[
Y = \sum_{i=0}^{\infty} 2^{-i} X_{N+i}.
\]
We consider \((TY)^n\). This will be a linear combination of monomials; each monomial being of the form

\[ M_{k_1}X_{N_1}^j M_{k_2}X_{N_2}^j \ldots M_{k_n}X_{N_n}^j, \]

where \(j_1, \ldots, j_n, k_1, \ldots, k_n \in \{0, 1, 2, 3, \ldots\}\). We fix attention on those monomials of the form

\[ M_0X_{N_1}^j M_0X_{N_2}^j \ldots M_0X_{N_n}^j \]

and we are worried lest, in computing \((TY)^n\), cancellation should occur between these and other monomials of form (1). However, if, in (1), any of the \(M_k\) should contain an \(X_r\) with \(r \geq N\), then the total number of such \(X_r\) in (1) would exceed \(n\), and so (1) could not equal (2), where the total number of such \(X_r\) is precisely \(n\). If, on the other hand,

\[ N > \max\{m: X_m \text{ occurs in } M_k\} \]

for every \(M_k\) occurring in (1), then (1) and (2) can only be equal if \(M_{k_i} = M_0\) \((1 \leq i \leq n)\). Thus monomials (2) are distinct from all other monomials in \((TY)^n\); and, of course, different sequences \((j_1, \ldots, j_n)\) give distinct monomials (2). Thus \(||(TY)^n||\) is not less than the modulus of the coefficient of one of the monomials of form (2), provided that the sequence \((j_1, \ldots, j_n)\) be chosen so that (2) does not vanish. Such a sequence is given by

\[ j_r = \max\{i: N^i \text{ divides } r\}, \]

and this yields

\[ ||(TY)^n|| \geq 2^{-t}|\lambda_0|^n, \]

where, if

\[ n = a_0N^s + \ldots + a_1N + a_0 \quad (0 \leq a_0, a_1, \ldots, a_s < N), \]

then

\[ t = \frac{a_s(N^s-1) + \ldots + a_1(N-1)}{N-1} \leq \frac{n}{N-1}. \]

Thus

\[ ||(TY)^n||^{1/n} \geq 2^{-1/(N-1)}|\lambda_0|, \]

so \(TY\) is not topologically nilpotent. This completes the proof.

REFERENCES


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