ANNALES
POLONICI MATHEMATICI
XXI (1968)

# Some applications of multidimensional integration by parts

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The object of the present note is to show how Abel's transformation and the formula for integration by parts of Stieltjes integrals which goes with it can be extended to an arbitrary finite number of dimensions and how, then, they can be applied to two hitherto incompletely solved problems. The first problem is this: An integral over a multidimensional unit cube is approximated by the average of integrand values at the points of a finite sequence; it is required to find a good upper bound for the absolute error in terms of properties of the integrand and of either of two measures, discussed below, of the equidistribution of the point sequence. The second problem is that of finding useful upper bounds for the Fourier coefficients of suitably periodic functions of several variables satisfying certain types of regularity conditions. It should be mentioned that the multidimensional Abel transformation introduced in § 1 was used in a somewhat less general form by Hlawka [4]; it is hoped that the different system of notation adopted here will make the transformation easily tractable in its full generality and capable of being applied to a variety of problems beyond the two treated presently.

**§ 1. Integration by parts in several dimensions.** Letters with subscripts $1, \ldots, s$ will denote coordinates of a point, the corresponding bold character denoting the point itself. $Q^s$ will denote the partly open $s$-dimensional cube

$$Q^s: \quad 0 \leqslant x_j < 1 \quad (j = 1, \ldots, s),$$

$\bar{Q}^s$ representing its closure.

Sets of $2s$ finite sequences $\langle x_j^{(0)}, \ldots, x_j^{(m(j))}\rangle$ and $\langle \xi_j^{(0)}, \ldots, \xi_j^{(m(j)+1)}\rangle$ $(j = 1, \ldots, s)$ will be said to generate a *double partition* of $\bar{Q}^s$ if they satisfy the relations

$$(1.1) \qquad 0 = \xi_j^{(0)} = x_j^{(0)} < \xi_j^{(1)} < x_j^{(1)} \leqslant \xi_j^{(2)} < \ldots < x_j^{(m(j))} = \xi_j^{(m(j)+1)} = 1 \qquad (j = 1, \ldots, s).$$

Given such a double partition, the operators $\Delta_j$ and $\Delta_j^*$ $(j = 1, \ldots, s)$, acting on arbitrary functions defined over $\bar{Q}^s$, will be defined respectively by the formulae

$$\Delta_j\varphi(x_1, \ldots, x_{j-1}, x_j^{(k)}, x_{j+1}, x_s)$$
$$= \varphi(x_1, \ldots, x_{j-1}, x_j^{(k+1)}, x_{j+1}, \ldots, x_s) - \varphi(x_1, \ldots, x_{j-1}, x_j^{(k)}, x_{j+1}, \ldots, x_s)$$

and

$$\Delta_j^*\varphi(x_1, \ldots, x_s)$$
$$= \varphi(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_s) - \varphi(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_s),$$

valid for $j = 1, \ldots, s$. The same definition will apply when the coordinates are denoted by the letter $\xi$ with similar upper and lower indices. Operators with different subscripts obviously commute, and

$$\Delta_{j(1),\ldots,j(k)} \quad \text{or} \quad \Delta^*_{j(1),\ldots,j(k)}$$

will stand for

$$\Delta_{j(1)}\ldots\Delta_{j(k)} \quad \text{or} \quad \Delta^*_{j(1)}\ldots\Delta^*_{j(k)}$$

respectively. Clearly, each of these operators commutes with summation applied to variables on which it does not act.

Given any expression $\Phi(r, \ldots, r+k-1; r+k, \ldots, s)$ depending only on the partition of the variables $j(r), \ldots, j(s)$ into the sets $\langle j(r), \ldots, j(r+k-1)\rangle$ and $\langle j(r+k), \ldots, j(s)\rangle$,

$$\sum_{r,\ldots,s;\,k}^{*} \Phi(r, \ldots, r+k-1; r+k, \ldots, s)$$

will denote the sum of all the expressions derived from $\Phi(r, \ldots, r+k-1;$ $r+k, \ldots, s)$ by replacing the given partition of $\langle j(1), \ldots, j(s)\rangle$ successively by all the other partitions of this set into a set of $k$ and a set of $s-r-k-1$, each partition being taken exactly once. The sum which has just been defined is meaningful only if $0 < k < s-r+1$. If either $k = 0$ or $k = s-$ $-r+1$, one of the sets becomes empty and there is, strictly speaking, no partition; however, in order to avoid troublesome exceptions, the sum will be interpreted, in such cases, as being reduced to one term.

Proposition 1. *If $s$ is any positive integer, if $f(x)$ and $g(x)$ are any functions defined on $\bar{Q}^s$, and if* (1.1) *is satisfied, then*

$$\sum_{l(1)=0}^{m(1)-1} \ldots \sum_{l(s)=0}^{m(s)-1} f(\xi_1^{(l(1)+1)}, \ldots, \xi_s^{(l(s)+1)})\Delta_{1,\ldots,s}g(x_1^{(l(1))}, \ldots, x_s^{(l(s))}) \tag{1.2}$$
$$= \sum_{k=0}^{s}(-1)^k \sum_{1,\ldots,s;\,k}^{*} \Delta^*_{k+1,\ldots,s} \sum_{l(1)=0}^{m(1)} \ldots \sum_{l(k)=0}^{m(k)} g(x_1^{(l(1))}, \ldots, x_k^{(l(k))}, x_{k+1}, \ldots, x_s) \times$$
$$\times \Delta_{1,\ldots,k} f(\xi_1^{(l(1))}, \ldots, \xi_k^{(l(k))}, x_{k+1}, \ldots, x_s);$$

*in the right-hand side, when $k = 0$, the summation signs referring to $l(1), \ldots, l(k)$, as well as $\Delta_{1,\ldots,k}$ are understood to disappear, and, similarly, when $k = s$, $\Delta^*_{k+1,\ldots,s}$ should be disregarded, the dummy variables $x_{k+1}, \ldots, x_s$ disappearing altogether.*

**Proof.** When $s = 1$, (1.2) reduces to

$$(1.3)\qquad \sum_{l(1)=0}^{m(1)-1} f(\xi_1^{(l(1)+1)})\Delta_1 g(x_1^{(l(1))}) = \Delta_1^*\{g(x)f(x)\} - \sum_{l(1)=0}^{m(1)} g(x_1^{(l(1))})\Delta_1 f(\xi_1^{(l(1))}),$$

which is nothing else but the well-known Abel transformation.

The proof is now completed by induction. Assume that the proposition holds for variables with subscripts $2, \ldots, s$. Substituting $\Delta_1 g$ for $g$, and taking the sum with respect to $j(1)$ from 0 to $m(1)-1$, we find

$$(1.4)\qquad \sum_{l(1)=0}^{m(1)-1} \cdots \sum_{l(s)=0}^{m(s)-1} f(\xi_1^{(l(1)+1)}, \ldots, \xi_s^{(l(s)+1)})\Delta_{1,\ldots,s} g(x_1^{(l(1))}, \ldots, x_s^{(l(s))})$$

$$= \sum_{k=0}^{s-1} (-1)^k \sum_{2,\ldots,s;\,k}^{*} \Delta^*_{k+2,\ldots,s} \times$$

$$\times \sum_{l(1)=0}^{m(1)-1} \sum_{l(2)=0}^{m(2)} \cdots \sum_{l(k+1)=0}^{m(k+1)} \Delta_1 g(x_1^{(l(1))}, \ldots, x_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s) \times$$

$$\times \Delta_{2,\ldots,k+1} f(\xi_1^{(l(1)+1)}, \xi_2^{(l(2))}, \ldots, \xi_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s).$$

Here, the left-hand side coincides with that of (1.2). On the other hand, (1.3) yields, after an obvious substitution,

$$\sum_{l(1)=0}^{m(1)-1} \Delta_1 g(x_1^{(l(1))}, \ldots, x_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s) \times$$

$$\times \Delta_{2,\ldots,k+1} f(\xi_1^{(l(1)+1)}, \xi_2^{(l(2))}, \ldots, \xi_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s)$$

$$= \Delta_1^*\{g(x_1, x_2^{(l(2))}, \ldots, x_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s) \times$$

$$\times \Delta_{2,\ldots,k+1} f(x_1, \xi_2^{(l(2))}, \ldots, \xi_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s)\} -$$

$$- \sum_{l(1)=0}^{m(1)} g(x_1^{(l(1))}, \ldots, x_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s) \times$$

$$\times \Delta_{1,\ldots,k+1} f(\xi_1^{(l(1))}, \ldots, \xi_{k+1}^{(l(k+1))}, x_{k+2}, \ldots, x_s).$$

This identity, understood in the same way as (1.2), holds with $k = 0, \ldots, s-1$. We can, therefore, apply to both sides of it the operator

$$\sum_{k=0}^{s-1} (-1)^k \sum_{2,\ldots,s;\,k}^{*} \Delta^*_{k+2,\ldots,s} \sum_{l(2)=0}^{m(2)} \cdots \sum_{l(k+1)=0}^{m(k+1)}.$$

In the new identity, the left-hand side coincides with the right-hand side of (1.4), and, consequently, is equal to the left-hand side of (1.2), while the right-hand side becomes

$$(1.5)\qquad \sum_{k=0}^{s-1}(-1)^k \sum_{2,\ldots,s;\,k}^{*} \Delta^{*}_{1,k+2,\ldots,s}\times \\ \times \sum_{l(2)=0}^{m(2)}\cdots\sum_{l(k+1)=0}^{m(k+1)} g(x_1, x_2^{(l(2))},\ldots,x_{k+1}^{(l(k+1))},x_{k+2},\ldots,x_s)\times \\ \times \Delta_{2,\ldots,k+1} f(x_1,\xi_2^{(l(2))},\ldots,\xi_{k+1}^{(l(k+1))},x_{k+2},\ldots,x_s)+ \\ +\sum_{k=0}^{s-1}(-1)^{k+1}\sum_{2,\ldots,s;\,k}^{*}\Delta^{*}_{k+2,\ldots,s}\times \\ \times\sum_{l(1)=0}^{m(1)}\cdots\sum_{l(k+1)=0}^{m(k+1)} g(x_1^{(l(1))},\ldots,x_{k+1}^{(l(k+1))},x_{k+2},\ldots,x_s)\times \\ \times\Delta_{1,\ldots,k+1} f(\xi_1^{(l(1))},\ldots,\xi_{k+1}^{(l(k+1))},x_{k+2},\ldots,x_s)\,.$$

This is nothing else but the right-hand side of (1.2) written in a slightly different form, Indeed, the terms corresponding to $k = 0$ in the right-hand side of (1.2) and in the first part of (1.5) are the same. Similarly, the term corresponding to $k = s$ in the right-hand side of (1.2) is equal to the term corresponding to $k = s-1$ in the second part of (1.5). Finally, if $0 < k < s$, the corresponding group of terms in (1.2) can be split into two parts in connection with the effect of the operator $\sum^*$, viz. the sum of all the terms in which 1 appears as a subscript of $\Delta^*$, and the sum of all the other terms; now the first part is identical with the term corresponding to the same value of $k$ in the first part of (1.5), while the second part does not differ from the term in $k-1$ in the second part of (1.5). Hence the proof is complete.

It is well known that in the one-dimensional case, the Riemann-Stieltjes integral

$$\int_0^1 f(x)\,dg(x)$$

exists whenever $f$ and $g$ are defined over $[0,1]$ and either of the two functions is continuous, the other being of bounded variation; moreover, such an integral can be integrated by parts. Essentially the same argument which leads to these conclusions can be applied to any finite number of dimensions using Proposition 1, provided that functions of bounded variation are suitably defined in the case of several variables.

For any function $f$ defined over $\bar{Q}^s$ and for any set of $s$ sequences $\langle x_j^{(0)},\ldots,x_j^{(m(j))}\rangle$ $(j = 1,\ldots,s)$ satisfying

$$(1.6)\qquad 0 = x_j^{(0)} < x_j^{(1)} < \ldots < x_j^{(m(j))} = 1 \qquad (j = 1,\ldots,s)$$

form the multiple sum

$$\sum_{j(1)=0}^{m(1)-1} \cdots \sum_{j(s)=0}^{m(s)-1} |\Delta_{1,\ldots,s} f(x_1^{(l(1))}, \ldots, x_s^{(l(s))})|;$$

the least upper bound of such sums for all the sets of sequences satisfying (1.6) is known as the *$s$-dimensional variation of $f$ over $\bar{Q}^s$ in the sense of Vitali* and is denoted by $V^{(s)}(f)$. If $V^{(s)}(f)$ is finite, $f$ is described as being *of bounded variation over $\bar{Q}^s$ in the sense of Vitali.* If the same function restricted to the various faces of $\bar{Q}^s$ with $1, \ldots, s$ dimensions is of bounded variation in the sense of Vitali over each of them, then $f$ is said to be *of bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause.*

When dealing with multidimensional Stieltjes integrals, we shall use the notation $d_{j(1),\ldots,j(k)}f$ to indicate that the integration applies only to the variables with subscripts $j(1), \ldots, j(k)$, the other variables being kept constant in the process of integration. Thus, for instance,

$$\Delta_3^* \int_{Q^2} g(x_1, x_2, x_3)\, d_{1,2}(x_1, x_2, x_3)$$
$$= \int_{Q^2} g(x_1, x_2, 1)\, df(x_1, x_2, 1) - \int_{Q^2} g(x_1, x_2, 0)\, df(x_1, x_2, 0)\,.$$

It is quite clear that if $f$ is continuous and $g$ is of bounded variation in the sense of Vitali over $\bar{Q}^s$, the Riemann-Stieltjes integral

$$\int_{Q^s} f(\boldsymbol{x})\, dg(\boldsymbol{x})$$

exists.

PROPOSITION 2. *If over $\bar{Q}^s$, one of the functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ is of bounded variation in the sense of Hardy and Krause, and the other is continuous, then the Riemann-Stieltjes integral*

$$\int_{Q^s} f(\boldsymbol{x})\, dg(\boldsymbol{x})$$

*exists and satisfies*

$$(1.7) \qquad \int_{Q^s} f(\boldsymbol{x})\, dg(\boldsymbol{x}) = \sum_{k=0}^{s} (-1)^k \sum_{1,\ldots,s;\,k}^{*} \Delta_{k+1,\ldots,s}^* \int_{Q^k} g(\boldsymbol{x})\, d_{1,\ldots,k} f(\boldsymbol{x})\,.$$

*If, on the other hand, $f$ and $g$ are periodic with a unit period in each of the $s$ coordinates of $\boldsymbol{x}$, it suffices to assume that one of these functions is continuous and the other is of bounded variation in the sense of Vitali over $\bar{Q}^s$, and* (1.7) *simplifies to*

$$(1.8) \qquad \int_{Q^s} f(\boldsymbol{x})\, dg(\boldsymbol{x}) = (-1)^s \int_{Q^s} g(\boldsymbol{x})\, df(\boldsymbol{x})\,.$$

**Proof.** If it is $g$ which is continuous, apply (1.2) and pass to the limit. When

$$\max_{0\leqslant l<m(j)} (x_j^{(l+1)}-x_j^{(l)})\to 0 \qquad (j=1,\dots,s)\,, \tag{1.9}$$

the right-hand side, clearly, tends to that of (1.7). Consequently, the left-hand side tends to a limit which is, by definition, the left-hand side of (1.7). Thus $\int f dg$ exists whenever $g$ is continuous and $f$ is bounded variation. In view of this, the passage to the limit is also legitimate when $f$ is continuous and $g$ is of bounded variation. The conclusion concerning the case of periodic functions follows from the fact that, in this case, (1.2) reduces to

$$\sum_{j(1)=0}^{m(1)-1}\dots\sum_{j(s)=0}^{m(s)-1} f(\xi_1^{(j(1)+1)},\dots,\xi_s^{(j(s)+1)})\Delta_{1,\dots,s}g(x_1^{(j(1))},\dots,x_s^{(j(s))})$$
$$=(-1)^s\sum_{j(1)=0}^{m(1)}\dots\sum_{j(s)=0}^{m(s)} g(x_1^{(j(1))},\dots,x_s^{(j(s))})\Delta_{1,\dots,s}f(\xi_1^{(j(1))},\dots,\xi_s^{(j(s))});$$

the parts played by $f$ and $g$ are then symmetric.

**§ 2. Applications to discrepancy and numerical integration.** Let $X=\langle \mathfrak{x}^{(0)},\dots,\mathfrak{x}^{(N-1)}\rangle$ with $\mathfrak{x}^{(k)}=\langle \mathfrak{x}_1^{(k)},\dots,\mathfrak{x}_s^{(k)}\rangle$ be any finite sequence of points of $Q^s$, and let $\nu(\boldsymbol{x})=\nu(x_1,\dots,x_s)$ be the number of points of this sequence satisfying $\mathfrak{x}_j^{(k)}<x_j$ $(j=1,\dots,s)$. It is proposed to call the function

$$g(\boldsymbol{x})=g(x_1,\dots,x_s)=N^{-1}\nu(x_1,\dots,x_s)-x_1\dots x_s$$

the *local discrepancy of* $X$. This function can obviously be regarded as describing the imperfection of the equidistribution of $X$ over $Q^s$. If a single number is wanted to measure this imperfection, it is natural to use one of the possible norms of $g(\boldsymbol{x})$. The norm

$$D(X)=\sup_{\boldsymbol{x}\in Q^s}|g(\boldsymbol{x})|$$

is known in the literature as the discrepancy of $X$. It is now proposed to replace this name by that of *extreme discrepancy* in order to distinguish it from another equally natural norm of $g(\boldsymbol{x})$ which is

$$T(X)=\Big\{\int_{Q^s}[g(\boldsymbol{x})]^2 d\boldsymbol{x}\Big\}^{1/2},$$

and will be called the *quadratic-mean discrepancy of* $X$. The two propositions which follow relate each of the two discrepancies of a sequence $X$ of points of $Q^s$ to the absolute value of the error committed when taking the value of an integral over $Q^s$ to be the average of integrand values at points of $X$.

PROPOSITION 3. *With the previous notations, if the mixed partial derivative*

$$\frac{\partial^s f}{\partial x_1 \ldots \partial x_s} = f_{x_1,\ldots,x_s}(x_1, \ldots, x_s)$$

*is continuous in $\bar{Q}^s$ and if $X_{i(1),\ldots,i(n)}$ denotes the projection of the sequence $X$ on the $s-n$-dimensional face of $\bar{Q}^s$ defined by $x_{i(1)} = \ldots = x_{i(s)} = 1$, then*

$$(2.1)\qquad \left|\frac{1}{N}\sum_{l=0}^{N-1} f(\mathfrak{x}^{(l)}) - \int_{Q^s} f(\boldsymbol{x})\,d\boldsymbol{x}\right|$$
$$\leqslant \sum_{k=1}^{s} \sum_{1,\ldots,s;\,k}^{*} T(X_{k+1,\ldots,s})\left\{\int_{Q^k} [f_{x_1,\ldots,x_k}(x_1, \ldots, x_k, 1, \ldots, 1)]^2 d\langle x_1, \ldots, x_k\rangle\right\}^{1/2},$$

*where $X$ should be substituted for $X_{k+1,\ldots,s}$ when $k = s$.*

**Proof.** Form a double subdivision of $\bar{Q}^s$ satisfying (1.1) and such that each coordinate of every point of $X$ should be an element of the corresponding sequence $\langle \xi_j^{(0)}, \ldots, \xi_j^{(m(j))}\rangle$, and pass to the limit with (1.9). Owing to this property of the subdivision of $\bar{Q}^s$, the left-hand side of (1.2) tends to

$$(2.2)\qquad \frac{1}{N}\sum_{l=0}^{N-1} f(\mathfrak{x}^{(l)}) - \int_{Q^s} f(\boldsymbol{x})\,d\boldsymbol{x}\,.$$

But since $g(\boldsymbol{x}) = 0$ whenever at least one coordinate of $\boldsymbol{x}$ vanishes, and since $g(1, \ldots, 1) = 0$, the right-hand side of (1.2) reduces to

$$(2.3)\qquad \sum_{k=1}^{s} (-1)^k \sum_{1,\ldots,s;\,k}^{*} \sum_{l(1)=0}^{m(1)} \cdots \sum_{l(s)=0}^{m(s)} g(x_1^{(l(1))}, \ldots, x_k^{(l(k))}, 1, \ldots, 1)\times$$
$$\times \Delta_{1,\ldots,k} f(\xi_1^{(l(1))}, \ldots, \xi_k^{(l(k))}, 1, \ldots, 1)\,,$$

tending to

$$\sum_{k=1}^{s} (-1)^k \sum_{1,\ldots,s;\,k}^{*} \int_{Q^k} g(x_1, \ldots, x_k, 1, \ldots, 1)\times$$
$$\times f_{x_1,\ldots,x_k}(x_1, \ldots, x_k, 1, \ldots, 1)\,d\langle x_1, \ldots, x_k\rangle\,,$$

since the integrals exist by Proposition 2. An application of the Schwarz inequality completes the proof.

PROPOSITION 4. *With the previous notations, if $f(\boldsymbol{x})$ is of bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause, then*

$$(2.4)\qquad \left|\frac{1}{N}\sum_{l=0}^{N-1} f(\mathfrak{x}^{(l)}) - \int_{Q^s} f(\boldsymbol{x})\,d\boldsymbol{x}\right| \leqslant \sum_{k=1}^{s} \sum_{1,\ldots,s;\,k}^{*} D(X_{k+1,\ldots,s}) V^{(k)}\big(f(\ldots, 1, \ldots, 1)\big)\,,$$

*where* $V^{(k)}(f(\ldots, 1, \ldots, 1))$ *denotes the* $k$*-dimensional variation of* $f(x_1, \ldots, x_k, 1, \ldots, 1)$ *over* $\bar{Q}^k$ *in the sense of Vitali, and where the term of the sum corresponding to* $k = s$ *is understood to be* $D(X)V^{(s)}(f)$.

Proof. We proceed as in the preceding proof as far as (2.3) and note that its absolute value will increase, if anything, when, omitting $(-1)^k$, we replace $\Delta_{1,\ldots,k} f(\xi_1^{(l(1))}, \ldots, \xi_k^{(l(n))}, 1, \ldots, 1)$ by its absolute value, and $g(x_1, \ldots, x_k, 1, \ldots, 1)$ by $D(X_{k+1,\ldots,s})$, obtaining the right-hand side of (2.4).

Proposition 4 differs but little from a theorem proved by Hlawka [4], but because of the shortness of our proof it was felt worth giving here. The difference between two propositions is twofold. In the first place, Hlawka allowed the points of $X$ to be anywhere in $\bar{Q}^s$ instead of restricting them to $Q^s$, but, to compensate for it, he had to assume the periodicity of $f$ with a unit period in each of the $s$ coordinates of $x$; it is easy to see that either restriction can be replaced by the other, but is seemed more natural to the present author to drop the requirement of periodicity and rather restrict the points of $X$ to $Q^s$, as would be done in any case if, for instance, the sequence was obtained by reducing modulo 1 the coordinates of the points of some other sequence (see [5] or [6]). Secondly, Hlawka's inequality is slightly weaker than (2.4) insomuch as he replaces the factor $D(X_{k+1,\ldots,s})$ in (2.4) by $D(X)$, which cannot be smaller, and indeed is likely to exceed $D(X_{k+1,\ldots,s})$.

Comparing Propositions 3 and 4, one notices immediately that the former requires more of $f$ by way of smoothness than the latter. However, when it can be applied, (2.1) is capable of yielding a more favourable upper bound for the absolute value of (2.2) than (2.4). Indeed, in any event

$$T(X) < D(X),$$

and it has been found that, with $N \to \infty$, $T(X)$ can even be of a smaller order of magnitude than $D(X)$ [2], which, incidentally, is a good reason for introducing $T(X)$ besides $D(X)$.

**§ 3. The coefficients of multiple Fourier series.** It is well known that in the case of one variable the Fourier coefficients of a function of bounded variation are $O(n^{-1})$, but a careful search of the existing literature produced only one paper [1] extending this property to multiple series. This paper, written by S. Faedo, treats only double series (although a rather tedious extension of his approach to functions of more variables would be possible), and considers only the case when all the subscripts of the coefficient are different from 0. Faedo's result will now be obtained, for any finite number of variables, as an immediate consequence of Proposition 2, and will also be extended to the case when some of the subscripts of the coefficient are zeros. Three propositions will lead to a final corollary which is more general than the theorem merely quoted by Hlawka [5] without references to the existing literature [1].

In what follows, $\boldsymbol{h} = \langle h_1, \ldots, h_s \rangle$ will always denote a lattice point, i.e. a vector with integral coordinates, and $R(\boldsymbol{h})$ will stand for the product

$$\max(1, |h_1|) \ldots \max(1, |h_s|) ,$$

a dot denoting scalar multiplication.

PROPOSITION 5. *If $f(\boldsymbol{x})$ is of bounded variation over $\bar{Q}^s$ in the sense of Vitali, and is periodic with a unit period in each of the $s$ coordinates of $\boldsymbol{x}$, then its Fourier coefficients can be expressed in terms of Riemann integrals*

$$c_{\boldsymbol{h}} = \int_{Q^s} f(\boldsymbol{x}) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x}) d\boldsymbol{x} \tag{3.1}$$

*and satisfy*

$$|c_{\boldsymbol{h}}| \leqslant (2\pi)^{-s} R(\boldsymbol{h})^{-1} V^{(s)}(f) \tag{3.2}$$

*whenever $h_1 \ldots h_s \neq 0$.*

Proof. If

$$g(\boldsymbol{x}) = i^s (2\pi)^{-s} R(\boldsymbol{h})^{-1} \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x}) ,$$

then (3.1) can be re-written, according to (1.8), in the form of

$$c_{\boldsymbol{h}} = \int_{Q^s} f(\boldsymbol{x}) dg(\boldsymbol{x}) = (-1)^s \int_{Q^s} g(\boldsymbol{x}) df(\boldsymbol{x}) ,$$

and (3.2) follows from the definition of $V^{(s)}(f)$ and from

$$|g(\boldsymbol{x})| = (2\pi)^{-s} R(\boldsymbol{h})^{-1} .$$

When some of the coordinates of $\boldsymbol{h}$ are equal to 0, the situation becomes slightly more complicated. There is no loss of generality in assuming that for some $k$

$$h_1 \ldots h_k \neq 0 \quad \text{and} \quad h_{k+1} = \ldots = h_s = 0 . \tag{3.3}$$

Now the variations of $f$ in the sense of Vitali over linear varieties with fewer than $s$ dimensions have to be brought in. It should be noted that

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(1) [Added on January 22, 1968] Essentially, the latter theorem can be found in N. M. Korobov, *Teoretiko-chislovye metody v priblizhennom analize* (*Number theoretical methods in numerical analysis*), Moscow 1963. It may be noted that, when compared with our Proposition 8 below, both this theorem and some interesting variations on it obtained by Korobov require the existence of partial derivatives of a higher order (as well as their continuity) to ensure that the Fourier coefficients of the function in question are of a given order of magnitude.

if $f$ is of bounded variation in the sense of Hardy and Krause over $\bar{Q}^s$, then

$$(3.4)\qquad \sup_{0\leqslant \xi_j\leqslant 1\,(j=k+1,\dots,s)} V^{(k)}\big(f(\dots,\xi_{k+1},\dots,\xi_s)\big) < +\infty \qquad (k=1,\dots,s-1);$$

more precisely, one finds by induction that this supremum does not exceed the sum of $V^{(s)}(f)$ and of the variations, in the sense of Vitali, of the function $f$ restricted to the various faces of $\bar{Q}^s$ with $s-k,\dots,s-1$ dimensions, obtained by equating some of the coordinates $x_{k+1},\dots,x_s$ to 1.

PROPOSITION 6. *If $f(\boldsymbol{x})$ is periodic with a unit period in each of the $s$ coordinates of $\boldsymbol{x}$ and is of bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause, then* (3.3) *entails*

$$(3.5)\qquad |c_{\boldsymbol{h}}| \leqslant (2\pi)^{-k}R(\boldsymbol{h})^{-1} \sup_{0\leqslant \xi_j\leqslant 1\,(j=k+1,\dots,s)} V^{(k)}\big(f(\dots,\xi_{k+1},\dots,\xi_s)\big)\,.$$

Proof. In view of (3.3), put

$$(3.6)\qquad g^{(k)}(x_1,\dots,x_k) = i^k(2\pi)^{-k}R(\boldsymbol{h})^{-1}\exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})\,,$$

and regard $x_{k+1},\dots,x_s$ and $\xi_{k+1},\dots,\xi_s$ provisionally as fixed, though arbitrary. Substituting $k$ for $s$ in (1.2), and taking into account the periodicity of the functions involved, we find

$$\sum_{l(1)=0}^{m(1)-1}\dots\sum_{l(k)=0}^{m(k)-1} f(\xi_1^{(l(1)+1)},\dots,\xi_k^{(l(k)+1)},\xi_{k+1},\dots,\xi_s)\,\Delta_{1,\dots,k}g^{(k)}(x_1^{(l(1))},\dots,x_k^{(l(k))})$$
$$= (-1)^k\sum_{l(1)=0}^{m(1)}\dots\sum_{l(k)=0}^{m(k)} g^{(k)}(x_1^{(l(1))},\dots,x_k^{(l(k))})\times$$
$$\times\,\Delta_{1,\dots,k}f(\xi_1^{(l(1))},\dots,\xi_k^{(l(k))},\xi_{k+1},\dots,\xi_s)\,,$$

and further

$$(3.7)\qquad \sum_{l(1)=0}^{m(1)-1}\dots\sum_{l(s)=0}^{m(s)-1} f(\xi_1^{(l(1)+1)},\dots,\xi_s^{(l(s)+1)})\times$$
$$\times\,\Delta_{1,\dots,k}g^{(k)}(x_1^{(l(1))},\dots,x_k^{(l(k))})\,\Delta_{k+1}x_{k+1}^{(l(k+1))}\dots\Delta_s x_s^{(l(s))}$$
$$= \sum_{l(1)=0}^{m(1)}\dots\sum_{l(k)=0}^{m(k)}\sum_{l(k+1)=0}^{m(k+1)-1}\dots\sum_{l(s)=0}^{m(s)-1} g^{(k)}(x_1^{(l(1))},\dots,x_k^{(l(k))})\times$$
$$\times\,\Delta_{1,\dots,k}f(\xi_1^{(l(1))},\dots,\xi_k^{(l(k))},\xi_{k+1}^{(l(k+1)+1)},\dots,\xi_s^{(l(s)+1)})\,\Delta_{k+1}x_{k+1}^{(l(k+1))}\dots\Delta_s x_s^{(l(s))}\,.$$

It is easily seen that, with (1.9), the left-hand side tends to the Riemann integral

$$\int_{Q^s} f(\boldsymbol{x})\exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})\,d\boldsymbol{x} = c_{\boldsymbol{h}}\,,$$

while, clearly, the modulus of the right-hand side cannot exceed the right-hand side of (3.5). Hence the proof is complete.

In what follows, $f$ will always be assumed to be of bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause; then Proposition 5 can be regarded as a special case of Proposition 6. An adaptation of the proof of the latter yields the following result.

PROPOSITION 7. *If $f(\boldsymbol{x})$ is periodic with a unit period in each of the $s$ coordinates of $\boldsymbol{x}$, if it has all the mixed partial derivatives*

$$\frac{\partial^n f}{\partial x_{j(1)}\dots\partial x_{j(n)}} = f_{x_{j(1)},\dots,x_{j(n)}}(\boldsymbol{x}) \qquad (j(1) < \dots < j(n);\ 1 \leqslant n \leqslant s)\,,$$

*and if these are bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause, then* (3.3) *entails*

$$c_{\boldsymbol{h}} = i^k (2\pi)^{-k} R(\boldsymbol{h})^{-1} c_{\boldsymbol{h}}^{(k)}\,, \tag{3.8}$$

*where $c_{\boldsymbol{h}}^{(k)}$ is the corresponding Fourier coefficient of $f_{x_1,\dots,x_k}(\boldsymbol{x})$.*

Proof. By an easy extension of the mean value theorem of the differential calculus, we find

$$\begin{aligned}\Delta_{1,\dots,k} f(\xi_1^{(l(1))},\dots,\xi_k^{(l(k))},\xi_{k+1}^{(l(k+1)+1)},\dots,\xi_s^{(l(s)+1)}) \\ = f_{x_1,\dots,x_k}(y_1^{(l(1))},\dots,y_k^{(l(k))},\xi_{k+1}^{(l(k+1)+1)},\dots,\xi_s^{(l(s)+1)})\Delta_1\xi_1^{(l(1))}\dots\Delta_k\xi_k^{(l(k))}\,,\end{aligned}$$

where

$$\xi_j^{(l(j))} < y_j^{(l(j))} < \xi_j^{(l(j)+1)} \qquad (j = 1,\dots,k)\,.$$

Substitute this in the right-hand side of (3.7) and note that, in view of the continuity of $g^{(k)}$, altering its arguments to $y_1^{(l(1))},\dots,y_k^{(l(k))}$ will make no difference in the limit. A similar transformation affecting $\Delta_{1,\dots,k}g^{(k)}$ in the left-hand side is even more straightforward owing to the continuity of

$$\frac{\partial^k g^{(k)}}{\partial x_1\dots\partial x_k} = \exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})\,.$$

We have, then, on both sides Riemann sums tending to integrals which exist, since both integrands are products of functions being either continuous or of bounded variation in the sense of Hardy and Krause. Taking into account (3.6), we obtain, therefore, in the limit

$$\int_{Q^s} f(\boldsymbol{x})\exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})\,d\boldsymbol{x} = i^k(2\pi)^{-k}R(\boldsymbol{h})^{-1}\int_{Q^s} f_{x_1,\dots,x_k}(\boldsymbol{x})\exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})\,d\boldsymbol{x}\,.$$

Hence the proof is complete.

Combining the last two propositions, we find by induction the following final result.

PROPOSITION 8. *If $f(\boldsymbol{x})$ is periodic with a unit period in each of the $s$ coordinates of $\boldsymbol{x}$, and admits all the partial derivatives*

$$\frac{\partial^{q(1)+\ldots+q(s)}f}{\partial x_1^{q(1)}\ldots\partial x_s^{q(s)}} \quad (0 \leqslant q(i) \leqslant r;\ i = 1, \ldots, s)$$

*for some positive integer $r$, these partial derivatives being of bounded variation over $\bar{Q}^s$ in the sense of Hardy and Krause, then the Fourier coefficients of $f$ satisfy*

$$|c_{\boldsymbol{h}}| \leqslant (2\pi)^{-(r+1)}R(\boldsymbol{h})^{-(r+1)}M \qquad (\boldsymbol{h} \neq \langle 0, \ldots, 0\rangle),$$

*where $M$ is a constant depending on $f$ and not exceeding the biggest of the sums, referring to each of the mixed partial derivatives*

$$\frac{\partial^{nr}f}{\partial x_{j(1)}^r\ldots\partial x_{j(n)}^r} \quad (j(1) < \ldots < j(n) \leqslant s),$$

*of their variations, in the sense of Vitali, on $\bar{Q}^s$ and on all its sides obtained by making some of the coordinates of $\boldsymbol{x}$ equal to* 1.

In this statement, the exponent $-(r+1)$ of $2\pi$ could clearly be replaced by $-k(r+1)$, where $k$ is the number of non-zero coordinates of $\boldsymbol{h}$ (see (3.5)), but the object of the exercise was to find for $|c_{\boldsymbol{h}}|$ an upper bound depending only on $f$ and on $R(\boldsymbol{h})$.

Remark. It may be noted that under the conditions of Proposition 8, the Fourier series of $f$ converges uniformly, and therefore, converges to $f$ (see e.g. [3], Theorem 8; the argument is essentially the same in $s$ dimensions).

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*Reçu par la Rédaction le 9. 11. 1967*