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On the representation of non-negative solutions of linear parabolic systems of partial differential equations

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Abstract. This article is a continuation of the author's investigations author (Ann. Polon. Math. 19 (1967), p. 193—197, and 22 (1970), p. 323—331) showing that if $\{u^i(t,x)\}$ $(i = 1, \ldots, N)$ is a non-negative solution of the parabolic system

$$\frac{\partial u^k}{\partial t} = \sum_{i=1}^{n} a_{ij}^k(t,x) u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(t,x) u_{x_i}^k + \sum_{l=1}^{N} c_l^k(t,x) u^k$$

in $(0, T] \times R_n$, then there exist non-negative Borel measures $\{\gamma^j\}$ such that

$$u^i(t,x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t,x;\ 0,y)\gamma^j(dy).$$

The purpose of this paper is to prove the uniqueness of the measures γ_j.

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Consider the system of equations

$$(1)\quad L^k(u^1, \ldots, u^N) = \sum_{i,j=1}^{n} a_{ij}^k(t,x) u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(t,x) u_{x_i}^k + \sum_{i=1}^{N} c_i^k(t,x) u^i - u_t^k = 0 \quad (k = 1, \ldots, N),$$

where the coefficients a_{ij}^k, b_i^k and c_i^k are defined and bounded in a strip $(0, T] \times R_n$. We take $\big(a_{ij}^k(t,x)\big)$ to be a symmetric matrix, i.e., $a_{ij}^k = a_{ji}^k$ for $k = 1, \ldots, N$.

Throughout this paper we shall assume that

(a) a_{ij}^k, $\dfrac{\partial a_{ij}^k}{\partial x_j}$, $\dfrac{\partial^2 a_{ij}^k}{\partial x_i \partial x_j}$, b_i^k, $\dfrac{\partial b_i^k}{\partial x_i}$ and c_i^k are Hölder-continuous with respect to (t,x) in $(0, T] \times R_n$;

(b) there exists a positive constant a such that for any real vector $\xi \in R_n$

$$\sum_{i,j=1}^{n} a_{ij}^k(t,x)\xi_i \xi_j \geqslant a|\xi|^2 \quad (k = 1, \ldots, N)$$

for all $(t, x) \in (0, T] \times R_n$;

(c) $c_i^k(t, x) \geqslant 0$ for all $(t, x) \in (0, T] \times R_n$ and $i \neq k$ $(i, k = 1, \ldots, N)$.

Under assumptions (a) and (b), it is well known that there exists a fundamental matrix $\{\Gamma_{ij}(t, x; \tau, y)\}$ $(i, j = 1, \ldots, N)$ of functions defined for (t, x), $(\tau, y) \in (0, T] \times R_n$, $\tau < t$ (see [4], Chapter 9). It follows from (c) that $\Gamma_{ij}(t, x; \tau, y) \geqslant 0$ $(i, j = 1, \ldots, N)$ for all (t, x), $(\tau, y) \in (0, T] \times R_n$ $(\tau < t)$ (see [2]).

In [3] it is shown that if $\{u^j(t, x)\}$ $(j = 1, \ldots, N)$ is a non-negative solution of (1) in $(0, T] \times R_n$, then there exist non-negative Borel measures γ^j and a number $T_1 > 0$ such that

$$u^i(t, x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y)\gamma^j(dy) \quad (i = 1, \ldots, N) \tag{2}$$

for all $(t, x) \in (0, T_1] \times R_n$. The purpose of this paper is to prove the uniqueness of the measures γ^j. We also obtain a necessary and sufficient condition for a system of functions defined by a formula such as (2) to be a non-negative solution of (1).

We shall need the following lemma:

LEMMA. *Let $f(x)$ be a continuous and bounded function in R_n. Then*

$$\lim_{t \to 0} \int_{R_n} \Gamma_{ij}(t, x; 0, y) f(x)\, dx = \delta_{ij} f(y) \quad (i, j = 1, \ldots, N).$$

Proof. Using the decomposition of Γ_{ij}, we get ([5], p. 252)

$$\begin{aligned}\Gamma_{ij}(t, x; \tau, y) &= Z_{ij}(t, x; \tau, y) + \sum_{j=1}^{N} \int_{\tau}^{t} d\sigma \int_{R_n} Z_{ik}(t, x; \sigma, z)\Phi_{kj}(\sigma, z; \tau, y)\, dy \\ &= Z_{ij}(t, x; \tau, y) + R_{ij}(t, x; \tau, y),\end{aligned}$$

where

$$Z_{ij}(t, x; \tau, y) = C_i(\tau, y) \exp\left\{-\frac{\sum_{r,s=1}^{n} A_{rs}^i(\tau, y)(x_s - y_s)(x_r - y_r)}{4(t-\tau)}\right\}\delta_{ij},$$

$$C_i(\tau, y) = (2\sqrt{\pi})^n \{\det[A_{rs}^i(\tau, y)]\}^{\frac{1}{2}},$$

$\{A_{rs}^i(\tau, y)\}$ denotes the inverse matrix to $\{a_{rs}^i(\tau, y)\}$, $\Phi_{kj}(\sigma, z; \tau, y)$ $(k, j = 1, \ldots, N)$ are continuous functions for (σ, z), $(\tau, y) \in [0, T] \times R_n$ $(\sigma > \tau)$ such that

$$|\Phi_{kj}(\sigma, z; \tau, y)| \leqslant C(\sigma - \tau)^{-(n+2-\alpha)/2} \exp\left(-c\frac{|x-y|^2}{\sigma - \tau}\right),$$

where $\alpha \in (0, 1)$. It follows from the last inequality (for details see [5], p. 20)

$$\lim_{t \to 0} \int_{R_n} R_{ij}(t, x; 0, y) f(x)\, dx = 0 \quad (i, j = 1, \ldots, N).$$

It is clear that

$$\lim_{t\to 0} \int_{R_n} \Gamma_{ii}(t, x;\ 0, y) f(x)\, dx = f(y) \qquad (i = 1, \ldots, N).$$

THEOREM 1. *If* $\{u^i(t, x)\}$ $(i = 1, \ldots, N)$ *is a non-negative solution of* (1) *in* $(0, T] \times R_n$, *then exist unique non-negative Borel measures* $\{\gamma^j\}$ $(j = 1, \ldots, N)$ *such that formula* (2) *is satisfied.*

Proof. The existence part of this theorem was proved in [3]. The uniqueness proof is similar to that of Aronson's [1]. Suppose that there are two systems of measures $\{\gamma_1^j\}$ and $\{\gamma_2^j\}$ $(j = 1, \ldots, N)$, each giving a representation of $\{u^i\}$ in the form (2) and such that $\gamma_1^{j_0} = \gamma_2^{j_0}$ for certain j_0. By Theorem 2.3 of [2], it follows that $\Gamma_{kk}(t, x;\ \tau, y)$ satisfies

$$\Gamma_k(t, x;\ \tau, y) \leqslant \Gamma_{kk}(t, x;\ \tau, y)$$

for (t, x), $(\tau, y) \in (0, T] \times R_n$ $(\tau < t)$, where Γ_k is a fundamental solution of the equation

$$\mathscr{L}v = \sum_{i,j=1}^{n} a_{ij}^k(t, x) v_{x_i x_j} + \sum_{i=1}^{n} b_i^k(t, x) v_{x_i} + c_k^k(t, x) v - v_t = 0.$$

It is known that there exist positive constants C_1 and C_2

$$C_1 (t-\tau)^{-n/2} \exp\left\{-C_2 \frac{|x-y|^2}{t-\tau}\right\} \leqslant \Gamma_k(t, x;\ \tau, y)$$

for (t, x), $(\tau, y) \in (0, T] \times R_n$ $(\tau < t)$ (see [1], Theorem 7). This estimate implies that there exist positive constants β and M such that

$$\int_{R_n} e^{-\beta|y|^2} \gamma_i^j(dy) \leqslant M \qquad (j = 1, \ldots, N,\ i = 1, 2). \tag{3}$$

Set $\sigma^j = \gamma_1^j - \gamma_2^j$. Then

$$0 = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0, y) \sigma^j(dy) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0, y) e^{\beta|y|^2} \lambda^j(dy) \tag{4}$$

$(i = 1, \ldots, N)$ for all $(t, x) \in (0, T_1] \times R_n$, where

$$\lambda^j(E) = \int_E e^{-\beta|x|^2} \sigma^j(dx)$$

for all Borel subsets E of R_n. By the Hahn–Jordan decomposition theorem there are Borel sets A_j $(j = 1, \ldots, N)$ such that $\lambda^j \geqslant 0$ on all Borel-measurable subsets of A_j and $\lambda^j \leqslant 0$ on all Borel-measurable subsets of $R_n - A_j = B_j$. Set $\lambda_+^j(E) = \lambda^j(E \cap A_j)$ and $\lambda_-^j(E) = -\lambda^j(E \cap B_j)$. It follows from (3) that $\lambda_\pm^j(R_n) < \infty$. Therefore the measures λ_+^j and λ_-^j are regular. Since $\sigma^{j_0} \not\equiv 0$, we can assume that $\lambda_+^{j_0}(A_{j_0}) = a > 0$. We will show that this leads to a contradiction. By the regularity of $\lambda_+^{j_0}$ and $\lambda_-^{j_0}$, there exist a compact set $K \subset A_{j_0}$ such that $\lambda_+^{j_0}(K) > \frac{3}{4}a$ and a bounded open set $E \supset K$ such that $\lambda_0^j(E-K) < a/4$. Let $\varphi = \varphi(x)$ be a continuous function in R_n with $\varphi(x) = 1$ on K and $\varphi(x) = 0$ on $R_n - E$ and $0 \leqslant \varphi(x) \leqslant 1$. Introduce the functions

$$v_{ij}(t, y) = \int_{R_n} \Gamma_{ij}(t, x;\ 0, y)\varphi(x)e^{-\beta|x|^2}dx.$$

From the lemma it follows that

$$\lim_{t\to 0} v_{ij}(t, y) = \varphi(y)e^{-\beta|y|^2} \quad \text{for } i = j$$

and

$$\lim_{t\to 0} v_{ij}(t, y) = 0 \quad \text{for } i \neq j.$$

It is well known that

$$\Gamma_{ij}(t, x;\ \tau, y) \leqslant C_3(t-\tau)^{-n/2}\exp\left\{-C_4\frac{|x-y|^2}{t-\tau}\right\}$$

for $(t, x), (\tau, y) \in (0, T] \times R_n$ $(\tau < t)$, where C_3 and C_4 are positive constants. Hence, if $\bar{E} \subset \{x;\ |x| \leqslant r\}$, then there exists C_5 and T_2 such that

(5) $$0 \leqslant e^{\beta|y|^2}v_{ij}(t, y) \leqslant C_5 e^{\beta r^2}$$

for $0 < t < T_2$. It follows from (3) and (5) that

(6) $$\int_{R_n} v_{ij}(t, y)|\sigma^j|(dy) = \int_{R_n} e^{\beta|y|^2}v_{ij}(t, y)|\lambda^j|(dy) < \infty.$$

Therefore v_{ij} is integrable with respect to σ^j for each $t \in (0, T_2)$ and

$$\int_{R_n} v_{ij}(t, y)\sigma^j(dy) = \int_{R_n} e^{\beta|y|^2}v_{ij}(t, y)\lambda^j(dy).$$

Since $|\lambda^j|(R_n) < \infty$, applying the dominated convergence theorem we get

(7) $$\lim_{t\to 0}\int_{R_n} v_{j_0j_0}(t, y)\sigma^{j_0}(dy) = \lambda^{j_0}(K) + \int_{E-K}\varphi\lambda_+^{j_0}(dy) - \int_{E-K}\varphi\lambda_-^{j_0}(dy) > \tfrac{3}{4}a - a/4 = a/2 > 0$$

and

(8) $$\lim_{t\to 0}\int_{R_n} v_{ij}(t, y)\sigma^j(dy) = 0 \quad \text{for } i \neq j.$$

In view of (6), for each $t \in (0, T_2)$ we have

$$\int_{R_n} v_{j_0i}(t, y)\sigma^i(dy) = \int_{R_n}\Big[\int_{R_n}\Gamma_{ji}(t, x;\ 0, y)\varphi(x)e^{-\beta|x|^2}dx\Big]\sigma^i(dy)$$
$$= \int_{R_n}\Big[\int_{R_n}\Gamma_{j_0i}(t, x;\ 0, y)\sigma^i(dy)\Big]\varphi(x)e^{-\beta|x|^2}dx.$$

Summing over i, we conclude that

$$\sum_{i=1}^{N}\int_{R_n} v_{j_0 i}(t,y)\sigma^i(dy) = \int_{R_n}\Big[\int_{R_n}\sum_{i=1}^{N}\Gamma_{j_0 i}(t,x;\ 0,y)\sigma^i(dy)\Big]\varphi(x)e^{-\beta|x|^2}dx = 0$$

and

$$\lim_{t\to 0}\sum_{j=1}^{N}\int_{R_n} v_{j_0 i}(t,y)\sigma^i(dy) = 0$$

in contradiction to (7) and (8).

THEOREM 2. *Let*

$$U^j(t,x) = \int_{R_n}\Gamma_{ji}(t,x;\ 0,y)\varrho(dy) \qquad (j=1,\ldots,N),$$

where ϱ *is a non-negative Borel measure.*

Then U^j *is a non-negative solution of* (1) *in* $(0,T_1]\times R_n$ *if and only if*

$$\int_{R_n} U^j(t,x)e^{-\mu|x|^2}dx < \infty \qquad (j=1,\ldots,N) \tag{9}$$

for $t\in(0,T_1]$, *where* μ *is a non-negative number.*

Proof. If $\{U^j\}$ is a non-negative solution of (1) in $(0,T_1]\times R_n$, then it follows from the maximum principle that U^j satisfies (9) (for details see [3]).

To prove the converse, consider the Cauchy problem

$$L^k(u^1,\ldots,u^N) = 0 \quad \text{for } (t,x)\in(t_\nu,T_1]\times R_n,$$

$$u^j(t_\nu,x) = U^j(t_\nu,x) \quad \text{for } x\in R_n\ (j=1,\ldots,N),$$

where $\{t_\nu\}$ is a sequence of points in $(0,T_1)$ such that $t_\nu\to 0$. It follows from Theorem 1.3 of [4] (p. 237) that this problem has a non-negative solution u^j which can be written in the form

$$u^j(t,x) = \sum_{k=1}^{N}\int_{R_n}\Gamma_{jk}(t,x;\ t_\nu,y)\,U^k(t_\nu,y)dy.$$

By the Kolmogorov identity, we obtain

$$\begin{aligned} u^j(t,x) &= \sum_{k=1}^{N}\int_{R_n}\Gamma_{jk}(t,x;\ t_\nu,y)\Big[\int_{R_n}\Gamma_{ki}(t_\nu,y;\ 0,z)\varrho(dz)\Big]dy \\ &= \int_{R_n}\Big[\int_{R_n}\sum_{k=1}^{N}\Gamma_{jk}(t,x;\ t_\nu,y)\Gamma_{ki}(t_\nu,y;\ 0,z)\Big]\varrho(dz) \\ &= \int_{R_n}\Gamma_{ji}(t,x;\ 0,z)\varrho(dz) = U^j(t,x) \end{aligned}$$

for $(t,x)\in(t_\nu,T_1]\times R_n$. Hence $\{U^j\}$ is a solution of (1).

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