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CHARACTERIZATIONS OF PERIODIC FUNCTION SPACES OF BESOV-SOBOLEV TYPE VIA APPROXIMATION PROCESSES AND RELATIONS TO THE STRONG SUMMABILITY OF FOURIER SERIES

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1. Introduction

This paper gives a survey on results obtained in the joint works with W. Sickel [26]–[29] (cf. also W. Sickel [34], [35] and [30, Chapter 3]. We are concerned with the following problems. Let f be a 2π-periodic function and let $\{M_\nu f\}_{\nu=1}^{\infty}$ with $M_\nu f(x) \to f(x)$ $(\nu \to \infty)$ be an approximation process. We measure the rate of convergence of $M_\nu f(x) \to f(x)$ $(\nu \to \infty)$ by

$$(1)\qquad \sum_{\nu=1}^{\infty} \nu^{sq-1} \|f(x)-M_\nu f(x)|L_p\|^q < \infty,$$

$$(2)\qquad \sum_{j=0}^{\infty} 2^{sjq} \Big\|\Big(2^{-j} \sum_{\nu=2^j}^{2^{j+1}-1} |f(x)-M_\nu f(x)|^u\Big)^{1/u} \Big| L_p\Big\|^q < \infty,$$

$$(3)\qquad \Big\|\Big(\sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x)-M_\nu f(x)|^q\Big)^{1/q} \Big| L_p\Big\| < \infty.$$

Here $0 < s < \infty$, $0 < u \leqslant \infty$, $0 < q \leqslant \infty$, $0 < p \leqslant \infty$. $\|\cdot|L_p\|$ stands for the usual L_p-norm on the n-torus, $0 < p \leqslant \infty$. If $q = \infty$ we have to replace $\sum_\nu \nu^{-1}(\ldots)^q$ by $\sup_\nu(\ldots)$ in (1) and (3). If $u = \infty$ in (2) the modification for $(2^{-j}\sum_\nu(\ldots)^u)^{1/u}$ is $\sup_\nu|\ldots|$. We are interested in equivalent characterizations of (1)–(3) for concrete approximation processes in terms of periodic function spaces. Many special and general results are known concerning (1) and many authors have dealt with this problem (at least if $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$). Let us refer to the investigations by A. Zygmund [47], S. M. Nikol'skii [22, Chapter 8], P. L. Butzer, H. Berens [5], P. L. Butzer, R. J. Nessel [6], H. S. Shapiro [32, 33], J. Löfström [19, 20], J. Boman, H. S. Shapiro [4], W.

Trebels [39], J. Bergh, J. Löfström [2, Chapter 7], J. Peetre [25, Chapter 8], J. Boman [3], and H. Triebel [44, 2.5.3]. Thus it is well known that for certain approximation processes (1) with $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < \sigma$ is satisfied if and only if f belongs to the periodic Besov (Nikol'skii, Lipschitz) space $B^s_{p,q}$ (cf. Section 2 for the definition). Here σ corresponds to the saturation order of the approximation process under consideration. As we shall see later (cf. Section 5). problems (2) and (3) are related to the so-called strong approximation by Fourier series (or strong summability) due to G. Alexits, D. Králik [1] (at least in the case $p = \infty$). For the historical background and an extensive treatment including references see the recent book by L. Leindler [15]. It turns out that (2) corresponds to the Besov spaces, too, whereas (3) leads to a different class of function spaces, the so-called Lizorkin–Triebel spaces $F^s_{p,q}$ generalizing the usual Sobolev spaces (cf. Section 2 for the definition). These spaces are due to P. I. Lizorkin [17], [18] and H. Triebel [40] and have been studied extensively in the last years. We refer to the book by H. Triebel [44] (non-periodic spaces) and to [30, Chapter 3] (periodic spaces). In this paper we restrict ourselves to the periodic case. Of course, at least (1) and (3) have non-periodic counterparts. From the point of view of the theory of function spaces we look for equivalent characterizations of the two scales of spaces $B^s_{p,q}$ and $F^s_{p,q}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < \infty$, via approximation. There are several other types of equivalent descriptions or representations, e.g. via the decomposition method (cf. 2.1), via derivatives and differences (cf. 2.2), via temperatures and harmonic functions, via maximal functions, via interpolation or via atomics and molecules. More information concerning the spaces can be found, for example, in S. M. Nikol'skii [22], M. H. Taibleson [38], E. M. Stein [36], J. Bergh, J. Löfström [2], J. Peetre [25], H. Triebel [42], [44], [45], and [30, Chapter 3]. In particular, the theory of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ could be extended to values $0 < p < 1$ and $0 < q < 1$ in a natural way using tools of Fourier analysis, such as maximal inequalities and inequalities of Plancherel–Polya–Nikol'skii type. This is due to J. Peetre [24], [25]. A systematic theory has been developed by H. Triebel, cf. [44]. We are interested in characterizations of the spaces by means of differences and derivatives on the one-side, thus classifying functions according to their smoothness properties, and characterizations of type (1)–(3) on the other side. We consider approximation processes of type

$$M_\nu f = M^\psi_\nu f = \sum_{k \in Z_n} \psi\left(\frac{k}{\nu}\right) \hat{f}(k) e^{ikx}, \qquad \nu = 1, 2, \ldots, \tag{4}$$

where ψ is an appropriate function defined on the euclidean n-space R_n with $\psi(0) = 1$. Z_n denotes the set of all points $k = (k_1, \ldots, k_n)$ with integer components. $\hat{f}(k)$ is the kth Fourier coefficient and $kx = k_1 x_1 + \ldots + k_n x_n$. Examples are approximation by partial sums or certain means such as de la

Vallée-Poussin, Fejér, Riesz or Abel–Poisson means. We shall establish sufficient conditions concerning ψ implying direct and inverse approximation theorems (and hence also equivalence theorems) for the problems (1)–(3) in the language of the spaces $B^s_{p,q}$ and $F^s_{p,q}$. These general results have the following consequences:

(i) We are able to give a unified approach to the approximation of functions via classical means in the sense of (1)–(3) including the cases $0 < p < 1$ and $0 < q < 1$.

(ii) We extend the well-known results concerning (1) to values $p < 1$.

(iii) We can show that it is quite natural to deal with the problem of strong approximation (summability) within the framework of the spaces $B^s_{p,q}$ and $F^s_{p,q}$. In particular, this reveals the significance of the spaces $F^s_{p,q}$ in approximation theory.

Detailed proofs can be found in [26]–[29] (for partial results see also W. Sickel [34], [35] and [30, Chapter 3]). To derive our results we use methods of Fourier analysis related to those ones used by H. S. Shapiro [31]–[33], P. L. Butzer, R. J. Nessel [6], J. Löfström [19], [20], J. Boman, H. S. Shapiro [4], J. Peetre [25, Chapter 8], J. Boman [3], and, in particular, by H. Triebel [43]–[45], where we can find the key to deal with the full range of parameters $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Let us mention that for this latter purpose we essentially need the vector-valued maximal inequality for the Hardy–Littlewood maximal function by C. Fefferman, E. M. Stein [9] as well as the extension of Nikol'skii's inequality for entire analytic functions to values less than 1, cf. R. J. Nessel, G. Wilmes [21] or H. Triebel [41].

The paper is organized as follows. In Section 2 we define the periodic spaces $B^s_{p,q}$ and $F^s_{p,q}$ as decomposition spaces, describe special cases and embeddings, and give characterizations via differences and derivatives. Section 3 and Section 4 deal with general direct and inverse approximation theorems in the above sense, respectively. The problem of strong summability of Fourier series is treated in Section 5. Here our general results are applied to partial sums of Fourier series. The final Section 6 is devoted to the characterization of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ in the sense of (1)–(3) via classical means, i.e. via de la Vallée-Poussin, Riesz, and Abel–Poisson means.

This survey is based on lectures given at the Stefan Banach International Mathematical Centre in Warsaw in April 1986. We take the opportunity to thank our colleagues for their great hospitality.

2. Periodic function spaces of Besov–Sobolev type

2.1. Definitions

Let us fix some notation which will be used throughout the sequel. By Z_n, T_n, R_n we denote the set of all lattice-points, the n-dimensional torus, and the Euclidean n-space, respectively. Their generic points are denoted by k

$=(k_1, \ldots, k_n) \in Z_n$ and $x=(x_1, \ldots, x_n) \in R_n$. As usually T_n is represented by $T_n = \{x | \; x \in R_n, \; -\pi \leqslant x_j \leqslant \pi, \; j=1, \ldots, n\}$, where opposite sides are identified. $S(R_n)$ and $S'(R_n)$ mean the Schwartz space of all rapidly decreasing functions on R_n and its dual the space of tempered distributions, respectively. $D(T_n)$ and $D'(T_n)$ stand for the infinitely differentiable functions on T_n and its dual space the distributions on T_n. $S'(R_n)$ and $D'(T_n)$ are equipped with the weak topologies. If $f \in S'(R_n)$ then Ff and $F^{-1}f$ denote the Fourier transform of f and its inverse, respectively. If $\varphi \in S(R_n)$ then

$$(F\varphi)(y) = (2\pi)^{-n/2} \int_{R_n} f(x) e^{-ixy} dx, \quad y \in R_n.$$

Here $xy = x_1 y_1 + \ldots + x_n y_n$, dx Lebesgue measure. Furthermore we use

$$\hat{f}(k) = (2\pi)^{-n} f(e^{-ikx}), \quad k \in Z_n, \tag{5}$$

being the Fourier coefficients of $f \in D'(T_n)$. Via the representation of $f \in D'(T_n)$ as its Fourier series

$$f = \sum_{k \in Z_n} \hat{f}(k) e^{ikx}$$

we identify $D'(T_n)$ with the subspace $S'_\pi(R_n) \subset S'(R_n)$ of 2π-periodic tempered distributions, cf. [30, Chapter 3] or R. E. Edwards [8, Chapter 12]. C and L_p, $0 < p \leqslant \infty$, denote the spaces of continuous an p-integrable (with respect to the Lebesgue measure) complex-valued functions on T_n, respectively. They are endowed with the (quasi-) norms

$$\begin{aligned} \|f \,|\, C\| &= \max_{x \in T_n} |f(x)|, \\ \|f \,|\, L_p\| &= \Big(\int_{T_n} |f(x)|^p dx\Big)^{1/p}, \quad 0 < p < \infty, \\ \|f \,|\, L_\infty\| &= \operatorname{ess-sup}_{x \in T_n} |f(x)|. \end{aligned} \tag{6}$$

Furthermore, $L_p(R_n)$ and $\|f \,|\, L_p(R_n)\|$ $(0 < p \leqslant \infty)$ are used for the obvious non-periodic counterparts. Finally, $f * g$ means the convolution of $f \in S'(R_n)$ and $g \in S'(R_n)$, whenever it exists. For example,

$$(g * f)(x) = \int_{R_n} g(y) f(x-y) dy \tag{7}$$

if $g \in L_1(R_n)$ and $f \in L_p$, $1 \leqslant p \leqslant \infty$ (identity in $L_p = L_p(T_n)$).

Let Φ be the class of systems $\{\varphi_j(y)\}_{j=0}^{\infty} \subset S(R_n)$ with the following properties:

$$\begin{aligned} &\varphi_j(y) = \varphi(2^{-j} y); \quad j = 1, 2, \ldots, \\ &\operatorname{supp} \varphi_0 \subset \{y | \; |y| \leqslant 2\}, \\ &\operatorname{supp} \varphi \subset \{y | \; 1/2 \leqslant |y| \leqslant 2\}, \\ &\sum_{j=0}^{\infty} \varphi_j(y) = 1 \quad \text{for all } y \in R_n. \end{aligned} \tag{8}$$

If $f \in S'(R_n)$ and $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$ then we put

$$(9) \qquad f_j(x) = (F^{-1}(\varphi_j Ff))(x) = (2\pi)^{-n/2}((F^{-1}\varphi_j) * f)(x);$$

$j = 0, 1, \ldots$ (9) makes sense. Because of the famous Paley–Wiener–Schwartz theorem $f_j(x)$ is an entire analytic function of exponential type. If $f \in D'(T_n)$ $(= S'_\pi(R_n))$ then (9) can be reformulated by

$$(9') \qquad f_j(x) = \sum_{k \in Z_n} \varphi_j(k) \hat{f}(k) e^{ikx}$$

and $f_j(x)$ is a trigonometric polynomial. We have

$$(10) \qquad f = \sum_{j=0}^{\infty} f_j(x) \quad \text{(convergence in } S'(R_n)(D'(T_n))).$$

Definition. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $-\infty < s < \infty$. Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$.

(i) We put

$$(11) \qquad \begin{aligned} B^s_{p,q} &= \{f \mid f \in D'(T_n), \|f \mid B^s_{p,q}\|^\varphi = \Big(\sum_{j=0}^{\infty} 2^{sjq} \|f_j(x) \mid L_p\|^q\Big)^{1/q} < \infty\}, \\ B^s_{p,\infty} &= \{f \mid f \in D'(T_n), \|f \mid B^s_{p,\infty}\|^\varphi = \sup_{j=0,1,\ldots} 2^{sj} \|f_j(x) \mid L_p\| < \infty\}. \end{aligned}$$

(ii) If additionally $p < \infty$ we put

$$(12) \qquad \begin{aligned} F^s_{p,q} &= \{f \mid f \in D'(T_n), \|f \mid F^s_{p,q}\|^\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{sjq} |f_j(x)|^q\Big)^{1/q} \mid L_p\Big\| < \infty\}, \\ F^s_{p,\infty} &= \{f \mid f \in D'(T_n), \|f \mid F^s_{p,\infty}\|^\varphi = \|\sup_{j=0,1,\ldots} 2^{sj} |f_j(x)| \mid L_p\| < \infty\}. \end{aligned}$$

Remark 1. The spaces $B^s_{p,q}$ are called periodic Besov (Lipschitz) spaces and the spaces $F^s_{p,q}$ are referred to as periodic Lizorkin–Triebel spaces. For historical remarks we refer to H. Triebel [44] and to [30, Chapter 3]. We defined them by the so-called decomposition method. Indeed, (10) yields a decomposition of f in a series of trigonometric polynomials according to an underlying (dyadic) smooth partition of unity. Its convergence is measured by (11) and (12). For more information concerning the background of this construction principle we refer to H. Triebel [44, 2.2] or J. Peetre [25, Chapter 1]. The spaces $B^s_{p,q}$ and $F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $\min(p, q) \geqslant 1$). They are independent of the choice of $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$ (equivalent quasi-norms). For a systematic study we refer to [30, Chapter 3] (cf. also W. Sickel [34], [35]).

Remark 2. It is obvious how to define the non-periodic counterparts. We obtain $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ by replacing $D'(T_n)$ by $S'(R_n)$ and L_p by $L_p(R_n)$, respectively, in (11) and (12). In its full generality the spaces on R_n are extensively studied in H. Triebel [44]. We should emphasize that our considerations concerning the periodic case are based on this model case.

Remark 3. In general, $B^s_{p,q}$ and $F^s_{p,q}$ are spaces of periodic distributions. They can be identified with function spaces if s is sufficiently large. For example, we have

$$B^s_{p,q} \subset L_p \cap L_1, \quad F^s_{p,q} \subset L_p \cap L_1 \tag{13}$$

if $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and $s > \max(0, n(1/p-1))$ ($p < \infty$ in the case of the F-spaces). For further embeddings we refer to Section 2.3.

Remark 4. Let us consider problem (1) for means $M^\psi_\nu f$ of type (4). Its "dyadic" version reads as

$$\Big(\sum_{j=0}^{\infty} 2^{sjq}\Big\|\sum_{k\in Z_n} (1-\psi)(2^{-j}k)\hat{f}(k)e^{ikx}|L_p\Big\|^q\Big)^{1/q} < \infty .$$

Having in mind (9), (9′) this corresponds formally to the definition of the spaces $B^s_{p,q}$ in (11) with $1-\psi$ instead of φ. Given a function ψ we ask whether such a replacement leads to an equivalent representation of $B^s_{p,q}$ for some values p, q, and s. Roughly speaking this is our approach to approximation theorems (cf. also J. Peetre [25, Chapter 8] and H. Triebel [44], [45]. Analogously we deal with problems (2) and (3). Thus it is our task to find criterions concerning ψ (or $1-\psi$) implying equivalent representations in the above sense.

2.2. Characterizations by means of differences and derivatives, special cases

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of non-negative integers. If $f \in D'(T_n)$ we denote by

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} f$$

the distributional derivative of f of order α ($|\alpha| = \alpha_1 + \ldots + \alpha_n$).

Proposition 1. *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $-\infty < s < \infty$. *Let* $m = 1, \ldots$

(i) *We have*

$$B^s_{p,q} = \{f \in D'(T_n) |\ D^\alpha f \in B^{s-m}_{p,q},\ 0 \leqslant |\alpha| \leqslant m\}. \tag{14}$$

(ii) *If additionally* $p < \infty$ *then*

$$F^s_{p,q} = \{f \in D'(T_n) |\ D^\alpha f \in F^{s-m}_{p,q},\ 0 \leqslant |\alpha| \leqslant m\}. \tag{15}$$

For the proof we refer to [30, Chapter 3] and its non-periodic counterpart in H. Triebel [44, Theorem 2.3.8].

PROPOSITION 2. *Let* $1 < p < \infty$.

(i) *We have*

(16) $$F^0_{p,2} = L_p.$$

(ii) *If* $m = 1, 2, \ldots$ *and* $-\infty < s < \infty$ *then*

(17) $$F^m_{p,2} = \{f \mid D^\alpha f \in L_p,\ 0 \leqslant |\alpha| \leqslant m\},$$
$$F^s_{p,2} = \{f \in D'(T_n) \mid \sum_{k \in Z_n} (1+|k|^2)^{s/2} \hat{f}(k) e^{ikx} \in L_p\}.$$

For the proof we refer to [30, 3.5.4]. (16) is the well-known Littlewood–Paley theorem. (17) follows from (15) with $s = m$ and (16). We put $F^m_{p,2} = W^m_p$, $m = 1, 2, \ldots$ (periodic Sobolev spaces) and $F^s_{p,2} = H^s_p$ (periodic Liouville or Bessel-potential spaces).

Let f be a function defined on R_n. We put

$$\Delta_h f(x) = f(x+h) - f(x), \quad (x, h \in R_n),$$
$$\Delta^l_h f(x) = \Delta_h(\Delta^{l-1}_h f)(x), \quad l = 1, 2, \ldots,$$
$$\omega^l(t, f)_p = \sup_{|h| \leqslant t} \|\Delta^l_h f(x) \mid L_p\|, \quad 0 < p \leqslant \infty,\ 0 < t \leqslant 1.$$

PROPOSITION 3. *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. *Let* $m = 0, 1, \ldots$, $l = 1, 2, \ldots$ *such that* $l > s - m > \max(0, n(1/p - 1))$. *Then,*

(18) $$B^s_{p,q} = \left\{ f \in L_p \mid \|f \mid B^s_{p,q}\|^{(1)}_{m,l} = \sum_{|\alpha| \leqslant m} \|D^\alpha f \mid L_p\| + \sum_{|\alpha| \leqslant m} \left(\int_{T_n} |h|^{-(s-m)q} \|\Delta^l_h D^\alpha f \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q} < \infty \right\},$$

(19) $$B^s_{p,q} = \left\{ f \in L_p \mid \|f \mid B^s_{p,q}\|^{(2)}_{m,l} = \sum_{|\alpha| \leqslant m} \|D^\alpha f \mid L_p\| + \sum_{|\alpha| \leqslant m} \left(\int_0^1 t^{-(s-m)q} \omega^l(t, D^\alpha f)^q_p \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

(modification if $q = \infty$*). All quasi-norms* $\|f \mid B^s_{p,q}\|^{(i)}_{m,l}$, $i = 1, 2$, *are equivalent to each other.*

For the proof we refer to the non-periodic counterpart in H. Triebel [44, Theorem 2.5.12 and Remarks 2.5.12/2, 3] (or in H. Triebel [45]) and the methods developed in [30, Chapter 3]. (18) and (19) show that the spaces $B^s_{p,q}$ with $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $0 < s < \infty$ coincide with the (nowadays classical) Besov (Nikol'skii if $q = \infty$, Hölder (Lipschitz)–Zygmund if $p = q = \infty$) spaces.

PROPOSITION 4. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$.

(i) *Let* $m = 0, 1, \ldots, l = 1, 2, \ldots$ *such that*

$$l > s-m > \max\left(0, n\left(\frac{1}{\min(p, q)}-1\right)\right).$$

Then

$$(20)\qquad \|f\,|F^s_{p,q}\|^{(1)}_{m,l} = \sum_{|\alpha|\leq m} \|D^\alpha f\,|L_p\| + \sum_{|\alpha|\leq m}\left\|\left(\int_0^1 t^{-(s-m)q}\Big(\int_{|h|\leq 1} |\Delta^l_{th} D^\alpha f(x)|\,dh\Big)^q \frac{dt}{t}\right)^{1/q} |L_p\right\|$$

is an equivalent quasi-norm in $F^s_{p,q}$ *(modified if* $q = \infty$*).*

(ii) *Let* $m = 0, 1, \ldots, l = 1, 2, \ldots$ *such that* $l > s-m > 1/\min(p, q)$. *Then*

$$(21)\qquad \|f\,|F^s_{p,q}\|^{(2)}_{m,l} = \sum_{|\alpha|\leq m} \|D^\alpha f\,|L_p\| + \sum_{|\alpha|\leq m}\left\|\left(\int_{T_n} |h|^{-(s-m)q} |\Delta^l_h D^\alpha f(x)|^q \frac{dh}{|h|^n}\right)^{1/q} |L_p\right\|$$

is an equivalent quasi-norm in $F^s_{p,q}$ *(modified if* $q = \infty$*).*

For the proof we refer again to the non-periodic counterpart in H. Triebel [44, Corollary 2.5.11 and Theorem 2.5.10] (or in H. Triebel [45]) and the methods developed in [30, Chapter 3]. The proposition shows that the spaces $F^s_{p,q}$ can also be characterized via differences (ball means of differences in (20)) and derivatives if $q \neq 2$ and/or $p < 1$. We also refer to G. A. Kalyabin [11].

Remark 5. As already mentioned the above assertions have non-periodic counterparts. The necessary modifications are obvious.

2.3. Embeddings

Obviously we have $B^s_{p,p} = F^s_{p,p}$, $-\infty < s < \infty$, $0 < p < \infty$. With this trivial exception the spaces $B^{s_0}_{p_0,q_0}$ and $F^{s_1}_{p_1,q_1}$ are always different. Also the spaces $B^{s_0}_{p_0,q_0}$ $(F^{s_0}_{p_0,q_0})$ and $B^{s_1}_{p_1,q_1}$ $(F^{s_1}_{p_1,q_1})$ are different for different triplets (s_0, p_0, q_0) and (s_1, p_1, q_1). This can be proved analogously to H. Triebel [44, 2.3.9]. Thus embeddings between these spaces are of peculiar interest. For illustration let us mention few of them (cf. also Remark 3). Proofs can be found in [30, Chapter 3].

If $0 < p < \infty$, $0 < q \leq \infty$, and $-\infty < s < \infty$ then

$$(22)\qquad B^s_{p,\min(p,q)} \subset F^s_{p,q} \subset B^s_{p,\max(p,q)}.$$

If $1 \leq p \leq \infty$ then

$$(23)\qquad B^0_{p,1} \subset L_p \subset B^0_{p,\infty}.$$

Moreover,

(24) $$B^0_{\infty,1} \subset C \subset B^0_{\infty,\infty}.$$

If $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $0 < s < \infty$, then

(25) $$B^{s+n/p}_{p,q} \subset B^s_{\infty,\infty}$$

and (if $p < \infty$)

(26) $$F^{s+n/p}_{p,q} \subset B^s_{\infty,\infty}.$$

Let us note that the spaces

$$C^m = \{f \mid D^\alpha f \in C,\ 0 \leqslant |\alpha| \leqslant m\} \quad \text{and} \quad W^m_1 = \{f \mid D^\alpha f \in L_1,\ 0 \leqslant |\alpha| \leqslant m\},$$

$m = 0, 1, \ldots$, are not contained in the two scales $B^s_{p,q}$ and $F^s_{p,q}$ as special cases. Furthermore, the space

$$\mathrm{Lip}\,1 = \{f \mid f \in C,\ \sup_h |h|^{-1} \|\Delta_h f(x) \mid C\| < \infty\}$$

is not included. We have the strict embeddings

(27) $$B^1_{\infty,1} \subset C^1 \subset \mathrm{Lip}\,1 \subset B^1_{\infty,\infty}.$$

3. Direct approximation theorems

3.1. Direct results for $F^s_{p,q}$

Let $f \in D'(T_n)$ and let $\psi = \psi(\xi) \in L_\infty(R_n)$ be defined for all $\xi \in R_n$, with $\psi(0) = 1$. We introduce the means

(28) $$M^\psi_\nu f(x) = \sum_{k \in Z_n} \psi\left(\frac{k}{\nu}\right) \hat{f}(k) e^{ikx}, \qquad \nu = 1, 2, \ldots$$

This makes sense at least in $D'(T_n)$. If ψ has compact support then the $M^\psi_\nu f$ are trigonometric polynomials, otherwise periodic functions (distributions). In this section we assume that

(29) $$F^{-1}\psi \in L_1(R_n).$$

Then (28) can be reformulated as

$$M^\psi_\nu f(x) = c\big((F^{-1}\psi(\nu^{-1}\cdot)) * f\big)(x)$$

(cf. (7) for $f \in L_p$, $1 \leqslant p \leqslant \infty$). Clearly, $M^\psi_\nu f \to f$ in $D'(T_n)$.

Let $h(\xi) \in S(R_n)$ and $H(\xi) \in S(R_n)$ be functions satisfying

$$h(\xi) = 1 \quad \text{if } |\xi| \leqslant 1,$$

$$\mathrm{supp}\, h \subset \{\xi \mid |\xi| \leqslant 2\},$$

$$H(\xi) = 1 \quad \text{if } 1/2 \leqslant |\xi| \leqslant 2,$$

$$\operatorname{supp} H \subset \{\xi \mid 1/4 \leqslant |\xi| \leqslant 4\}.$$

THEOREM 1 ([28, Theorem 1]). *Let ψ be defined on R_n such that $\psi(0) = 1$ and $F^{-1}\psi \in L_1(R_n)$. Let $\sigma > 0$ and $\lambda > n$ such that*

$$|F^{-1}(|\xi|^{-\sigma}(1-\psi(\xi))h(\xi))(y)| \leqslant c(1+|y|)^{-\lambda} \tag{30}$$

and

$$\sup_{l=-L,-L+1,\ldots} |F^{-1}(\psi(2^l\xi)H(\xi))(y)| \leqslant c_L(1+|y|)^{-\lambda} \tag{31}$$

for all $y \in R_n$ and natural numbers L with positive constants c, c_L independent of y. If $n/\lambda < p < \infty$, $n/\lambda < q \leqslant \infty$, and $0 < s < \sigma$ then there exists a positive constant c' such that

$$\|(\sum_{\nu=1}^{\infty} \nu^{sq-1}|f(x)-M_\nu^\psi f(x)|^q)^{1/q} | L_p\| \leqslant c\|f \mid F_{p,q}^s\| \tag{32}$$

for all $f \in F_{p,q}^s \cap L_1$ (modified if $q = \infty$ by $\sup_\nu \nu^s|\ldots|$).

Remark 6. Let us discuss the conditions (30) and (31). If (30) is satisfied for some $\lambda > n$ then the function on the left-hand side belongs to $L_1(R_n)$ and hence $|x|^{-\sigma}(1-\psi(x))$ is continuous in a neighbourhood of 0. Consequently,

$$1-\psi(x) = O(|x|^\sigma) \qquad (|x| \to 0)$$

is a necessary condition to apply our theorem. Moreover, if $1-\psi(x) \asymp |x|^{\sigma_0}$ then σ_0 corresponds to the saturation order of the approximation process $\{M_\nu^\psi\}$ in L_p $(1 \leqslant p \leqslant \infty)$, cf. P. L. Butzer, R. J. Nessel [6, 12.4]. This shows that we cannot expect (33) for $s > \sigma_0$. Note that we are not able to deal with the case $s = \sigma_0$ by our methods. Conditions (30) and (31) can be replaced by

$$|\xi|^{-\sigma}(1-\psi(\xi))h(\xi) \in B_{1,\infty}^\lambda(R_n) \tag{30'}$$

and

$$\sup_{l=-L,-L+1,\ldots} \|\psi(2^l\xi)H(\xi) | B_{1,\infty}^\lambda(R_n)\| < \infty, \tag{31'}$$

respectively, where $B_{1,\infty}^\lambda(R_n)$ denotes the non-periodic Besov space. (30), (31), (30'), (31') stand for regularity conditions of the involved functions. The larger the number λ the larger the admissible range of parameters p and q. Moreover, (31) and (31') are satisfied if ψ is infinitely differentiable in $R_n - \{0\}$ and

$$\sup_{|x| \geqslant \delta} |x|^{|\alpha|}|D^\alpha\psi(x)| \leqslant c_\delta < \infty \tag{31''}$$

for all α, $|\alpha| \leqslant \lambda+1$ and all $\delta, \delta > 0$. If the generating function ψ has additionally compact support then

$$(33)\qquad |(F^{-1}\psi)(y)| \leqslant c(1+|y|)^{-\lambda}$$

implies (31) (and $F^{-1}\psi \in L_1(R_n)$). If $\psi(\xi) = 1$ in a neighbourhood of 0 then (33) implies also (30) for all $\sigma > 0$. Moreover, (33) is satisfied if $\psi \in B^{\lambda+\varepsilon}_{1,\infty}(R_n)$ for some $\varepsilon > 0$.

3.2. Direct results for $B^s_{p,q}$

THEOREM 2 ([28, Theorem 3]). *Let $\psi(\xi)$ be a function satisfying the assumptions of Theorem 1 with real numbers $\sigma > 0$ and $\lambda > n$.*

(i) *If $n/\lambda < p \leqslant \infty$, $0 < q \leqslant \infty$, and $0 < s < \sigma$ then there exists a positive constant c such that*

$$(34)\qquad \Big(\sum_{\nu=1}^{\infty} \nu^{sq-1} \|f(x) - M^{\psi}_{\nu} f(x)| L_p\|^q\Big)^{1/q} \leqslant c\|f|B^s_{p,q}\|$$

(modified if $q = \infty$ by $\sup_{\nu} \nu^s\|\ldots\|$) holds for all $f \in B^s_{p,q} \cap L_1$. Moreover,

$$(35)\qquad \Big(\sum_{j=0}^{\infty} 2^{sjq} \sup_{\nu=2^j,\ldots,2^{j+1}-1} \|f(x) - M^{\psi}_{\nu} f(x)| L_p\|^q\Big)^{1/q} \leqslant c\|f|B^s_{p,q}\|$$

holds for all $f \in B^s_{p,q} \cap L_1$.

Remark 7. Of course (34) is a trivial consequence of (35). Furthermore, let us note that

$$(36)\qquad \Big\|\Big(2^{-j} \sum_{\nu=2^j}^{2^{j+1}-1} |f(x) - M^{\psi}_{\nu} f(x)|^u\Big)^{1/u} \Big| L_p\Big\| \leqslant \sup_{\nu=2^j,\ldots,2^{j+1}-1} \|f(x) - M^{\psi}_{\nu} f(x)| L_p\|$$

if $0 < u \leqslant p \leqslant \infty$.

Remark 8. Assertions of type (34) (with less restrictive assumptions concerning ψ) are well known for $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$. We refer to the literature mentioned in the introduction. However, one of our main goals is the extension to values $p < 1$ (cf. also J. Peetre [25], p. 258). This requires the stronger conditions (30) and (31).

4. Inverse approximation theorems

4.1. Inverse results for $F^s_{p,q}$

THEOREM 3 ([27, Theorem 1], [28, Theorem 2]). *Let $\psi(\xi)$ be defined on R_n, continuous at the point 0 and let $\psi(0) = 1$.*

(i) *Let* $\operatorname{supp}\psi$ *be compact and let* ψ *be infinitely differentiable in a neighbourhood of* 0, *except at the point* 0 *itself. If* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < \infty$ *then there exists a positive constant c such that*

$$\|f \mid F^s_{p,q}\| \leqslant c\Big(\|f \mid L_p\| + \Big\|\Big(\sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x) - M^\psi_\nu f(x)|^q\Big)^{1/q} \Big| L_p\Big\|\Big) \tag{37}$$

holds for all $f \in L_p \cap D'(T_n)$.

(ii) *Let* $F^{-1}\psi \in L_1(R_n)$ *and let* (31) *be satisfied for some number* $\lambda > n$. *Let* ψ *be infinitely differentiable in* $R_n - \{0\}$ *and let d be a natural number such that*

$$|\psi(\xi/d) - \psi(\xi)| \neq 0 \quad \text{if } 1 \leqslant |\xi| \leqslant 2.$$

If $n/\lambda < p < \infty$, $n/\lambda < q \leqslant \infty$ *and* $\max\big(0, n(1/\min(p, q) - 1)\big) < s < \infty$ *then* (37) *holds true for all* $f \in F^s_{p,q}$.

Remark 9. If $1 < p < \infty$ and $1 < q \leqslant \infty$ then assumption (31) can be omitted in part (ii). Then (37) holds true for all $1 < p < \infty$, $1 < q \leqslant \infty$, $0 < s < \infty$ and all $f \in L_p \cap D'(T_n)$ with finite right-hand side (cf. [28, Remark 6]).

4.2. Inverse results for $B^s_{p,q}$

THEOREM 4 ([27, Theorem 2] and [28, Theorem 4]).

(i) *Let* ψ *be a function satisfying the assumptions of part* (i) *of Theorem* 3. *If* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < \infty$ *then there exists a positive constant c such that*

$$\|f \mid B^s_{p,q}\| \leqslant c\Big(\|f \mid L_p\| + \Big(\sum_{\nu=1}^{\infty} \nu^{sq-1} \|f(x) - M^\psi_\nu f(x) | L_p\|^q\Big)^{1/q}\Big) \tag{38}$$

holds for all $f \in L_p \cap D'(T_n)$. *Furthermore,*

$$\|f \mid B^s_{p,q}\| \leqslant c\Big(\|f \mid L_p\| + \Big(\sum_{j=0}^{\infty} 2^{sjq} \Big\|\Big(2^{-j} \sum_{\nu=2^j}^{2^{j+1}-1} |f(x) - M^\psi_\nu f(x)|^u\Big)^{1/u} \Big| L_p\Big\|^q\Big)^{1/q}\Big) \tag{39}$$

for all $0 < u < \infty$ *(modified if* $q = \infty$*).*

(ii) *Let* ψ *be a function satisfying the assumptions of part* (ii) *of Theorem* 3. *If* $n/\lambda < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $n\big(1/\min(1, p) - 1\big) < s < \infty$ *then* (38) *holds for all* $f \in B^s_{p,q}$. *If* $n/\lambda < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < u < \infty$, *and* $n\big(1/\min(1, p, u) - 1\big) < s < \infty$ *then* (39) *holds for all* $f \in B^s_{p,q}$ *(modified if* $q = \infty$*).*

Of our special interest are estimates of type (38) and (39) in the case of $p = \infty$. The following corollary is an immediate consequence of (39).

COROLLARY. *Let* ψ *be a function satisfying the assumptions of part* (i) *of Theorem* 3. *If* $0 < q < \infty$, $0 < s < \infty$, *then there exist positive constants c and* *c' such that*

$$(40)\quad \|f\,|B^s_{\infty,\infty}\| \leqslant c\Big(\|f\,|L_\infty\| + \sup_{j=0,1,\ldots} 2^{sj}\Big\|\Big(2^{-j}\sum_{\nu=2^j}^{2^{j+1}-1} |f(x)-M^\psi_\nu f(x)|^q\Big)^{1/q}\Big|L_p\Big\|\Big)$$

$$\leqslant c'\Big(\|f\,|L_\infty\| + \Big\|\Big(\sum_{\nu=1}^{\infty} \nu^{sq-1}|f(x)-M^\psi_\nu f(x)|^q\Big)^{1/q}\Big|L_\infty\Big\|\Big).$$

Remark 10. Besides the regularity conditions the Tauberian conditions "$\psi(\xi) > 0$ in a neighbourhood of 0" in part (i) and "$|\psi(\xi/d) - \psi(\xi)| \neq 0$ for $1/2 \leqslant |\xi| \leqslant 2$" in part (ii) of Theorems 3 and 4 are crucial assumptions in our proofs. For the use of Tauberian conditions in this sense we refer to H. S. Shapiro [31], J. Peetre [25, Chapter 1] and H. Triebel [43], [45].

Remark 11. For an improvement of the last part of (40) in the case $1 < q < \infty$ we refer to W. Sickel [35, Theorem 7] (cf. [30, Theorem 3.7.3/2]).

5. Approximation and strong summability by partial sums

5.1. Equivalence theorems

Let $f \in D'(T_n)$. We put

$$(41)\qquad S^Q_\nu f(x) = \sum_{|k_1|<\nu} \ldots \sum_{|k_n|<\nu} \hat{f}(k)e^{ikx}, \qquad \nu = 1, 2, \ldots,$$

$$(42)\qquad S^B_\nu f(x) = \sum_{|k|<\nu} \hat{f}(k)e^{ikx}, \qquad \nu = 1, 2, \ldots$$

Clearly, $S^Q_\nu f$ and $S^B_\nu f$ denote partial sums of the Fourier series of f with respect to summation on cubes and balls. Choosing ψ in (28) as the characteristic function of the open cube $Q = \{x|\ |x_i| < 1,\ i = 1, \ldots, n\}$ or the unit ball $B = \{x|\ |x| < 1\}$ we obtain $S^Q_\nu f$ and $S^B_\nu f$ as special cases, respectively. Unfortunately, the direct results of Section 3 can not be applied, because $F^{-1}\psi$ does not belong to $L_1(R_n)$. However, the assumptions of parts (i) of Theorems 3 and 4 are satisfied. Hence the inverse statements (37)–(40) hold true for S^B_ν and S^Q_ν. Moreover, using the Fourier multiplier properties of characteristic functions on cubes we can prove direct results for $1 < p < \infty$, $0 < s < \infty$. We have the following

THEOREM 5 ([26] or [30, Theorem 3.7.1]). *Let* $1 < p < \infty$ *and* $0 < s < \infty$.

(i) *If* $0 < q < \infty$, *then*

$$(43)\qquad f \in B^s_{p,q} \Leftrightarrow \|f\,|L_p\| + \Big(\sum_{\nu=1}^{\infty} \nu^{sq-1}\|f(x)-S^Q_\nu f(x)|L_p\|^q\Big)^{1/q} < \infty.$$

(ii) *If* $1 < q < \infty$, *then*

$$(44) \qquad f \in F^s_{p,q} \Leftrightarrow \|f\,|\,L_p\| + \Big\|\Big(\sum_{v=1}^{\infty} v^{sq-1}\,|f(x) - S^Q_v f(x)|^q\Big)^{1/q}\,\Big|\,L_p\Big\| < \infty .$$

Furthermore, the right-hand sides of (43) *and* (44) *are equivalent* (*quasi-*) *norms in* $B^s_{p,q}$ *and* $F^s_{p,q}$, *respectively.*

5.2. Strong approximation (summability)

Now, we are concerned with the one-dimensional case and with the case $p = \infty$. We put $S_v f = S^Q_v f = S^B_v f$. Let $f \in L_\infty$, $0 < \beta$, $u < \infty$. We introduce the so-called strong means

$$(45) \qquad h_n(f, \beta, u, x) := \Big(n^{-\beta} \sum_{v=1}^{n} v^{\beta-1}\,|S_v f(x) - f(x)|^u\Big)^{1/u}.$$

Strong approximation deals with the rate of convergence of

$$h_n(f, \beta, u, x) \to 0 \quad \text{if } n \to \infty .$$

For the historical background we refer to the book by L. Leindler [15]. Obviously,

$$\|h_n(f, \beta, u, x)\,|\,L_\infty\| = O(n^{-\beta/u})$$

is equivalent to

$$\Big\|\Big(\sum_{v=1}^{\infty} v^{\beta-1}\,|S_v f - f(x)|^u\Big)^{1/u}\,\Big|\,L_\infty\Big\| < \infty .$$

This justifies to call (3) a problem of strong approximation (summability). In particular, (44) can be considered as an equivalence theorem concerning strong summability in the L_p-norm, $1 < p < \infty$. In the case $p = \infty$ we are able to establish the following results.

THEOREM 6 ([27, Theorem 5 and Corollary 4]). *Let* $0 < u < \infty$, $0 < s < \infty$, *and let* $f \in L_\infty(T_1)$.

(i) *If* $0 < q < \infty$, *then*

$$(46) \qquad f \in B^s_{\infty,q}(T_1)$$

$$\Leftrightarrow \|f\,|\,L_\infty(T_1)\| + \Big(\sum_{j=0}^{\infty} 2^{sjq}\,\Big\|\Big(2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |(f - S_v f)(x)|^u\Big)^{1/u}\,\Big|\,L_\infty(T_1)\Big\|^q\Big)^{1/q} < \infty .$$

(ii) *If* $q = \infty$, *then*

$$(47) \qquad f \in B^s_{\infty,\infty}(T_1)$$

$$\Leftrightarrow \|f\,|\,L_\infty(T_1)\| + \sup_{j=0,1,\dots} 2^{sj}\,\Big\|\Big(2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |(f - S_v f)(x)|^u\Big)^{1/u}\,\Big|\,L_\infty(T_1)\Big\| < \infty .$$

Remark 12. The theorem is due to L. Leindler [12, Lemma 6] (cf. also [14, Lemma 1]). The inverse parts in (46) and (47) are also consequences of the Corollary and (39). They hold true in the higher dimensional case, too.

Remark 13. Let us add few remarks on relations to classical results. As a consequence of (47) we obtain (after some calculation)

(48) $$\|h_n(f, \beta, u, x) | L_\infty(T_1)\| = O(n^{-s}) \Leftrightarrow f \in B^s_{\infty,\infty}(T_1)$$

if $0 < s < \beta/u$, $0 < u < \infty$, $0 < \beta < \infty$. Furthermore, (47) implies

(49) $$\|h_n(f, \beta, u, x) | L_\infty(T_1)\| = O(n^{-\beta/u}) \Rightarrow f \in B^{\beta/u}_{\infty,\infty}(T_1).$$

These are well-known statements which can be found in L. Leindler [15]. The converse of (49) is not true (L. Leindler [13]). However, (46) with $q = u$ and $s = \beta/u$ shows

(50) $$f \in B^{\beta/u}_{\infty,u}(T_1) \Rightarrow \|h_n(f, \beta, u, x) | L_\infty(T_1)\| = O(n^{-\beta/u}).$$

It is proved in W. Sickel [35, Theorem 7] (cf. also [30, Theorem 3.7.3/2]) that (49) can be improved if $u > 1$. Namely, we have

(51) $$\|h_n(f, \beta, u, x) | L_\infty(T_1)\| = O(n^{-\beta/u}) \Rightarrow f \in F^{\beta/u}_{\infty,u}(T_1).$$

Here $F^{\beta/u}_{\infty,u}(T_1)$ stands for the dual space of $F^{-\beta/u}_{1,u'}(T_1)$, $1/u + 1/u' = 1$. Note that

(52) $$B^{\beta/u}_{\infty,u}(T_1) \subset F^{\beta/u}_{\infty,u}(T_1) \subset B^{\beta/u}_{\infty,\infty}(T_1),$$

cf. [30, Theorem 3.5.6] and the embedding (22). If we choose $q = 1$ in (46) then we obtain

(53) $$\sum_{j=0}^{\infty} \|(\sum_{\nu=2^j}^{2^{j+1}-1} \nu^{su-1} |(f - S_\nu f)(x)|^u)^{1/u} | L_\infty(T_1)\| < \infty \Leftrightarrow f \in B^s_{\infty,1}(T_1)$$

for all u, $0 < u < \infty$, and s, $0 < s < \infty$. Having in mind the embeddings (27) and Proposition 1 this is an improvement of L. Leindler [12, Theorem 5] (cf. also L. Leindler [14, Theorem A]). Finally, let us mention that the above inverse results ((49), (51), and "$\Rightarrow$" in (48) and (53)) carry over to the multiple case. This follows from (39), (40), and Remark 11.

6. Characterizations of $B^s_{p,q}$ and $F^s_{p,q}$ via means

$M^\psi_\nu f(x)$, $\nu = 1, 2, \ldots$, has the meaning of (28). In the following examples we shall apply the results of Sections 3 and 4 to derive approximation theorems of Jackson and Bernstein type for classical means. We are interested in the following equivalences

(54) $$f \in B^s_{p,q} \Leftrightarrow \|f | L_p\| + (\sum_{\nu=1}^{\infty} \nu^{sq-1} \|f(x) - M^\psi_\nu f(x) | L_p\|^q)^{1/q} < \infty,$$

(55) $$f \in B^s_{p,q} \Leftrightarrow \|f \mid L_p\| + \Big(\sum_{j=0}^{\infty} 2^{sjq} \Big\|\Big(2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M^{\psi}_v f(x)|^u\Big)^{1/u} \Big| L_p\Big\|^q\Big)^{1/q} < \infty,$$

(56) $$f \in F^s_{p,q} \Leftrightarrow \|f \mid L_p\| + \Big\|\Big(\sum_{v=1}^{\infty} v^{sq-1} |f(x) - M^{\psi}_v f(x)|^q\Big)^{1/q} \Big| L_p\Big\| < \infty$$

(modified if $q = \infty$ and/or $u = \infty$).

6.1. Classical de la Vallée-Poussin means

Let $f \in D'(T_1)$. We put

(57) $$V_v f(x) = \frac{1}{v} \sum_{\mu=v}^{2v-1} S_\mu f(x), \qquad v = 1, 2, \ldots$$

It is not difficult to see that $V_v f(x) = M^{\psi}_v f(x)$, $v = 1, 2, \ldots$, where

(58) $$\psi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leqslant 1, \\ 2 - |\xi| & \text{if } 1 < |\xi| < 2, \\ 0 & \text{if } |\xi| \geqslant 2. \end{cases}$$

One can show that $\psi(\xi) \in B^2_{1,\infty}(R_1)$ (cf. [27, 5.2]). Hence by Remark 6 Theorem 1, and Theorem 2 apply with $\lambda = 2$ and $0 < \sigma < \infty$. Trivially, the assumptions of parts (i) of Theorems 3 and 4 are satisfied. Using Remark 7 we obtain the following equivalences.

THEOREM 7 ([27, Theorem 6]). *$\psi(\xi)$ has the meaning of* (58). *Let $f \in L_1(T_1)$.*

(i) *If $1/2 < p \leqslant \infty$, $0 < q \leqslant \infty$, and $0 < s < \infty$ then* (54) *holds.*

(ii) *If $1/2 < p \leqslant \infty$, $0 < u \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, and $0 < s < \infty$ then* (55) *holds.*

(iii) *If $1/2 < p < \infty$, $1/2 < q \leqslant \infty$, and $0 < s < \infty$ then* (56) *holds.*

Remark 14. For characterizations of type (54) and (56) for the means V_v we refer also to P. Oswald [23]. Further characterizations of the spaces $B^s_{p,q}$, $1 \leqslant p \leqslant \infty$, with the help of de la Vallée-Poussin means can be found in S. M. Nikol'skii [22, Chapter 8] (cf. also P. L. Butzer, R. J. Nessel [6], W. Trebels [39]). (55) with $p = \infty$ corresponds to (46) and (47) with V_v instead of S_v. Hence, Remark 13 applies to V_v. Results of type (49) with V_v instead of S_v have been obtained by L. Leindler, A. Meir [16].

6.2. Generalized de la Vallée-Poussin means

Let $f \in D'(T_n)$. Let $\psi = \psi(\xi)$ be a continuous function defined on R_n, such that

$$(59)\qquad \begin{cases} \psi(\xi) = 1 & \text{if } |\xi| \leqslant 1, \\ \operatorname{supp}\psi \subset \{\xi \mid |\xi| \leqslant 2\}. \end{cases}$$

Then $M_\nu^\psi f$, $\nu = 1, 2, \ldots$, are called *generalized de la Vallée–Poussin means*. In the same way as in 6.1 we derive the following equivalences.

THEOREM 8 ([27, Theorem 7]). *Let* $\psi \in B^{\lambda}_{1,\infty}(R_n)$, $\lambda > n$, *such that* (59) *is satisfied. Let* $f \in L_1(T_n)$.

(i) *If* $n/\lambda < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $0 < s < \infty$ *then* (54) *holds.*

(ii) *If* $n/\lambda < p \leqslant \infty$, $0 < u \leqslant p \leqslant \infty$, $0 < q < \infty$, *and* $0 < s < \infty$ *then* (55) *holds.*

(iii) *If* $n/\lambda < p < \infty$, $n/\lambda < q \leqslant \infty$, *and* $0 < s < \infty$ *then* (56) *holds.*

Remark 15. Theorem 8 is a generalization of Theorem 7. Using smooth means instead of partial sums (cf. Theorem 5) we are able to extend our considerations to values p (and q) less than 1. In particular, if ψ is C^∞ with (59) then the full range $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ is covered (cf. also P. Oswald [23] and [30, Chapter 3]).

6.3. Riesz means

Let $0 < \beta < \infty$ and $0 < \gamma < \infty$ be real numbers. We put

$$(60)\qquad \psi_{\beta,\gamma}(\xi) = (1 - |\xi|^\gamma)^\beta_+ = \begin{cases} (1 - |\xi|^\gamma)^\beta & \text{if } |\xi| < 1, \\ 0 & \text{if } |\xi| \geqslant 1. \end{cases}$$

The related means $M_\nu^{\psi_{\beta,\gamma}} f$, $\nu = 1, 2, \ldots$, are called the *Riesz means of f* (*Bochner–Riesz* if $\gamma = 2$, *Fejér* if $\beta = \gamma = n = 1$). The functions $\psi_{\beta,2k}(\xi)$, $k = 1, 2, \ldots$, satisfy the assumptions (33) (and hence (31)) as well as (30) with $\lambda = \beta + (n+1)/2$ and $\sigma = 2k$ (cf. [27, Lemma 4] and J. Peetre [25, p. 215]). Therefore we can apply the results of Section 3 and Section 4 if $\beta > (n-1)/2$. This leads to the following theorem.

THEOREM 9 ([27, Theorem 8]). *Let* $(n-1)/2 < \beta < \infty$, $k = 1, 2, \ldots$, ψ *has the meaning of* (60) *with* $\gamma = 2k$. *Let* $f \in L_1(T_n)$.

(i) *If* $2n/(2\beta + n + 1) < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < s < 2k$ *then* (54) *holds.*

(ii) *If* $2n/(2\beta + n + 1) < p \leqslant \infty$, $0 < u \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $0 < s < 2k$ *then* (55) *holds.*

(iii) *If* $2n/(2\beta + n + 1) < p < \infty$, $2n/(2\beta + n + 1) < q \leqslant \infty$, $0 < s < 2k$ *then* (56) *holds.*

Remark 16. In general, the function $|x|^{-\gamma}(1 - \psi_{\beta,\gamma}(x))$ is not C^∞ at the point 0. Hence the claim in [27, Lemma 4] can not be applied to arbitrary numbers γ. However, one proves that $|x|^{-\gamma}(1 - \psi_{\beta,\gamma}(x)) \in B^{n+\gamma}_{1,\infty}(R_n)$ in a neighbourhood of 0. Moreover, $\psi_{\beta,\gamma}$ satisfies (30) and (31) for $\lambda < \min(\beta + n/2, \gamma + n)$ and $\sigma = \gamma$. Hence the assertions of the theorem remain true for general $M_\nu^{\psi^{\beta,\gamma}}$ if $1 \leqslant p \leqslant \infty$ ($1 \leqslant q \leqslant \infty$ in the case of (iii)) with γ instead of $2k$. For values p (and q in (iii)) less than 1 we obtain additional restrictions, which seem to be unnatural. Furthermore, it is easy to see that the theorem applies to $\gamma = \beta = 1$ and $n = 1, 2$.

Remark 17. By our methods we can not deal with the case $s = \gamma$ which corresponds to the saturation order of the Riesz means (at least if $p \geqslant 1$), cf. P. L. Butzer, R. J. Nessel [6, Chapters 12, 13], W. Trebels [39, Theorem 4.7]. Moreover, $\beta = (n-1)/2$ corresponds to Bochner's critical index, cf. E. M. Stein, G. Weiss [37, Chapter 7]. The question arises whether Theorem 9 remains true for some $\beta \leqslant (n-1)/2$ if $1 < p < \infty$. Partial results concerning (54) can be found in W. Trebels [39, 5.2] and the references given there. For characterizations of type (54) (with $1 \leqslant p \leqslant \infty$) for Riesz means we refer also to J. Löfström [20], R. M. Trigub [46], P. L. Butzer, R. J. Nessel [6], and to the comparison theorems by H. S. Shapiro [32], [33] and J. Boman, H. S. Shapiro [4].

6.4. Abel–Poisson means

Let $0 < \gamma < \infty$. We put

$$\psi_\gamma(\xi) = e^{-|\xi|^\gamma}, \qquad \xi \in R_n. \tag{61}$$

The related means $M_\nu^{\psi_\gamma} f$, $\nu = 1, 2, \ldots$, are called *Abel–Poisson* or *Abel–Cartwright means* (*Abel* if $\gamma = 1$ and *Gauss–Weierstrass* if $\gamma = 2$). If $\gamma = 2k$, $k = 1, 2, \ldots$ then ψ_γ satisfies (30) and (31) with $\sigma = \gamma$ for all $\lambda > 0$. In the general case we obtain $\sigma = \gamma$ and $\lambda < n+\gamma$. Furthermore we can apply parts (ii) of Theorems 3 and 4. This leads to the following theorem.

THEOREM 10 ([28, Theorem 5]). *Let* $0 < \gamma < \infty$. ψ. *has the meaning of* (61).

(i) *If* $n/(n+\gamma) < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $n(1/\min(1, p)-1) < s < \gamma$ *then the quasi-norm on the right-hand side of* (54) *is equivalent to* $\|f \,|\, B^s_{p,q}\|$.

(ii) *If* $n/(n+\gamma) < p \leqslant \infty$, $0 < u \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $n(1/\min(1, u) - 1) < s < \gamma$ *then the quasi-norm on the right-hand side of* (55) *is equivalent to* $\|f \,|\, B^s_{p,q}\|$.

(iii) *If* $n/(n+\gamma) < p < \infty$, $n/(n+\gamma) < q \leqslant \infty$, $n(1/\min(1, p, q)-1) < s < \gamma$ *then the quasi-norm on the right-hand side of* (56) *is equivalent to* $\|f \,|\, F^s_{p,q}\|$.

(iv) *If* $\gamma = 2k$, $k = 1, 2, \ldots$, *then* $n/(n+\gamma)$ *can be replaced by* 0 in (i)–(iii).

Remark 18. The theorem, Remark 9, and Remark 16 show that Abel–Poisson and Riesz means (for sufficiently large numbers β) with fixed γ have similar approximation properties in the sense of (54)–(56). For $1 \leqslant p \leqslant \infty$ and assertions of type (54) this is known by the so-called comparison theorems (cf. H. S. Shapiro [32], [33], P. L. Butzer, R. J. Nessel [6], W. Trebels [39] and the literature cited there). However, note that there are negative results for pointwise comparison (cf. W. Dickmeis, R. J. Nessel, E. van Wickeren [7]). For the characterization of $B^s_{p,\infty}$, $1 \leqslant p \leqslant \infty$, in the sense of (54) by Abel–Poisson means we refer also to B. I. Golubov [10].

Remark 19. Concerning results on approximation by Bessel-potential means ($\psi(\xi) = (1+|\xi|^2)^{-\beta/2}$, $\beta > 0$) we refer to [28, Theorem 6]. In this case (30) and (31) are satisfied for $\sigma = 2$ and $0 < \lambda < \infty$. Characterizations via (C, α)-means can be found in [29].

Added in proof: Theorem 9 holds true for all γ, $0 < \gamma < \infty$. Theorem 10 is true for all p, q with $0 < p, q < \infty$. This follows from the revised version of [28].

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