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On the maximum term of a class of integral functions and its derivatives

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Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{2}$$

be two integral functions. We consider the class of functions defined as

$$f(z) \ast g(z) = \sum_{n=0}^{\infty} a_n b_n z^n . \tag{3}$$

It has been proved that this class is contained in the class of integral functions.

It is known [1] that

$$f^{(s)}(z) \ast g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2 \dots (n-s+1)^2 a_n b_n z^{n-s} \tag{4}$$

is an integral function of the same order as $f(z) \ast g(z)$. Let $[f(z) \ast g(z)]^{(s)}$ denote the s-th derivative of $f(z) \ast g(z)$. Then

$$[f(z) \ast g(z)]^{(s)} = \sum_{n=s}^{\infty} n(n-1) \dots (n-s+1) a_n b_n z^{n-s} , \tag{5}$$

$[f(z) \ast g(z)]^{(s)}$ is an integral function of the same order as $f(z) \ast g(z)$ ([3], p. 35).

We shall denote by $\mu(r, s)$, $\mu^*(r, s)$ the maximum terms in the Taylor's expansion of $f^{(s)}(z) \ast g^{(s)}(z)$ and $[f(z) \ast g(z)]^{(s)}$ and by $\nu(r, s)$, $\nu^*(r, s)$ their ranks for $|z| = r$. We shall also denote by $\mu_1(r, s)$, $\mu_1^*(r, s)$ the derivatives of $\mu(r, s)$, $\mu^*(r, s)$ with respect to r, respectively.

The object of this paper is to obtain some relations between $\mu(r, s)$ and $\mu^*(r, s)$ which give us more information about the class of integral functions defined by (3).

THEOREM 1. *If $f(z) \ast g(z)$ is an integral function of finite order ϱ and lower order λ, then for almost all values of $r > r_0$,*

$$\mu^*(r, s) r^{(s+2)\lambda-\varepsilon-1} < \mu(r, s+1) < r^{(s+2)\varrho+\varepsilon-1}\mu^*(r, s) .$$

Proof. It is known [1] that

$$\log \mu(r, s) = \log \mu(r_1, s) + \int_{r_1}^{r} \frac{\nu(\varkappa, s) - s}{\varkappa} d\varkappa \tag{6}$$

for $0 < r_1 < r$.

Differentiating (6), we get

$$\frac{\mu_1(r, s)}{\mu(r, s)} = \frac{\nu(r, s) - s}{r} \tag{7}$$

for almost all values of $r > r_0$.

If $\nu(r, s)$, $\nu(r, s+1)$ denote the rank of the maximum term of $f^{(s)}(z) \ast g^{(s)}(z)$ and $f^{(s+1)}(z) \ast g^{(s+1)}(z)$ for $|z| = r$, respectively, then we have [1]

$$\{\nu(r, s) - s\}^2 \leqslant r\frac{\mu(r, s+1)}{\mu(r, s)} \leqslant \{\nu(r, s+1) - s\}^2 . \tag{8}$$

From (7) and (8) we get

$$\{\nu(r, s) - s\}\mu_1(r, s) \leqslant \mu(r, s+1) \tag{9}$$

for almost all values of $r > r_0$.

Further,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu(r, 0)}{\log r} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(\nu(r, s) - s\big)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix} .$$

Therefore by (7)

$$\frac{\mu_1(r, s)}{\mu(r, s)} > r^{(\lambda-\varepsilon-1)}$$

for almost all values of $r > r_0$ and any $\varepsilon > 0$.

Hence

$$\begin{aligned}\mu(r, s+1) &> \mu(r, s) r^{(\lambda-\varepsilon-1)}\big(\nu(r, s) - s\big) \\ &\geqslant \nu^*(r, s) \ldots \big(\nu^*(r, s) - s + 1\big) \mu^*(r, s) r^{(\lambda-\varepsilon-1)}\big(\nu(r, s) - s\big) \\ &\geqslant \mu^*(r, s) r^{(s+2)\lambda-\varepsilon-1} .\end{aligned}$$

For the second inequality, consider the right-hand inequality of (8),

$$\mu(r, s+1) \leqslant \frac{\mu(r, s)}{r}\big(\nu(r, s+1)-s\big)^2$$

$$\leqslant \frac{[\nu(r, s)]^s}{r}[\nu(r, s+1)]^2\mu^*(r, s)$$

$$\leqslant r^{(s+2)\varrho+\varepsilon-1}\mu^*(r, s) ,$$

for any $\varepsilon > 0$ and $r > r_0$.

Hence the theorem follows.

THEOREM 2. *If* $0 < r_1 < r_2$, *then*

$$\left(\frac{r_2}{r_1}\right)^{\nu^*(r_1,s)-s} \leqslant \frac{\mu(r_2, s)}{\mu^*(r_1, s)} \leqslant \{\nu(r_1, s)\}^s\left(\frac{r_2}{r_1}\right)^{\nu(r_2,s)-s} . \tag{10}$$

Proof. From (6) we get

$$\log\mu(r_2, s) \leqslant \log\mu(r_1, s)+\{\nu(r_2, s)-s\}\log\frac{r_2}{r_1}$$

for $0 < r_1 < r_2$.

Therefore,

$$\log\mu(r_2, s) \leqslant s\log\nu(r_1, s)+\log\mu^*(r_1, s)+\big(\nu(r_2, s)-s\big)\log\frac{r_2}{r_1} .$$

From the above inequality, the second inequality of (10) follows.

To prove the first inequality we note that for all r

$$\log\mu(r_2, s) \geqslant \log\mu^*(r_2, s) . \tag{11}$$

Further,

$$\log\mu^*(r, s) = \log\mu^*(r_1, s)+\int_{r_1}^{r}\frac{\nu^*(x, s)-s}{x}dx \tag{12}$$

for $0 < r_1 < r$. Therefore,

$$\log\mu^*(r_2, s) \geqslant \log\mu^*(r_1, s)+\{\nu^*(r_1, s)-s\}\log\frac{r_2}{r_1} . \tag{13}$$

From (11) and (13) the first inequality of (10) follows.

COROLLARY. *If $f(z)$ and $g(z)$ are two integral functions $f\times g$ is not polynomials and a is a constant* $(0 < a < 1)$, *then*

$$\lim_{r\to\infty}\frac{\mu^*(ar, s)}{\mu(r, s)} = 0 .$$

For from relation (10) with $r_1 = ar$ and $r_2 = r$ we obtain

$$a^{\nu^*(ar,s)-s} \geqslant \frac{\mu^*(ar, s)}{\mu(r, s)} \geqslant \frac{1}{\{\nu(ar, s)\}^s}a^{\nu(r,s)-s} .$$

Hence the result follows letting $r \to \infty$.

THEOREM 3. *If the order* ϱ *of* $f(z) \ast g(z)$ *is less than* $1/(s+2)$, *then*

$$\mu^*(r, 0) > \mu(r, 1) > \mu^*(r, 1) > \mu(r, 2) > \dots$$
$$> \mu^*(r, s) > \mu(r, s+1) > \mu^*(r, s+1)$$

for almost all values $r > r_0 > 1$.

Proof. From the second inequality in Theorem 1, we have for $\varrho < 1/(s+2)$, $\mu(r, s+1) < \mu^*(r, s)$ for almost all values $r > r_0 \geqslant 1$. Again, $\mu^*(r, s+1) < \mu(r, s+1)$. Therefore,

$$\mu^*(r, s) > \mu(r, s+1) > \mu(r, s+1)\,.$$

Similarly, $\mu^*(r, 1) > \mu(r, 2)$ for almost all values $r > r_0 \geqslant 1$ if $\varrho < 1/(s+2) < 1/(s+1)$. Repeating the process for $s-2, s-3, \dots$ we get the required result.

COROLLARY 1. *If* $f(z)$ *and* $g(z)$ *are two integral functions of orders* ϱ_1 $(0 < \varrho_1 < 1)$ *and* ϱ_2 $(0 < \varrho_2 < 1/(s+1))$, *respectively, then*

$$\mu(r, 0) > \mu^*(r, 0) > \mu(r, 1) > \mu^*(r, 1) > \dots$$
$$> \mu^*(r, s) > \mu(r, s+1) > \mu^*(r, s+1)$$

for almost all values $r > r_0 \geqslant 1$.

For, we have ([4], p. 421) $1/\varrho \geqslant 1/\varrho_1 + 1/\varrho_2$, where ϱ $(0 < \varrho < \infty)$ is the order of $f(z) \ast g(z)$. Thus from the given condition $\varrho < 1/(s+2)$. Hence the result follows from Theorem 3.

THEOREM 4. *If* $\nu(r, s)$ *and* $\nu^*(r, s)$ *denote the rank of the maximum term of the power series of* $f^{(s)}(z) \ast g^{(s)}(z)$ *and* $[f(z) \ast g(z)]^{(s)}$ *for* $|z| = r$, *respectively, then*

$$[\nu^*(r, s+1) - s]^{s+1} \nu^*(r, s) \leqslant \frac{r\mu(r, s+1)}{\mu^*(r, s)} \leqslant [\nu(r, s+1)]^{s+1} \nu^*(r, s+1)\,.$$

Proof. From

$$f^{(s)}(z) \ast g^{(s)}(z) = \sum_{n=s}^{\infty} n^2 (n-1)^2 \dots (n-s+1)^2 a_n b_n z^{n-s}$$

we have

$$\begin{aligned}(14)\qquad \mu(r, s+1) &= \big[\nu(r, s+1)\big(\nu(r, s+1)-1\big)\dots\big(\nu(r, s+1)-s\big)\big]^2 \times \\ &\quad \times |a_{\nu(r,s+1)} b_{\nu(r,s+1)}|\, r^{\nu(r,s+1)-s-1} \\ &\leqslant \big[\nu(r, s+1)\dots\big(\nu(r, s+1)-s\big)\big]\mu^*(r, s+1) \\ &\leqslant \big[\nu(r, s+1)\dots\big(\nu(r, s+1)-s\big)\big]\frac{\mu^*(r, s)\nu^*(r, s+1)}{r}\,.\end{aligned}$$

Again

$$(15)\qquad \mu(r, s+1) \geqslant \left[\nu^*(r, s+1)\ldots\left(\nu^*(r, s+1)-s\right)\right]\mu^*(r, s+1)$$
$$\geqslant \left[\nu^*(r, s+1)\ldots\left(\nu^*(r, s+1)-s\right)\right]\frac{\nu^*(r, s)\,\mu^*(r, s)}{r}.$$

From (14) and (15) the results follows.

Applications. If $f(z) \ast g(z)$ is of order ϱ and lower order λ, then

$$\text{(a)}\qquad \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left\{r\dfrac{\mu(r, s+1)}{\mu^*(r, s)}\right\}^{1/(s+2)}}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

$$\text{(b)}\qquad \limsup_{r\to\infty}\frac{\log\{[\nu^*(r, s+1)-s]^{s+1}\nu^*(r, s)\}^{1/(s+2)}}{\log r}$$
$$= \varrho \leqslant \frac{1}{s+2} + \limsup_{r\to\infty}\frac{\log\left\{\dfrac{\mu(r, s+1)}{\mu^*(r, s)}\right\}^{1/(s+2)}}{\log r}$$
$$\leqslant \varrho = \limsup_{r\to\infty}\frac{\log[\{\nu(r, s+1)\}^{s+1}\nu^*(r, s+1)]^{1/(s+2)}}{\log r}.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log\left\{\dfrac{\mu(r, s+1)}{\mu^*(r, s)}\right\}^{1/(s+2)}}{\log r} = \varrho - \frac{1}{s+2}.$$

(c) Similarly

$$\liminf_{r\to\infty}\frac{\log\left\{\dfrac{\mu(r, s+1)}{\mu^*(r, s)}\right\}^{1/(s+2)}}{\log r} = \lambda - \frac{1}{s+2}.$$

(d) If $f(z)$ and $g(z)$ are of regular growth, then

$$\lim_{r\to\infty}\frac{\log\left\{\dfrac{\mu(r, s+1)}{\mu^*(r, s)}\right\}^{1/(s+2)}}{\log r} = \varrho - \frac{1}{s+2}.$$

Let $\Phi(r)$ be any function non-decreasing and positive for $r > r_0$ and $\log\Phi(r) = 0(\log r)$ for $r\to\infty$.

If $\mu(r, s+1) \geqslant \dfrac{1}{\Phi(r)}\mu^*(r, s)$ for a sequence of values $r\to\infty$, then from (b) it follows that $\varrho \geqslant 1/(s+2)$. Again if the hypothesis holds for all $r > r_0$, then from (c) we get $\lambda \geqslant 1/(s+2)$.

If

$$\frac{1}{\Phi(r)} \leqslant \frac{\mu(r, s+1)}{\mu^*(r, s)} \leqslant \Phi(r)$$

for all $r > r_0$, then $\varrho = \lambda = 1/(s+2)$.

From the above discussions it follows: if $\varrho < 1/(s+2)$, then

$$\mu^*(r, 0) > \Phi(r)\mu(r, 1) > \Phi(r)\mu^*(r, 1) > \Phi^2(r)\mu(r, 2) > \dots$$
$$> \Phi^s(r)\mu^*(r, s) > \Phi^{s+1}(r)\mu(r, s+1) > \Phi^{s+1}(r)\mu^*(r, s+1) .$$

for all $r > r_0$.

This includes the results of Theorem 3.

THEOREM 5. *If $f(z) \ast g(z)$ is of order ϱ, then*

$$\limsup_{r\to\infty} \frac{\log\left\{\frac{\mu(r, s)}{\mu^*(r, s)}\right\}^{1/s}}{\log r} \leqslant \varrho .$$

Proof. We have

$$\mu(r, s) \leqslant \nu(r, s)\big(\nu(r, s)-1\big)\dots\big(\nu(r, s)-s+1\big)\mu^*(r, s) .$$

Therefore,

$$\limsup_{r\to\infty} \frac{\log[\mu(r, s)/\mu^*(r, s)]}{\log r} \leqslant \limsup_{r\to\infty} \frac{\log[\nu(r, s)]^s}{\log r} = s\varrho .$$

Hence the theorem follows.

COROLLARY. *If $f(z)$ and $g(z)$ are of regular growth, then*

$$\lim_{r\to\infty} \frac{\log[\mu(r, s)/\mu^*(r, s)]^{1/s}}{\log r} = \varrho .$$

For

$$\mu^*(r, s)\nu^*(r, s)\dots\big(\nu^*(r, s)-s+1\big) \leqslant \mu(r, s) .$$

Therefore,

$$\liminf_{r\to\infty} \frac{\log[\mu(r, s)/\mu^*(r, s)]^{1/s}}{\log r} \geqslant \lambda , \tag{16}$$

where λ is the lower order of $f(z) \ast g(z)$. Now it is known that if $f(z)$ and $g(z)$ are of regular growth, then $f(z) \ast g(z)$ is also of regular growth.

Hence applying the inequality (16) and the result of Theorem 5, the corollary follows.

THEOREM 6. *If $f(z)$ and $g(z)$ are two integral functions of regular growth, $\mu(r, s)$ is the maximum term and $\nu(r, s)$ is its rank in the Taylor series of $f^{(s)}(z) \ast g^{(s)}(z)$, then*

$$\mu(r, s) \sim \mu(r, 0) \frac{[\nu(r, 0)]^{2s}}{r^s},$$

as $r \to \infty$ with the exception of a set of values r of measure zero.

Proof. Differentiating (6) and (12) we have

$$\frac{\mu_1(r, s)}{\mu(r, s)} = \frac{\nu(r, s)-s}{r}, \quad \text{and} \quad \frac{\mu_1^*(r, s)}{\mu^*(r, s)} = \frac{\nu^*(r, s)-s}{r},$$

for almost all values of r except at a set of measure zero.

Now

$$\lim_{r\to\infty} \frac{\log\{\nu(r, s)-s\}}{\log r} = \varrho = \lim_{r\to\infty} \frac{\log(\nu^*(r, s)-s)}{\log r};$$

hence

$$\frac{\mu_1(r, s)}{\mu(r, s)} \sim \frac{\mu_1^*(r, s)}{\mu^*(r, s)}.$$

Again it is known ([2], p. 107) that

$$\frac{\mu_1^*(r, s)}{\mu^*(r, s)} \sim \frac{\mu^*(r, s+1)}{\mu^*(r, s)} \sim \frac{\nu^*(r, s)}{r} \sim \frac{\nu^*(r, 0)}{r}.$$

Therefore,

$$\frac{\mu_1(r, s)}{\mu(r, s)} \sim \frac{\nu^*(r, 0)}{r} = \frac{\nu(r, 0)}{r}. \tag{17}$$

Also we have [1]

$$\mu_1(r, s) \sim \left[\frac{\mu(r, s)\mu(r, s+1)}{r}\right]^{1/2}.$$

Hence

$$\frac{\mu_1(r, s)}{\mu(r, s)} \sim \left[\frac{\mu(r, s+1)}{\mu(r, s)r}\right]^{1/2}.$$

Therefore,

$$\left[\frac{\mu(r, s+1)}{\mu(r, s)}\right]^{1/2} \sim \frac{\nu(r, 0)}{r^{1/2}}.$$

Taking in place of s the values $0, 1, 2, \ldots, s-1$ and multiplying the formulae thus obtained we find

$$\left[\frac{\mu(r, s)}{\mu(r, s)}\right] \sim r^{-s}[\nu(r, 0)]^{2s}.$$

Application.

$$\lim_{r\to\infty} \frac{\log r\left[\dfrac{\mu(r,s)}{\mu(r,0)}\right]^{1/s}}{\log r} = \lim_{r\to\infty} \frac{\log[\nu(r,0)]^2}{\log r} = 2\varrho\,.$$

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