ANNALES
POLONICI MATHEMATICI
XXX (1974)

A uniqueness criterion for fractional iteration*

by B. A. Reznick (Pasadena, Calif.)

Abstract. The uniqueness of solutions to the functional equation $f^N(x) = h(x)$. Let hypothesis (H) denote the following assumptions on a real function h:

(H) *h is of class C^1 in $[0, a)$, $0 < a \leqslant \infty$, $h'(0) = s$ with $0 < s \leqslant 1$, $h'(x) > 0$ in $(0, a)$ and $0 < h(x) < x$ in $(0, a)$.*

Theorem 1. *Suppose the functions f, g, and h each satisfy hypothesis* (H), *and that, further, h' is monotone in a subinterval $[0, b)$ of $[0, a)$. If $f^N(x) = g^N(x) = h(x)$, then $f = g$ on $[0, a)$.*

Theorem 2. *Suppose $f^N(x) = h(x)$, where f and h satisfy hypothesis* (H). *Suppose further that $h \in C^3$, $h''(x) < 0$ and $(h''(x))^2 > h'(x)h'''(x)$ on $[0, a)$. Then f' is decreasing.*

The proofs to Theorems 1 and 2 are both elementary.

1. Introduction. The question of uniqueness in the solution of the functional equation

$$f^N(x) = h(x) \tag{1}$$

is of some interest. Let h be a real function with the following properties:

(H) *h is of class C^1 in $[0, a)$, $0 < a \leqslant \infty$, $h'(0) = s$ with $0 < s \leqslant 1$, $h'(x) > 0$ in $(0, a)$ and $0 < h(x) < x$ in $(0, a)$.*

Kuczma [1] proved that if h fulfils hypothesis (H), and f is a strictly increasing C^1 solution of equation (1), then f satisfies in $(0, a)$ the differential equation:

$$f'(x) = \sqrt{s}\cdot G\big(x, f(x)\big), \tag{A}$$

where $s = f'(0)$ and

$$G(x, y) = \prod_{i=0}^{\infty} \frac{h'\big(h^i(x)\big)}{h'\big(h^i(y)\big)}.$$

Using a uniqueness theorem for solutions of equation (A), Kuczma and Smajdor [3] proved that if h satisfies hypothesis (H), then equation (1) has a unique increasing C^1 solution provided h satisfies one of three additional hypotheses.

* Work supported by NSF Grants No. GY8936 and GY9754.

1° $0 < s < 1$ and the limit

$$\lim_{n\to\infty} \frac{s^n}{\prod_{i=0}^{n-1} h'\big(h^i(x)\big)}$$

exists, is continuous and different from zero in $(0, a)$.

2° $0 < s < 1$ and $h'(x) = s + O(x^\delta)$, $x \to 0$, $\delta > 0$.

3° $s = 1$ and $h'(x) = 1 - A(\mu+1)x^\mu + O(x^{\mu+\delta})$, $x \to 0+$, where A, δ, μ are positive constants, $\delta < \mu$.

In this paper, a theorem closely related to that quoted from [1] will be discussed regarding uniqueness of solutions to equation (1). Additionally, a criterion for the preservation of concavity under fractional iteration will be described.

2. Uniqueness.

THEOREM 1. *Suppose the functions f, g, and h each satisfy hypothesis* (H) *and that, further, h' is monotone in a subinterval $[0, b)$ of $[0, a)$. If $f^N(x) = g^N(x) = h(x)$, then $f = g$ on $[0, a)$.*

Proof. Note first that the conditions $f \in C^1[0, a)$ and $0 < f(x) < x$ for $x \in [0, a)$ imply $f(0) = 0$, and similarly, $g(0) = h(0) = 0$. Taking the derivative of equation (1), for f and g, we obtain:

$$\prod_{i=0}^{N-1} f'\big(f^i(x)\big) = h'(x), \tag{2}$$

$$\prod_{i=0}^{N-1} g'\big(g^i(x)\big) = h'(x). \tag{3}$$

Putting $x = f(x)$ into equation (2) and dividing

$$\frac{f'\big(h(x)\big)}{f'(x)} = \frac{h'\big(f(x)\big)}{h'(x)}. \tag{4}$$

Similarly,

$$\frac{g'\big(h(x)\big)}{g'(x)} = \frac{h'\big(g(x)\big)}{h'(x)}. \tag{5}$$

There is no difficulty in dividing, as all derivatives are non-zero. Note also, that as $0 < h(t) < t$ for all $t \in [0, a)$, $\lim_{n\to\infty} h^n(t) = 0$.

Now consider the subinterval $[0, b)$ on which h' is monotone. If $f(x) > g(x)$ for all $x \in (0, b)$, then $h(x) = f^N(x) > g\big(f^{N-1}(x)\big) > g^2\big(f^{N-2}(x)\big) > \ldots > g^{N-1}\big(f(x)\big) > g^N(x) = h(x)$. A similar contradiction results from the assumptions that $g(x) > f(x)$ for all $x \in (0, b)$. Thus, there exists $x_0 \in (0, b)$ for which $f(x_0) = g(x_0)$.

Suppose now that $f(y) = g(y)$. From equations (4) and (5), we conclude that:

$$\frac{f'(h(y))}{f'(y)} = \frac{g'(h(y))}{g'(y)}. \tag{6}$$

But, as $f(y) = g(y)$, $h(f(y)) = h(g(y))$, and since h commutes with f and g, $f(h(y)) = g(h(y))$. Thus, $f(y) = g(y)$ implies that $f(h^m(y)) = g(h^m(y))$ for all m. Hence,

$$\frac{f'(h^{m+1}(y))}{f'(h^m(y))} = \frac{g'(h^{m+1}(y))}{g'(h^m(y))}. \tag{7}$$

Multiplying equation (7) for $m = 0, 1, \ldots, k-1$, we obtain

$$\frac{f'(h^k(y))}{f'(y)} = \frac{g'(h^k(y))}{g'(y)}. \tag{8}$$

Taking the limit as $k \to \infty$, and cross-multiplying,

$$\frac{f'(y)}{g'(y)} = \frac{f'(0)}{g'(0)}. \tag{9}$$

But, from (2), (3), and the hypotheses, when $x = 0$, $(f'(0))^N = (g'(0))^N = h'(0)$. As $f'(0) > 0$ and $g'(0) > 0$, we see that $f'(0) = g'(0)$. Thus, if $f(y) = g(y)$, then $f'(y) = g'(y)$.

For all x, by combining equations (4) and (5), we obtain

$$\frac{f'(h(x))}{g'(h(x))} = \frac{h'(f(x))}{h'(g(x))} \cdot \frac{f'(x)}{g'(x)}. \tag{10}$$

This is true, in particular, for $x = h^m(y)$, so, as h commutes with f and g,

$$\frac{f'(h^{m+1}(y))}{g'(h^{m+1}(y))} = \frac{h'(h^m(f(y)))}{h'(h^m(g(y)))} \cdot \frac{f'(h^m(y))}{g'(h^m(y))}. \tag{11}$$

By multiplying these together for $m = 0, \ldots, k$, we obtain

$$\frac{f'(h^{k+1}(y))}{g'(h^{k+1}(y))} = \frac{f'(y)}{g'(y)} \cdot \prod_{m=0}^{k} \frac{h'(h^m(f(y)))}{h'(h^m(g(y)))}. \tag{12}$$

Suppose that for $y \in [0, b)$, $f'(y) = g'(y)$, but $f(y) \neq g(y)$, without loss of generality, $f(y) > g(y)$. Then, as $y < b$, $h^m(f(y))$ and $h^m(g(y))$ fall within the domain of monotonicity of h'. Further, as h is an increasing function, $h^m(f(y)) > h^m(g(y))$. Thus, either every term in the product in (12) is greater than 1, or every term is less than 1. Taking $k \to \infty$ in (12), the left-hand side converges to $f'(0)/g'(0)$, which we have already established to be 1. Further, by assumption, $f'(y) = g'(y)$, so that the infinite product must converge to 1. But, as the terms are uniformly either greater than 1, or less than 1, we get a contradiction: Hence, if $f'(y) = g'(y)$, then $f(y) = g(y)$.

Combining the above, we have that for $x \in [0, b]$, $f(x) = g(x)$ if and only if $f'(x) = g'(x)$. Further, there is an $x_0 \neq 0$ for which $f(x_0) = g(x_0)$. Let $S = \{x : x \in [0, x_0] \text{ and } f(x) = g(x)\}$. As f and g are continuous, S is closed, and as $0 \in S$, $x_0 \in S$, the set $T = [0, x_0] \setminus S$ is open. Suppose $T \neq \emptyset$, then as T is open, write $T = \bigcup (c_i, d_i)$, a union of disjoint open intervals. Let (c, d) be one of those intervals. Then $c \notin T$, $d \notin T$ as the intervals are disjoint. Hence $c \in S$, $d \in S$, so $f(c) = g(c)$ and $f(d) = g(d)$. But then, by Rolle's Theorem, $\exists e$, $c < e < d$, such that $f'(e) = g'(e)$. Hence $f(e) = g(e)$, so $e \in S$, but $e \in (c, d) \subset T$, a contradiction. Thus, T is void, and $S = [0, x_0]$.

Now let $K = \min[a, \inf(\{x : f(x) \neq g(x), x \geqslant 0\})]$. If $K < a$, then $f(x) = g(x)$ for $0 \leqslant x < K$, and $f(K) = g(K)$ by continuity. As $f(K) < K$ and $g(K) < K$, by continuity again, there is $\varepsilon > 0$ such that $f(t) < K$ and $g(t) < K$ when $0 \leqslant t \leqslant K + \varepsilon$. As $K < a$, by its definition, there is $u \in (K, K + \varepsilon)$ for which $f(u) \neq g(u)$, say, $f(u) > g(u)$. But, on $[0, K]$, $f = g$, so $f^{N-1} = g^{N-1}$, and both functions are increasing. Thus, as $f(U) < K$ and $g(U) < K$, $f^{N-1}(F(u)) > F^{N-1}(g(u))$. But, $f^{N-1}(g(u)) = g^{N-1}(g(u))$. That is, $f^N(u) > g^N(u)$, or $h(u) > h(u)$, a contradiction. Therefore, $K = a$, and the theorem is proved.

3. Concavity. In case the functions f and h satisfy equation (1) and hypothesis (H), and if, further, f' is decreasing on $[0, a)$, then h' is decreasing as well. For, by equation (2), $h'(x)$ is the product of N terms, each of which is positive and decreasing. As for the converse, Kuczma and Smajdor [2] proved the uniqueness of concave solutions to equation (1), given that h is concave, that there is a unique concave increasing solution. (Actually, they proved it for h convex as well.) The following is a partial solution to the existence question:

THEOREM 2. *Suppose $f^N(x) = h(x)$, where f and h satisfy hypothesis* (H). *Suppose further that $h \in C^3$, $h''(x) < 0$ and $(h''(x))^2 > h'(x)h'''(x)$ on $[0, a)$. Then, f' is decreasing.*

Proof. Note that the hypotheses of Theorem 2 imply those of Theorem 1, so that the function f is unique. By equation (4), for any x and y, we have

$$\frac{f'(h(x))}{f'(x)} = \frac{h'(f(x))}{h'(x)}, \quad \frac{f'(h(y))}{f'(y)} = \frac{h'(f(y))}{h'(y)}. \tag{13}$$

Thus, as $f(x) < x$ and h' decreasing, $f'(h(x)) > f'(x)$ for all x. Repeating for $h(x), h^2(x), \ldots, h^m(x)$, we obtain $f'(h^m(x)) > f'(x)$. Taking m to infinity, $s = f'(0) > f'(x)$. Hence $1 > f'(x)$.

Let

$$\varphi(x) = \frac{h'(f(x))}{h'(x)}. \tag{14}$$

As $h \in C^3, f \in C^1$, and $h' \neq 0$, we see that $\varphi \in C^1$, indeed,

$$\varphi'(x) = \frac{h'(x)h''(f(x))f'(x) - h''(x)h'(f(x))}{(h'(x))^2}. \tag{15}$$

Now, $h'(x) > 0$, $h''(f(x)) < 0$, and $f'(x) < 1$, so,

$$\varphi'(x) > \frac{h'(x)h''(f(x)) - h''(x)h'(f(x))}{(h'(x))^2}. \tag{16}$$

That is,

$$\varphi'(x) > \left(\frac{h'(x)}{h''(x)} - \frac{h'(f(x))}{h''(f(x))}\right) \cdot \frac{h''(x)h''(f(x))}{(h'(x))^2}. \tag{17}$$

Consider $\psi(x) = \dfrac{h'(x)}{h''(x)}$. Then,

$$\psi'(x) = \frac{(h''(x))^2 - h'(x)h'''(x)}{(h''(x))^2}. \tag{18}$$

But, by hypothesis, $(h''(x))^2 > h'(x)h'''(x)$; therefore $\psi'(x) > 0$. Hence, by (17),

$$\varphi'(x) > \big(\psi(x) - \psi(f(x))\big) \cdot \frac{h''(x)h''(f(x))}{(h'(x))^2}. \tag{19}$$

And, as $x > f(x)$, $h''(x) < 0$, $h''(f(x)) < 0$, the right-hand side is positive, hence $\varphi'(x) > 0$. Now pick x_0 and x_1 in $[0, a)$ with $x_0 < x_1$. Then, by the above $\varphi(x_0) < \varphi(x_1)$. That is,

$$\frac{h'(f(x_0))}{h'(x_0)} < \frac{h'(f(x_1))}{h'(x_1)}. \tag{20}$$

But, by equation (13), and transposing,

$$\frac{f'(x_1)}{f'(x_0)} < \frac{f'(h(x_1))}{f'(h(x_0))}. \tag{21}$$

As $x_0 < x_1$, $h^k(x_0) < h^k(x_1)$ for $k = 1, 2, \ldots,$ and so (21) becomes

$$\frac{f'(x_1)}{f'(x_0)} < \frac{f'(h^m(x_1))}{f'(h^m(x_0))}. \tag{22}$$

Let $m \to \infty$, $h^m(x_0) \to 0$, $h^m(x_1) \to 0$, therefore,

$$\frac{f'(x_1)}{f'(x_0)} \leqslant \frac{f'(0)}{f'(0)} = 1. \tag{23}$$

So, if $x_0 < x_1$, then $f'(x_1) \leqslant f'(x_0)$. Hence, f' is decreasing.

The author wishes to thank his undergraduate research advisor, Prof. W. A. J. Luxemburg, for much wise counsel during the period of his research, and the reviewer for several valuable comments.

References

[1] M. Kuczma, *Fractional iteration of differentiable functions*, Ann. Polon. Math. 22 (1969), p. 217–227.

[2] – and A. Smajdor, *Fractional iteration in the class of convex functions*, Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys. 16 (1968), p. 717–720.

[3] – and A. Smajdor, *Regular fractional iteration*, ibidem 19 (1971), p. 203–207.

CALIFORNIA INSTITUTE OF TECHNOLOGY

Reçu par la Rédaction le 5. 5. 1973