ZASTOSOWANIA MATEMATYKI
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G. LOIZOU (London)

DIAGONAL TRANSFORMATIONS

A result of Stoer and Witzgall, concerning transformations by diagonal matrices (in a normed linear space), is generalized in the sense that it applies to a wider class of norms as well as to pairs of vectors that are not strictly positive.

1. Introduction. In 1962 Stoer and Witzgall [6] proved the following theorem:

THEOREM 1. *Given an absolute norm and two positive vectors $y > 0$ and $x > 0$, there exists one, and up to positive multiples only one, nonnegative nonsingular diagonal matrix $D \geqslant 0$ such that $y^T D$ and $D^{-1}x$ form a dual pair.*

A few years later, i.e., in 1967, Gries and Stoer [4] generalized the above theorem in the sense that it applies to other than absolute norms as well as to pairs of vectors that are not strictly positive.

In this paper we obtain further generalizations of Theorem 1 in the afore-said sense; this we achieve by judiciously utilizing a fundamental result of Zenger [7], namely, Theorem 2 therein. Finally, we summarize in a very compact form the results obtained herein and elsewhere ([4], [6]) in the case where y and x are positive and nonnegative (with coinciding zeros), respectively, and appropriately comment upon them.

2. Background and preliminaries. A *norm* $\|\cdot\|$ in a finite n-dimensional Euclidean real or complex space ($\boldsymbol{R}^n$ or $\boldsymbol{C}^n$) is a real function with the following three properties:

$$\|x\| > 0 \quad \text{for all } x \neq 0,\ x \in \boldsymbol{R}^n \text{ (or } \boldsymbol{C}^n\text{)},$$
$$\|\alpha x\| = \alpha\|x\| \quad \text{for all real numbers } \alpha \geqslant 0,$$
$$\|x+y\| \leqslant \|x\|+\|y\| \quad \text{for all } x, y \in \boldsymbol{R}^n \text{ (or } \boldsymbol{C}^n\text{)}.$$

Such a norm is known as a *weakly homogeneous* norm.

The norm $\|\cdot\|^D$ (defined in the space of all row vectors y^H) *dual* to the norm $\|\cdot\|$ is defined by

$$\|y^H\|^D = \sup_{x\neq 0} \frac{\operatorname{Re} y^H x}{\|x\|}.$$

(Note that y^H denotes the conjugate transpose of y.)

A pair of nonzero vectors, y^H and x, is *dual* with respect to the norm $\|\cdot\|$ if

$$\|y^H\|^D\|x\| = \operatorname{Re} y^H x. \tag{1}$$

Symbolically, this will be written as $y^H\|x$.

A norm $\|\cdot\|$ is *strictly homogeneous* if

$$\|\alpha x\| = |\alpha|\,\|x\| \quad \text{for all real numbers } \alpha,\ x\in \boldsymbol{R}^n$$

or

$$\|\beta x\| = |\beta|\,\|x\| \quad \text{for all complex numbers } \beta,\ x\in \boldsymbol{C}^n.$$

A norm $\|\cdot\|$ is *absolute* if

$$\|x\| = \||x|\|, \quad x\in \boldsymbol{R}^n \text{ (or } \boldsymbol{C}^n),$$

where, as usual,

$$|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T \quad \text{if } x = (x_1, x_2, \ldots, x_n)^T.$$

(Note that x^T denotes the transpose of x.)

If the norm $\|\cdot\|$ is strictly homogeneous, then the right-hand side of (1) is replaced by $|y^H x|$, and, in general, for any norm,

$$\|y^H\|^D\|x\| \geqslant \operatorname{Re} y^H x,$$

known as Hölder's inequality.

A norm $\|\cdot\|$ is *monotonic* if

$$|x| \leqslant |y| \text{ implies } \|x\| \leqslant \|y\|,$$

where $|x| \leqslant |y|$ means $|x_i| \leqslant |y_i|$, $i = 1, 2, \ldots, n$.

The *least upper bound* norm of an $(n \times n)$-matrix A, denoted by lub(A), with respect to the norm $\|\cdot\|$, is given by

$$\operatorname{lub}(A) = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

It can be shown [1] that for absolute norms

$$\operatorname{lub}(D) = \max_i |d_{ii}|, \tag{2}$$

where

$$D = \begin{bmatrix} d_{11} & & \\ & d_{22} & \\ & & \ddots d_{nn} \end{bmatrix} = \operatorname{diag}[d_{11}, d_{22}, \ldots, d_{nn}].$$

The following result now follows:

LEMMA 1. *A norm* $\|\cdot\|$ *is absolute if and only if*

$$\operatorname{lub}(D) = 1 \qquad \forall D \text{ with } |D| = I,$$

where $|D|$ *is the matrix whose* (i, j)*-th element is* $|d_{ij}|$.

Proof. If $\|\cdot\|$ is absolute, then the result follows from (2).

Conversely, for any $x \in \boldsymbol{R}^n$ (or $\boldsymbol{C}^n$) let D be such that $|x| = Dx$, and therefore $|D| = 1$. Then

$$\||x|\| \leqslant \operatorname{lub}(D)\|x\| = \|x\| \leqslant \operatorname{lub}(D^{-1})\||x|\| = \||x|\|;$$

therefore $\|x\| = \||x|\|$.

Throughout the rest of the paper $\mathscr{D}$ will denote the set of all $n \times n$ nonnegative nonsingular diagonal matrices.

A norm $\|\cdot\|$ in $\boldsymbol{R}^n$ is *orthant-monotonic* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all x, y satisfying $x_i y_i \geqslant 0$, $i = 1, 2, \ldots, n$, where

$$x = (x_1, x_2, \ldots, x_n)^T \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n)^T.$$

Orthant-monotonic norms were first introduced in [4] and extensively investigated in [3].

Given $x \in \boldsymbol{C}^n$, where

$$x = (x_1' + ix_1'', x_2' + ix_2'', \ldots, x_n' + ix_n'')^T, \qquad x_j', x_j'' \text{ real},$$

let

$$x^{\boldsymbol{R}} = (x_1', x_2', \ldots, x_n', x_1'', x_2'', \ldots, x_n'')^T = x' \oplus x'',$$

where $\oplus$ denotes the direct sum of the real vectors x', x'' with

$$x' = (x_1', x_2', \ldots, x_n')^T \quad \text{and} \quad x'' = (x_1'', x_2'', \ldots, x_n'')^T.$$

Now, given a norm $\|\cdot\|$ in $\boldsymbol{C}^n$, a corresponding norm $\|\cdot\|_{\boldsymbol{R}}$ in $\boldsymbol{R}^{2n}$ is defined by

$$\|x' \oplus x''\|_{\boldsymbol{R}} = \|x' + ix''\| = \|x\|$$

for all vectors $x^{\boldsymbol{R}}$ of $\boldsymbol{R}^{2n}$.

A norm $\|\cdot\|$ in $\boldsymbol{C}^n$ is *orthant-monotonic* if and only if the corresponding norm $\|\cdot\|_{\boldsymbol{R}}$ in $\boldsymbol{R}^{2n}$ is orthant-monotonic.

We observe that absolute norms (or, equivalently, monotonic norms [1]) are orthant-monotonic; moreover, all absolute norms are strictly homogeneous, however this is not true for all orthant-monotonic norms.

A norm $\|\cdot\|$ is *symmetric* [2] if, for all permutation matrices P and all vectors $x \in \boldsymbol{R}^n$ (or $\boldsymbol{C}^n$),

$$\|Px\| = \|x\|.$$

Symbolically, this will be denoted by $\|x\|_P$.

In the sequel, P will always denote a permutation matrix irrespective of whether this is explicitly stated or not, and P^T will denote the transpose of P.

A *coordinate subspace* is a subspace spanned by a proper subset of the set of axis vectors $\{e_1, e_2, \dots, e_n\}$. Hereinafter u_1 and u_2 as well as v_1 and v_2 will always denote elements of coordinate subspaces of $\mathbf{R}^n$ or $\mathbf{C}^n$.

A norm $\|\cdot\|$ is *subspace monotonic* if, for every permutation matrix P and any $x = P(u_1 \oplus u_2) \in \mathbf{R}^n$ (or $\mathbf{C}^n$),

$$\|u_1 \oplus 0\|_P \leqslant \|x\| = \|u_1 \oplus u_2\|_P.$$

(Cf. *monotone* in [5].)

We observe that all absolute (monotonic) norms are subspace monotonic.

We next present certain results relating to properties of the classes of norms we have considered herewith.

The following lemma establishes the relationship between orthant-monotonic and subspace-monotonic norms.

LEMMA 2. *A norm* $\|\cdot\|$ *in* $\mathbf{R}^n$ *is orthant-monotonic if and only if it is subspace monotonic.*

Proof. Assume that $\|\cdot\|$ is orthant-monotonic. Then the result follows, since $|u_1 \oplus 0| \leqslant |u_1 \oplus u_2|$ and $u_1 \oplus 0$ and $u_1 \oplus u_2$ lie in one common orthant at least.

Conversely, suppose that the norm $\|\cdot\|$ is subspace monotonic and, furthermore, let $x, y \in \mathbf{R}^n$ be in the same orthant, with $|x| \leqslant |y|$. If $x = y$, then the result is trivially true. Assume, therefore, that x and y differ in exactly k components, $1 \leqslant k \leqslant n$. Then there exists a sequence of vectors

$$x = x^{(0)},\ x^{(1)}, \dots, x^{(k-1)},\ x^{(k)} = y$$

with $x^{(j)}$ differing from $x^{(j+1)}$ in exactly one component and, in addition, $|x^{(j)}| \leqslant |x^{(j+1)}|$ for all j in $\{0, 1, \dots, k-1\}$.

In order to prove the result, it is necessary to show that, for all j, $\|x^{(j)}\| \leqslant \|x^{(j+1)}\|$; therefore, it suffices to show that $\|x\| \leqslant \|y\|$ when x and y differ in only one component. Assume now that

$$x = P(\hat{x} \oplus \eta), \quad y = P(\hat{x} \oplus \theta), \quad \hat{x} \in \mathbf{R}^{n-1},\ \eta, \theta \in \mathbf{R},$$

with $|\eta| < |\theta|$ and $\eta\theta \geqslant 0$. Then, on setting $\lambda = \eta/\theta$,

$$x = \lambda P(\hat{x} \oplus \theta) + (1-\lambda) P(\hat{x} \oplus 0), \quad \lambda \in [0, 1),$$

and, since the norm $\|\cdot\|$ is subspace monotonic,

$$\|x\| \leqslant \lambda \|y\| + (1-\lambda)\|\hat{x} \oplus 0\|_P \leqslant \|y\|.$$

The above lemma does not hold if the norm $\|\cdot\|$ is defined in $\mathbf{C}^n$. The ensuing example illustrates this point.

EXAMPLE 1. The norm $\|\cdot\|$ defined by

$$\|x\| = \max\{|x_1|, |x_2|, |x_1 - x_2|\}$$

in $\mathbf{C}^2$ is subspace monotonic but not orthant-monotonic.

From the necessity part of the proof of Lemma 2 it is easily seen that if the norm $\|\cdot\|$ in C^n is orthant-monotonic, then it is subspace monotonic also.

The next two lemmas deal with properties of subspace monotonic norms; these results were originally stated in [2], p. 313, without formal proofs. These are now provided herewith for the sake of completeness.

LEMMA 3. *If a norm $\|\cdot\|$ in $\boldsymbol{R}^n$ (or $\boldsymbol{C}^n$) is subspace monotonic, then the dual norm $\|\cdot\|^D$ is subspace monotonic also.*

Proof. For any permutation matrix P, consider any vector

$$y = P(v_1 \oplus v_2) \in \boldsymbol{R}^n \text{ (or } \boldsymbol{C}^n).$$

If $v_1 = 0$, then $\|(v_1 \oplus 0)^H\|_P^D \leqslant y^H\|^D$, so suppose that $v_1 \neq 0$ and let x be dual to $(v_1 \oplus 0)^H P^T$. Next, let $x = P(u_1 \oplus u_2)$, where u_1 and v_1 are in the same coordinate subspace. Then

$$\|(v_1 \oplus 0)^H\|_P^D \|x\| = \operatorname{Re} v_1^H u_1 \leqslant \|y^H\|^D \|u_1 \oplus 0\|_P \leqslant \|y^H\|^D \|x\|$$

on using Hölder's inequality and the fact that the norm $\|\cdot\|$ is subspace monotonic. Since $x \neq 0$, it follows immediately from the above that $\|\cdot\|^D$ is subspace monotonic.

Next we present a characterization of subspace-monotonic norms in terms of dual pairs of vectors.

LEMMA 4. *Let the norm $\|\cdot\|$ be subspace monotonic and let $x = P(u_1 \oplus 0) \neq 0$ in $\boldsymbol{R}^n$ (or $\boldsymbol{C}^n$) be given. Then, for any vector y^H which is dual to x, where $y^H = (v_1 \oplus v_2)^H P^T$, with u_1 and v_1 in the same coordinate subspace, the vector $(v_1 \oplus 0)^H P^T$ is also dual to x.*

Proof. Let $y^H = (v_1 \oplus v_2)^H P^T$ be any vector dual to $x = P(u_1 \oplus 0) \neq 0$. By the duality of the vectors y^H and x, the Hölder inequality and Lemma 3, it follows that

$$\|y^H\|^D \|x\| = \operatorname{Re} v_1^H u_1 \leqslant \|(v_1 \oplus 0)^H\|_P^D \|x\| \leqslant \|y^H\|^D \|x\|;$$

therefore

$$\|(v_1 \oplus 0)^H P^T\|^D \|x\| = \operatorname{Re} v_1^H u_1.$$

The following lemma generalizes Theorem 9 in [1].

LEMMA 5. *If the norm $\|\cdot\|$ is subspace monotonic, then the axis vectors e_i, $i = 1, 2, \ldots, n$, are self-dual, namely,*

$$\|e_i^T\|^D \|e_i\| = 1.$$

The lemma follows immediately from Lemma 4.

Finally, we introduce another norm which will feature prominently in the final section of the paper where we summarize the results presented here and elsewhere ([4], [6]) and discuss them in some detail.

Consider any (weakly homogeneous) norm in C^n with the property that for all x

$$\|x\| \geqslant \|\mathrm{Re}\, x\|, \qquad x \in C^n. \tag{3}$$

The following two lemmas now follow immediately.

LEMMA 6. *If any (weakly homogeneous) norm* $\|\cdot\|$ *in* C^n *satisfies* (3), *then the dual norm* $\|\cdot\|^D$ *satisfies* (3) *also.*

The proof is analogous to that of Lemma 3.

LEMMA 7. *If any (weakly homogeneous) norm in* C^n *satisfies* (3), *then the dual of any real vector is also real.*

The proof is analogous to that of Lemma 4.

From the above, if $\|\cdot\|_{\mathrm{Re}}$ denotes the restriction of $\|\cdot\|$ (in C^n) to R^n, it follows that $\|\cdot\|_{\mathrm{Re}}^D$ is identical to the restriction of $\|\cdot\|^D$ to R^n; therefore,

$$y^H \|_{\mathrm{Re}} x \Leftrightarrow y^H \| x \qquad \forall y, x \in R^n. \tag{4}$$

3. Generalizations. We now present various generalizations of Theorem 1 in the sense that it applies to other than absolute norms and to pairs of vectors that are not strictly positive.

In [4] Gries and Stoer proved that Theorem 1 obtains for all norms in R^n, including weakly homogeneous norms. Then, by using this result, they proved the following generalization of Theorem 1:

THEOREM 2. *Given an orthant-monotonic norm in* R^n *and two nonzero nonnegative vectors* $y, x \in R^n$, *with matching zeros, there exists* $D \in \mathscr{D}$ *such that* $y^T D \| D^{-1} x$. *Furthermore, the diagonal elements* d_{ii} *of* D *for which* y_i, x_i *are positive are uniquely determined up to positive multiples, whilst those for which* y_i, x_i *are zero are arbitrary numbers.*

We observe that, because of Lemma 2, Theorem 2 obtains for subspace monotonic norms in R^n.

From the definition of orthant-monotonic norms in C^n, given in the previous section, it is easy to show that Theorem 2 implies Theorem 1, and, furthermore, that Theorem 2 obtains for orthant-monotonic norms in C^n (see [4]).

It is noted here that the said generalizations obtain for weakly homogeneous norms also, provided that the norm is orthant-monotonic.

We now drop the requirement that D be nonnegative, and proceed to prove that the afore-said theorems obtain for even wider classes of norms in C^n. First, we state the following fundamental result, due to Zenger [7]:

THEOREM 3. *Let* $\alpha_i > 0$, $i = 1, 2, \ldots, r$, $r \leqslant n$, *be any numbers, and let* $p_1, \ldots, p_r \in C^n$ *be a set of* r *linearly independent vectors. Then, given any (weakly homogeneous) norm in* C^n, *there exists a dual pair* $w^H \| z$ *such that*

$$\|w^H\|^D \|z\| = w^H z = \sum_{i=1}^{r} \alpha_i,$$

where $z = \sum \beta_i p_i$ *and* $w^H \beta_i p_i = \alpha_i$ *for* $i = 1, 2, \ldots, r$.

In [7] the numbers α_i are scaled so that

$$\sum_{i=1}^{r} \alpha_i = 1.$$

Two further generalizations of Theorem 1 are now established in the form of Theorems 4 and 5.

THEOREM 4. *Given any (weakly homogeneous) norm in* C^n *and two nonzero vectors* $y, x \in C^n$, *with* $|y|, |x| > 0$ *and* $\arg(x_i) = \arg(y_i)$, $i = 1, 2, \ldots, n$, *there exists* D *with* $|D| \in \mathcal{D}$ *such that* $y^H D \| D^{-1} x$.

Proof. Let the set of linearly independent vectors referred to in Theorem 3 be axis vectors, namely,

$$\{p_i |\ i = 1, 2, \ldots, r\} = \{e_j |\ j \in H\},$$

where H is a subset of $\{1, 2, \ldots, n\}$ and the cardinality of H is r. Correspondingly, renumber, if necessary, the α_i referred to in Theorem 3 so that $\alpha_i > 0$ for $i \in H$. Next, let $\alpha_i = 0$ for $i \notin H$. Then, from Theorem 3 it follows that there exists a dual pair, w^H and z, such that

$$\|w^H\|^D \|z\| = w^H z = \sum_{i=1}^{n} \alpha_i, \qquad \bar{w}_i z_i = \alpha_i,\ i = 1, 2, \ldots, n, \tag{5}$$

where, as usual, $\bar{w}_i$ denotes the complex conjugate of w_i, the i-th component of the vector w.

Now let y and x be nonzero vectors in C^n with matching zeros and with $\arg(x_i) = \arg(y_i)$, $i = 1, 2, \ldots, n$. Furthermore, let

$$H = \{i |\ y_i, x_i \neq 0\} \quad \text{and} \quad \alpha_i = \bar{y}_i x_i, \quad i = 1, 2, \ldots, n,$$

so that $\alpha_i > 0$ for $i \in H$. Then, from (5) it follows that

$$\|w^H\|^D \|z\| = w^H z = y^H x, \qquad \bar{w}_i z_i = \bar{y}_i x_i,\ i = 1, 2, \ldots, n. \tag{6}$$

Next, let $D = (d_{ii})$ be a nonsingular diagonal matrix defined by

$$d_{ii} = \begin{cases} x_i/z_i & \text{for } i \in H, \\ 1 & \text{for } i \notin H. \end{cases}$$

Then $z = D^{-1}x$, and from (6) it follows that $\bar{w}_i = d_{ii}\bar{y}_i$ for $i \in H$. If the cardinality of H is n, then $w^H = y^H D$ and the result follows.

THEOREM 5. *Given any (weakly homogeneous) norm in* C^n, *which is subspace monotonic, and two nonzero vectors* $y, x \in C^n$ *with matching zeros and* $\arg(x_i) = \arg(y_i)$, $i = 1, 2, \ldots, n$, *there exists* D *with* $|D| \in \mathcal{D}$ *such that* $y^H D \| D^{-1} x$.

Proof. This is exactly the same as that of the previous theorem except that the cardinality of H is less than n, in which case $w^H = y^H D + q$, where q is any vector in C^n with $q_i = 0$ for $i \in H$. Since $w^H \| z$, it follows from Lemma 4 that $y^H D \| z$ if $\|\cdot\|$ is subspace monotonic.

4. Examples, discussion and conclusions. We begin this section by giving two examples which in a sense, as we shall see, complement the results presented herein and elsewhere ([4], [6]).

EXAMPLE 2. Let $\|\cdot\|$ be the strictly homogeneous norm on C^2 defined by

$$\|x\| = \max\{|x_1 - x_2|, |x_2|\}.$$

It can then be verified that its dual is given by

$$\|y^H\|^D = |y_1| + |y_1 + y_2|, \tag{7}$$

and it is easily seen that (3) obtains; so, for real y, $\|y^T\|^D_{\mathrm{Re}}$ is also given by the expression on the right-hand side of (7). Since

$$\|(2, 0)^T\| = 2 > \|(2, 1)^T\| = 1,$$

it is clear that $\|\cdot\|$ and $\|\cdot\|_{\mathrm{Re}}$ are not subspace monotonic.

By virtue of the remarks in the last paragraph of this section, Theorem 1 holds for $\|\cdot\|$, and it is easy to derive explicitly the scaling matrix D. However, Theorems 2 and 5 fail to hold for $\|\cdot\|$ and $\|\cdot\|_{\mathrm{Re}}$, since these norms are not subspace monotonic.

Finally, this example shows that, in general, Lemma 5 does not hold for nonsubspace-monotonic norms.

EXAMPLE 3. Let $\|\cdot\|$ be the strictly homogeneous norm on C^2 defined by

$$\|x\| = \max\{|x_1|, |x_2|, |x_1 - ix_2|\}. \tag{8}$$

For real x, $\|x\|_{\mathrm{Re}} = \sqrt{|x_1|^2 + |x_2|^2}$, so $\|\cdot\|_{\mathrm{Re}}$ is absolute. From (8), putting either $x_1 = 0$ or $x_2 = 0$, we immediately see that $\|\cdot\|$ is subspace monotonic. However, it is not orthant-monotonic and even fails to satisfy (3), since

$$\|(1+i, 2)^T\| = 2 < \|(1, 2)^T\| = \sqrt{5}.$$

As $\|\cdot\|$ is subspace monotonic, Theorem 5 holds; however, Theorems 1 and 2 do not hold. In order to see this, let $x = y = (1, 1)^T$ and let $D = \mathrm{diag}[d_{11}, d_{22}]$ be *real* and nonsingular. Then

$$\|D^{-1}x\| = \{d_{11}^{-2} + d_{22}^{-2}\}^{1/2}$$

and

$$\|y^T D\|^D \geq \frac{|(\pm 1 + i\sqrt{3})d_{11} + 2d_{22}|}{\|(\pm 1 + i\sqrt{3}, 2)^T\|} = \{d_{11}^2 + d_{22}^2 \pm d_{11}d_{22}\}^{1/2}.$$

Hence, for all real D,

$$\|y^T D\|^D \|D^{-1}x\| \geq \left(\frac{|d_{11}|}{|d_{22}|} + \frac{|d_{22}|}{|d_{11}|} + 1\right)^{1/2} \left(\frac{|d_{11}|}{|d_{22}|} + \frac{|d_{22}|}{|d_{11}|}\right)^{1/2} \geq \sqrt{6}.$$

So, as $|y^T x| = 2$, there is no real D such that $y^T D \| D^{-1} x$. For this particular norm Theorem 2 does in fact hold if y and x are not strictly positive, since by Lemma 5 the axis vectors are self-dual. However, if we define a norm $\|\cdot\|_t$ on C^3 by

$$\|x\|_t = \max\{|x_1|, |x_2|, |x_1 - ix_2|, |x_3|\},$$

it is not difficult to see that this norm has the same properties as $\|\cdot\|$ and, furthermore, there is no real D such that

$$(1, 1, 0)D\|_t D^{-1}(1, 1, 0)^T.$$

Next we note that Theorems 4 and 5 hold for y and x being positive and nonnegative (with coinciding zeros), respectively. For y and x nonnegative, the

TABLE 1

R^n	$y, x > 0$	$y, x \geqslant 0, \neq 0$ $x_i = 0 \Leftrightarrow y_i = 0$
weakly homogeneous	$\exists D \geqslant 0$	$\nexists D$, Ex. 2
orthant-/subspace monotonic	$\exists D \geqslant 0$	$\exists D \geqslant 0$

results obtained herein and elsewhere ([4], [6]) are now summarized in Tables 1 and 2 for the real and complex cases, respectively, and then commented upon.

TABLE 2

C^n	$y, x > 0$	$y, x \geqslant 0, \neq 0$ $x_i = 0 \Leftrightarrow y_i = 0$
weakly homogeneous	$\exists D$, $\nexists D \geqslant 0$ Ex. 3	$\nexists D$, Ex. 2
subspace monotonic	$\exists D$, $\nexists D \geqslant 0$ Ex. 3	$\exists D$, $\nexists D \geqslant 0$ Ex. 3
orthant-monotonic	$\exists D \geqslant 0$	$\exists D \geqslant 0$

For the various classes of norms and the vectors y and x being positive or nonnegative (with coinciding zeros), the tables indicate whether or not there always exists D such that $y^T D \| D^{-1} x$ (denoted in the tables by $\exists D$ or $\nexists D$, respectively) and, in the former case, whether or not there always exists such a D which is *nonnegative* (denoted by $\exists D \geqslant 0$ or $\nexists D \geqslant 0$, respectively). For those cases in which there may exist no D or no nonnegative D, the number of example illustrating this is given.

Since Tables 1 and 2 are complete, we conclude that, for the classes of norms and nonnegative vectors under consideration, the results presented are the best possible. We also conclude that, since the norms of Examples 2 and 3 are strictly homogeneous, the entries for strictly homogeneous norms would be exactly the same as those for weakly homogeneous norms, and therefore the best possible.

Notwithstanding the afore-said best possible results, it is possible to obtain further improved results if other classes of norms are considered. For example, consider the class of weakly homogeneous norms with the property that (3) is satisfied for all $x \in \boldsymbol{C}^n$. Then, by considering the restrictions of these norms to $\boldsymbol{R}^n$, it follows from (4) that Theorem 1 holds for all weakly homogeneous norms on $\boldsymbol{C}^n$ that satisfy (3). Likewise, Theorem 2 holds for those weakly homogeneous norms on $\boldsymbol{C}^n$ that satisfy (3) and whose restrictions to $\boldsymbol{R}^n$ are orthant-monotonic. It is noted here that all orthant-monotonic norms on $\boldsymbol{C}^n$ satisfy (3), but not all subspace-monotonic norms on $\boldsymbol{C}^n$ have this property, nor vice versa (see Examples 2 and 3). Additionally, not all norms on $\boldsymbol{C}^n$ that satisfy (3) and whose restrictions to $\boldsymbol{R}^n$ are orthant-monotonic (subspace monotonic) are themselves subspace monotonic. For example, the weakly homogeneous norm $\|\cdot\|$ on $\boldsymbol{C}^2$ defined by

$$\|x\| = \max\{|\mathrm{Re}\, x_1|, |\mathrm{Im}\, x_1|, |\mathrm{Re}\, x_2|, |\mathrm{Im}(x_1 - x_2)|\}$$

satisfies (3), $\|\cdot\|_{\mathrm{Re}}$ is orthant-monotonic, yet $\|\cdot\|$ is not subspace monotonic.

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DEPARTMENT OF COMPUTER SCIENCE
BIRKBEÇK COLLEGE
UNIVERSITY OF LONDON
LONDON, WC1E 7HX
UNITED KINGDOM

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