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Third note on the general solution of a functional equation

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In our two previous papers ([5], [6]) we described the general solution of the functional equation

$$\varphi[f(x)] = G(x, \varphi(x)), \tag{1}$$

where φ is the unknown function: in [5] under the assumption that $f[f(x)] = x$, in [6] under hypothesis that $f(x)$ is invertible. Now we are going to take up this subject again. We shall drop the condition of the invertibility of $f(x)$, making instead more restrictive assumptions regarding the function $G(x, y)$.

Let E and Ξ be two arbitrary non-empty sets, independent of each other. In the whole of this paper we shall assume that:

(i) The function $f(x)$ is defined in the set E and

$$f(E) \subset E. \tag{2}$$

(ii) The function $G(x, y)$ is defined in the set $E \times \Xi$ and, for every fixed $x_0 \in E$, $G(x_0, y)$ maps the set Ξ onto itself in a one-to-one manner.

Although the condition on G is rather strong, it is fulfilled in many important cases, such as for instance the case of the linear equation

$$\varphi[f(x)] = b_0(x)\varphi(x) + b_1(x)$$

provided that $b_0(x) \neq 0$ in E.

In § 1 we shall study the iteration of the function $f(x)$. In § 2 we shall introduce a family of functions ${}_\lambda g_n(x, y)$ and we shall establish some properties of these functions. § 3 contains the main result: the construction of the general solution of equation (1) in the class Φ of functions that are defined in E and assume values from Ξ. In § 4 we shall apply this result to some special cases: the Abel equation, the Schröder equation and the equation of automorphic functions.

§ 1. The iterates of $f(x)$ are defined by

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \quad n = 0, 1, 2, \ldots, \quad x \in E.$$

According to (2) the functions $f^n(x)$ are defined in E for every integer $n \geqslant 0$.

Let $\{f_\lambda^{-1}(x)\}$, $\lambda \in \Lambda$, be the family of all inverse functions to $f(x)$, i.e. the family of all the functions defined in $f(E)$ and fulfilling the condition

$$f[f_\lambda^{-1}(x)] = x \quad \text{for} \quad x \in f(E)\,. \tag{3}$$

The condition $f_\lambda^{-1}[f(x)] = x$ in general is not fulfilled.

We define $f_\lambda^{-n}(x)$, $n > 0$, as the nth iterate of the function $f_\lambda^{-1}(x)$. The function $f_\lambda^{-n}(x)$ is defined in a suitable subset of E, which may be empty ([1]). In order to be able to write some formulae uniformly, we shall sometimes write $f_\lambda^n(x)$ instead of $f^n(x)$ if $n \geqslant 0$. Of course, if $n \geqslant 0$, then $f_\lambda^n(x) = f^n(x)$ does not depend on λ.

If $f^n(x_1) = x_2$, $n > 0$, there always exists an index $\lambda \in \Lambda$ (in general not unique) such that $f_\lambda^{-n}(x_2) = x_1$. It is enough to put $f_\lambda^{-1}[f^j(x_1)] = f^{j-1}(x_1)$ for $j = 1, \ldots, n$, and then to extend f_λ^{-1} onto $f(E)$ to an inverse function to $f(x)$.

For $x_1, x_2 \in E$ we write $x_1 \iota x_2$ whenever there exist integers $n \geqslant 0$, $m \geqslant 0$ such that $f^n(x_1) = f^m(x_2)$. The relation ι is reflexive, symmetric and transitive, and therefore the set E can be split into disjoint sets such that x_1, x_2 belong to the same set if and only if $x_1 \iota x_2$. These sets will be called *cycles*. The cycle containing an $x_0 \in E$ will be denoted by $C(x_0)$, i.e.

$$C(x_0) = \{x\colon\ x \in E,\ x_0 \iota x\}\,.$$

Consequently

$$C(x_1) = C(x_2) \quad \text{if and only if } x_1 \iota x_2\,. \tag{4}$$

For any positive integer k we denote by E_k the set of those $x \in E$ for which there exists an integer $i \geqslant 0$ such that

$$f^{i+k}(x) = f^i(x) \tag{5}$$

(here i may depend on x) and (5) does not hold for a smaller k with any integer i. Further we put

$$E_0 = E - \bigcup_{k=1}^{\infty} E_k\,. \tag{6}$$

Consequently E_0 is the set of those $x \in E$ for which

$$f^{i+k}(x) \neq f^i(x) \quad \text{for all } i \geqslant 0\,,\ k \geqslant 1\,.$$

Suppose that $x \in E_k$, $k \geqslant 1$. Among integers $i \geqslant 0$ such that (5) holds there exists a smallest. It will be denoted by $J(x)$. Thus the func-

([1]) E.g. if $E = (-\infty, 0) \cup (0, +\infty)$, $f(x) = x^2$, then $f_\lambda^{-1}(x) = e_\lambda(x)\sqrt{x}$, $x \in (0, \infty)$, where $[e_\lambda(x)]^2 = 1$. (The family of functions $e_\lambda(x)$, and consequently also that of functions $f_\lambda^{-1}(x)$, has cardinality $2^{\mathfrak{c}}$.) For $e_{\lambda_0}(x) \equiv -1$ the function $f_{\lambda_0}^{-2}(x)$ is not defined for any $x \in E$.

tion $J(x)$ is defined in the set $\bigcup_{k=1}^{\infty} E_k$ and is characterized by the following property.

LEMMA 1. *If* $x \in E_k$, $k \geqslant 1$, *then*

$$f^{i+k}(x) = f^i(x) \quad \textit{for} \quad i \geqslant J(x),$$
$$f^{i+k}(x) \neq f^i(x) \quad \textit{for} \quad i < J(x).$$

Proof. The second of the above relations results from the definition of $J(x)$, the first from the fact that for $i \geqslant J = J(x)$ we have $f^{i+k}(x) = f^{i-J}[f^{J+k}(x)] = f^{i-J}[f^J(x)] = f^i(x)$.

Now we shall prove the following

LEMMA 2. *If* $x_0 \in E_k$, $k \geqslant 0$, *then* $C(x_0) \subset E_k$.

Proof. Suppose that $x_0 \in E_k$, $k \geqslant 1$. Thus there exists an i such that

$$f^{i+k}(x_0) = f^i(x_0)\,. \tag{7}$$

Let $x \in C(x_0)$. Consequently there exist integers $n \geqslant 0$, $m \geqslant 0$ such that $f^n(x_0) = f^m(x)$. Thus we have $f^{i+m+k}(x) = f^{i+k}[f^m(x)] = f^{i+k}[f^n(x_0)] = f^n[f^{i+k}(x_0)] = f^n[f^i(x_0)] = f^i[f^n(x_0)] = f^i[f^m(x)] = f^{i+m}(x)$, i.e. $f^{i+m+k}(x) = f^{i+m}(x)$. Here k cannot be replaced by a smaller one, for otherwise a similar argument would show that also k in (7) is not minimal. Thus $x \in E_k$, which proves that $C(x_0) \subset E_k$.

Now, if $x_0 \in E_0$ and $x \in C(x_0)$, then $x \in E_0$, for otherwise x would have to belong to an E_k, $k \geqslant 1$; and on account of what has already been proved we would have by (4) $C(x_0) = C(x) \subset E_k$, which is impossible in view of (6). Consequently $C(x_0) \subset E_0$, which completes the proof.

LEMMA 3. *If for an* $x_0 \in E$ *and for an* $n > 0$ $f_\lambda^{-n}(x_0) = f_\mu^{-n}(x_0)$, *then* $f_\lambda^{-i}(x_0) = f_\mu^{-i}(x_0)$ *for all* $i \leqslant n$.

Proof. It results from the fact that $f[f_\lambda^{-i}(x)] = f_\lambda^{-i+1}(x)$ (cf. formula (3)).

LEMMA 4. *If* $x_1, x_2 \in E_0$ *and for some non-negative integers* n, m, p, q, *we have*

$$f^n(x_1) = f^m(x_2) \quad \textit{and} \quad f^p(x_1) = f^q(x_2)\,, \tag{8}$$

then $n - m = p - q$.

Proof. For argument's sake let us suppose that $p \geqslant n$. Then we have

$$f^q(x_2) = f^p(x_1) = f^{p-n}[f^n(x_1)] = f^{p-n}[f^m(x_2)]\,,$$

i.e.

$$f^{p-n+m}(x_2) = f^q(x_2)\,. \tag{9}$$

Since $x_2 \in E_0$, we obtain hence $p - n + m = q$, i.e. $n - m = p - q$.

LEMMA 5. *If $x_1, x_2 \in E_k$, $k \geqslant 1$, and for some non-negative integers n, m, p, q relation* (8) *holds, then there exists an integer r such that*

$$(n-m)-(p-q) = rk\,. \tag{10}$$

Proof. As in the proof of Lemma 4 we obtain relation (9). We may write

$$p-n+m-q = rk+s\,,$$

where r is an integer (positive, negative, or zero) and

$$0 \leqslant s < k\,. \tag{11}$$

Thus (9) becomes

$$f^{q+rk+s}(x_2) = f^q(x_2)\,,$$

whence we get for $i = J(x_2)$

$$f^{i+q+rk+s}(x_2) = f^{i+q}(x_2)\,.$$

By Lemma 1 we obtain hence

$$f^{i+q+s}(x_2) = f^{i+q}(x_2)\,,$$

which, in view of (11) and of the definition of the sets E_k, implies $s = 0$. Hence (10) results.

If $x_1 \,\iota\, x_2$ and $x_1, x_2 \in E_0$, we put

$$D(x_1, x_2) = n-m\,,$$

where n, m are such that $f^n(x_1) = f^m(x_2)$. According to Lemma 4 the function $D(x_1, x_2)$ is unambiguously defined.

§ 2. It follows from hypothesis (ii) that there exists a unique function $G^{-1}(x, y)$ inverse to $G(x, y)$ with respect to y. The function $G^{-1}(x, y)$ is defined in $E \times \Xi$, like $G(x, y)$. Now we define functions ${}_\lambda g_n(x, y)$ by the relations

$$ {}_\lambda g_0(x, y) = y\,,$$

$$ {}_\lambda g_{n+1}(x, y) = G[f_\lambda^n(x), {}_\lambda g_n(x, y)]\,, \qquad n = 0, 1, 2, \ldots \tag{12}$$

$$ {}_\lambda g_{n-1}(x, y) = G^{-1}[f_\lambda^{n-1}(x), {}_\lambda g_n(x, y)]\,, \qquad n = 0, -1, -2, \ldots$$

For $n \geqslant 0$ instead of ${}_\lambda g_n(x, y)$ we shall often write simply $g_n(x, y)$, since in this case ${}_\lambda g_n(x, y)$ is independent of λ.

The function ${}_\lambda g_n(x, y)$ is defined for $x \in E$, $y \in \Xi$ whenever $f_\lambda^n(x)$ is defined. For $n \geqslant 0$ ${}_\lambda g_n(x, y) = g_n(x, y)$ is defined in the whole of $E \times \Xi$.

As an easy consequence of Lemma 3 we obtain the following

LEMMA 6. *If for an $x_0 \in E$ and for an integer n we have $f_\lambda^n(x_0) = f_\mu^n(x_0)$, then ${}_\lambda g_n(x_0, y) = {}_\mu g_n(x_0, y)$ for $y \in \Xi$.*

Let us note also the following lemmas.

LEMMA 7. *If $a \geqslant 0$, $b \geqslant 0$, or $a \leqslant 0$, $b \leqslant 0$, then*

$$ {}_\lambda g_{a+b}(x, y) = {}_\lambda g_a[f_\lambda^b(x), {}_\lambda g_b(x, y)] , \tag{13} $$

provided that one of the terms is defined (²).

Proof. We shall prove (13) for $a \leqslant 0$, $b \leqslant 0$; the proof in the other case is similar. For $a = 0$ (13) is obvious. Suppose that (13) holds for an $a \leqslant 0$ and every $b \leqslant 0$ and that ${}_\lambda g_{a+b-1}(x, y)$ is defined. We have by (12)

$$ \begin{aligned} {}_\lambda g_{a+b-1}(x, y) &= G^{-1}[f_\lambda^{a+b-1}(x), {}_\lambda g_{a+b}(x, y)] \\ &= G^{-1}\big[f_\lambda^{a+b-1}(x), {}_\lambda g_a\big(f_\lambda^b(x), {}_\lambda g_b(x, y)\big)\big] \\ &= G^{-1}\big[f_\lambda^{a-1}\big(f_\lambda^b(x)\big), {}_\lambda g_a\big(f_\lambda^b(x), {}_\lambda g_b(x, y)\big)\big] , \end{aligned} $$

and again by (12) we obtain

$$ {}_\lambda g_{a+b-1}(x, y) = {}_\lambda g_{a-1}[f_\lambda^b(x), {}_\lambda g_b(x, y)] , $$

i.e. relation (13) for $a-1, b$. Now, the left-hand side of (13) is defined if $f_\lambda^{a+b}(x)$ is defined, and the right-hand side of (13) is defined if $f_\lambda^a\big(f_\lambda^b(x)\big)$ is defined, which ammounts to the same.

This completes the proof.

LEMMA 8. *If $a \geqslant 0$ and $f_\lambda^{-a}[f^a(x)] = x$, then*

$$ {}_\lambda g_{-a}[f^a(x), g_a(x, y)] = y . $$

Proof. For $a = 0$ the lemma is trivial. Suppose it true for an $a \geqslant 0$ and let

$$ f_\lambda^{-a-1}[f^{a+1}(x)] = x . \tag{14} $$

Hence (cf. (3)) $f_\lambda^{-a}[f^{a+1}(x)] = f(x)$, i.e.

$$ f_\lambda^{-a}\big[f^a\big(f(x)\big)\big] = f(x) . \tag{15} $$

Now we have

$$ \begin{aligned} (16) \quad {}_\lambda g_{-a-1}&[f^{a+1}(x), g_{a+1}(x, y)] \\ &= G^{-1}\big[f_\lambda^{-a-1}\big(f^{a+1}(x)\big), {}_\lambda g_{-a}\big(f^{a+1}(x), g_{a+1}(x, y)\big)\big] . \end{aligned} $$

But by Lemma 7 and formulae (12)

$$ g_{a+1}(x, y) = g_a[f(x), g_1(x, y)] = g_a[f(x), G(x, y)] $$

(²) This last restriction is essential only if $a < 0$, $b < 0$ and should be understood as follows: if one side of (13) is defined, then the other side is also defined and both are equal.

and thus by the induction hypothesis and in view of (15)

$$(17)\quad {}_\lambda g_{-a}\big(f^{a+1}(x),\, g_{a+1}(x,y)\big) = {}_\lambda g_{-a}\Big(f^{a}\big(f(x)\big),\, g_a[f(x),\, G(x,y)]\Big) = G(x,y)\,.$$

Finally we obtain by (14), (17) and (16)

$${}_\lambda g_{-a-1}[f^{a+1}(x),\, g_{a+1}(x,y)] = G^{-1}[x,\, G(x,y)] = y\,,$$

which was to be proved.

For every $x \in E$ we define a set $V[x] \subset \Xi$. For $x \in E_0$ we put $V[x] = \Xi$. For $x \in E_k$, $k \geqslant 1$, $V[x]$ is the set of y fulfilling

$$g_{i+k}(x,y) = g_i(x,y)\,,$$

where $i = J(x)$. Because of (6) $V[x]$ is defined for all $x \in E$.

The significance of the sets $V[x]$ will be shown in the next section. Now we shall prove the following

LEMMA 9. *Suppose that for an $x_0 \in E$, for some non-negative integers n, m, p, q and for some parameters $\lambda, \mu \in \Lambda$ we have*

$$(18)\qquad f_\lambda^{-m}[f^n(x_0)] = f_\mu^{-q}[f^p(x_0)]\,.$$

Then for every $y \in V[x_0]$

$$(19)\qquad {}_\lambda g_{-m}[f^n(x_0),\, g_n(x_0,y)] = {}_\mu g_{-q}[f^p(x_0),\, g_p(x_0,y)]\,.$$

Proof. I. At first we suppose that we have

$$(20)\qquad n - m = p - q\,.$$

Without loss of generality we may assume that $p \geqslant n$. Then we have also $q \geqslant m$ and

$$f_\mu^{-q}[f^p(x_0)] = f_\mu^{-m}\big[f_\mu^{-(q-m)}\big(f^p(x_0)\big)\big]\,.$$

By (18) we get hence

$$(21)\qquad f_\mu^{-(q-m)}\big(f^p(x_0)\big) = f^n(x_0)\,,$$

i.e.

$$f_\mu^{-q}[f^p(x_0)] = f_\mu^{-m}[f^n(x_0)]$$

and

$$f_\lambda^{-m}[f^n(x_0)] = f_\mu^{-m}[f^n(x_0)]\,.$$

On account of Lemma 6 we obtain

$$(22)\qquad {}_\lambda g_{-m}[f^n(x_0),\, g_n(x_0,y)] = {}_\mu g_{-m}[f^n(x_0),\, g_n(x_0,y)]\,.$$

Now we have by Lemma 7 and by (21)

$$ {}_{\mu}g_{-q}[f^{p}(x_0), g_p(x_0, y)] = {}_{\mu}g_{-m}\big(f_{\mu}^{-(q-m)}[f^{p}(x_0)], {}_{\mu}g_{-(q-m)}[f^{p}(x_0), g_p(x_0, y)]\big) $$
$$ = {}_{\mu}g_{-m}\big(f^{n}(x_0), {}_{\mu}g_{-(q-m)}[f^{p}(x_0), g_p(x_0, y)]\big) . $$

Applying again Lemma 7 we obtain in view of (20)

$$ g_p(x_0, y) = g_{p-n}[f^{n}(x_0), g_n(x_0, y)] = g_{q-m}[f^{n}(x_0), g_n(x_0, y)] . $$

Further we obtain from (21)

$$ f^{p}(x_0) = f^{q-m}[f^{n}(x_0)] $$

and

$$ f_{\mu}^{-(q-m)}\big[f^{q-m}\big(f^{n}(x_0)\big)\big] = f^{n}(x_0) . $$

Hence we obtain by Lemma 8

$$ {}_{\mu}g_{-(q-m)}[f^{p}(x_0), g_p(x_0, y)] $$
$$ = {}_{\mu}g_{-(q-m)}\big(f^{q-m}[f^{n}(x_0)], g_{q-m}[f^{n}(x_0), g_n(x_0, y)]\big) = g_n(x_0, y) , $$

and finally

$$ {}_{\mu}g_{-q}[f^{p}(x_0), g_p(x_0, y)] = {}_{\mu}g_{-m}[f^{n}(x_0), g_n(x_0, y)] , $$

whence in view of (22) we obtain relation (19).

II. Now let us suppose that (20) does not hold. Then by Lemma 4 there exists a $k \geqslant 1$ such that $x_0 \in E_k$ and by Lemma 5 there exists an integer r such that (10) holds. Without loss of generality we may assume that $r > 0$.

Let us put $i = J(x_0)$ and let us choose an index $\omega \in \Lambda$ such that

$$ f_{\mu}^{-q}[f^{p}(x_0)] = f_{\omega}^{-(q+i)}[f^{p+i}(x_0)] . \tag{23} $$

Since $p-q = (p+i)-(q+i)$, we have on account of the first part of the proof

$$ {}_{\mu}g_{-q}[f^{p}(x_0), g_p(x_0, y)] = {}_{\omega}g_{-(q+i)}[f^{p+i}(x_0), g_{p+i}(x_0, y)] . \tag{24} $$

Now we shall prove that for every integer $s \geqslant 0$ we have

$$ g_{p+i+sk}(x_0, y) = g_{p+i}(x_0, y) . \tag{25} $$

For $s = 0$ relation (25) is trivial. Suppose it true for an $s \geqslant 0$. We have by Lemma 7

$$ g_{p+i+(s+1)k}(x_0, y) = g_{p+sk}[f^{i+k}(x_0), g_{i+k}(x_0, y)] . \tag{26} $$

But since $i = J(x_0)$ and $y \in V[x_0]$, we have $f^{i+k}(x_0) = f^i(x_0)$ and $g_{i+k}(x_0, y) = g_i(x_0, y)$. Thus applying again Lemma 7 and making use of the induction hypothesis, we obtain from (26)

$$g_{p+i+(s+1)k}(x_0, y) = g_{p+sk}[f^i(x_0), g_i(x_0, y)] = g_{p+i+sk}(x_0, y) = g_{p+i}(x_0, y),$$

which proves that (25) is valid for all integers $s \geqslant 0$.

Since $i = J(x_0)$, we have

$$f^{p+i+rk}(x_0) = f^{p+i}(x_0) \tag{27}$$

(cf. Lemma 1), and hence

$$f_\omega^{-(q+i)}[f^{p+i+rk}(x_0)] = f_\omega^{-(q+i)}[f^{p+i}(x_0)]. \tag{28}$$

Relations (18), (23) and (28) give

$$f_\lambda^{-m}[f^n(x_0)] = f_\omega^{-(q+i)}[f^{p+i+rk}(x_0)].$$

Now, we have by (10)

$$(p+i+rk)-(q+i) = p-q+rk = n-m,$$

and in view of the first part of the proof

$${}_\lambda g_{-m}[f^n(x_0), g_n(x_0, y)] = {}_\omega g_{-(q+i)}[f^{p+i+rk}(x_0), g_{p+i+rk}(x_0, y)],$$

whence by (27), (25) for $s = r$, and (24) we obtain relation (19). This completes the proof.

Let us write

$$E^* = \{x\colon V[x] \neq \emptyset\}.$$

We shall prove the following

LEMMA 10. *We have $f(E^*) \subset E^*$.*

Proof. We must prove that if for an x_0 we have $V[x_0] \neq \emptyset$, then also $V[f(x_0)] \neq \emptyset$. If $x_0 \in E_0$, this is trivial (cf. Lemma 2). So let us assume that $x_0 \in E_k$, $k \geqslant 1$, and that there exists a $y_0 \in V[x_0]$. We shall show that then $y_1 = g_1(x_0, y_0)$ belongs to $V[f(x_0)]$.

Since $x_0 \in E_k$ and $y_0 \in V[x_0]$, we have

$$g_{i+k}(x_0, y_0) = g_i(x_0, y_0), \tag{29}$$

where $i = J(x_0)$. We distinguish two cases.

1. $i = 0$. Then we have by Lemma 1 $f^k[f(x_0)] = f(x_0)$, which shows that $J(f(x_0)) = 0$. Further, according to Lemma 7 and relation (29),

$$g_k(f(x_0), y_1) = g_k[f(x_0), g_1(x_0, y_0)] = g_{k+1}(x_0, y_0)$$
$$= g_1[f^k(x_0), g_k(x_0, y_0)] = g_1(x_0, y_0) = y_1,$$

which means that $y_1 \in V[f(x_0)]$.

2. $i > 0$. Then, by Lemma 1, $f^{i-1+k}[f(x_0)] = f^{i-1}[f(x_0)]$ and for $j < i-1$ $f^{j+k}[f(x_0)] \neq f^j[f(x_0)]$. Consequently $J(f(x_0)) = i-1$. Thus, according to Lemma 7 and relation (29),

$$g_{i-1+k}(f(x_0), y_1) = g_{i-1+k}[f(x_0), g_1(x_0, y_0)] = g_{i+k}(x_0, y_0)$$
$$= g_i(x_0, y_0) = g_{i-1}[f(x_0), g_1(x_0, y_0)] = g_{i-1}(f(x_0), y_1),$$

which means that $y_1 \in V[f(x_0)]$. This completes the proof.

§ 3. In the present section we shall describe a construction of the general solution of equation (1) in E under the assumption that the functions $f(x)$ and $G(x, y)$ fulfil conditions (i) and (ii).

At first we shall prove the following

LEMMA 11. *Suppose that a function $\varphi(x)$ satisfies equation* (1) *in E. Then for every $n \geqslant 0$ and $\lambda \in \Lambda$ we have*

$$\varphi[f^n(x)] = g_n(x, \varphi(x)), \tag{30}$$

$$\varphi[f_\lambda^{-n}(x)] = {}_\lambda g_{-n}(x, \varphi(x)), \tag{31}$$

provided that $f_\lambda^{-n}(x)$ is defined.

Proof. We prove only relation (31); the proof of (30) is somewhat simpler. For $n = 0$ relation (31) is obvious. Suppose it true for an $n \geqslant 0$. Then we have on account of (1)

$$\varphi[f_\lambda^{-n}(x)] = \varphi\big[f\big(f_\lambda^{-n-1}(x)\big)\big] = G\big(f_\lambda^{-n-1}(x), \varphi[f_\lambda^{-n-1}(x)]\big),$$

i.e. by (31) (induction hypothesis)

$$G\big(f_\lambda^{-n-1}(x), \varphi[f_\lambda^{-n-1}(x)]\big) = {}_\lambda g_{-n}(x, \varphi(x)).$$

Hence, in view of hypothesis (ii) and relations (12)

$$\varphi[f_\lambda^{-n-1}(x)] = G^{-1}\big[f_\lambda^{-n-1}(x), {}_\lambda g_{-n}(x, \varphi(x))\big] = {}_\lambda g_{-n-1}(x, \varphi(x)),$$

which proves that (31) is valid for all $n \geqslant 0$.

LEMMA 12. *Suppose that a function $\varphi \in \Phi$ satisfies equation* (1) *in E. Then for every $x_0 \in E$ we have $\varphi(x_0) \in V[x_0]$.*

Proof. For $x_0 \in E_0$ this is obvious, so suppose that $x_0 \in E_k$, $k \geqslant 1$. Put $i = J(x_0)$. Consequently $f^{i+k}(x_0) = f^i(x_0)$ and

$$\varphi[f^{i+k}(x_0)] = \varphi[f^i(x_0)].$$

Hence we obtain according to Lemma 11

$$g_{i+k}(x_0, \varphi(x_0)) = g_i(x_0, \varphi(x_0)),$$

which means that $\varphi(x_0) \in V[x_0]$.

Corollary. *The condition*

$$V[x] \neq \emptyset \quad \text{for} \quad x \in E \tag{32}$$

is necessary for the existence of a solution belonging to the class Φ of equation (1) *in E.*

If condition (32) is not fulfilled, we must replace the set E by the set E^*, in which $V[x] \neq \emptyset$. Lemma 10 guarantees that if the function $f(x)$ fulfils hypothesis (i) in E, then it fulfils this hypothesis also in E^*. Therefore in the sequal we may assume that $V[x] \neq \emptyset$ in E.

For an arbitrary subset F of the set E we define $\Psi[F]$ as the class of functions $\varphi(x)$ which are defined in F and such that for arbitrary $x_0 \in F$ $\varphi(x_0) \in V[x_0]$. In view of Lemma 12 every solution of equation (1) in E belongs to the class $\Psi[E]$.

Let A be a set which contains exactly one element [3] of every cycle contained in E. We write

$$a(x) = A \cap C(x)\,. \tag{33}$$

The function $a(x)$ is unambiguously defined in E whenever the set A has been fixed.

Theorem 1. *Suppose that hypotheses* (i) *and* (ii) *are fulfilled and $V[x] \neq \emptyset$ for $x \in E$. Let A be a set containing exactly one element* [3] *of every cycle contained in E. Then to every function $\varphi_0(x)$ belonging to the class $\Psi[A]$ there exists exactly one function $\varphi(x)$ which belongs to the class Φ, satisfies equation* (1) *and fulfils the condition*

$$\varphi(x) = \varphi_0(x) \quad \text{for} \quad x \in A\,. \tag{34}$$

This function is given by the formula

$$\varphi(x) = {}_{\lambda}g_{-m}\Big(f^n[a(x)],\, g_n\big[a(x),\, \varphi_0\big(a(x)\big)\big]\Big)\,, \tag{35}$$

where the function $a(x)$ is defined by (33), *and the integers $n, m \geqslant 0$ and the index $\lambda \in \Lambda$ are chosen in such a manner that*

$$f_\lambda^{-m}\big[f^n\big(a(x)\big)\big] = x\,. \tag{36}$$

Proof. Since by (33) $x \,\iota\, a(x)$, there exist n, m and λ fulfilling (36). In view of Lemma 9 the right-hand side of (35) is independent of the choice of n, m, λ fulfilling (36). Consequently the function $\varphi(x)$ is by formula (35) unambiguously defined in the whole of E and evidently belongs to the class Φ. We must prove that $\varphi(x)$ satisfies equation (1) in E, fulfils condition (34) and is the unique function with these properties.

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(3) Here we make use of the axiom of choice.

Let us fix an $x \in E$ and n, m, λ such that (36) holds. Since $x \iota f(x)$, we have (cf. (4) and (33)) $a(x) = a[f(x)]$. Hence, according to (36)

$$f_\lambda^{-m+1}[f^n(a[f(x)])] = f(x)$$

and consequently

$$\varphi[f(x)] = {}_\lambda g_{-m+1}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big) . \tag{37}$$

On the other hand

$$G(x, \varphi(x)) = G\big\{f_\lambda^{-m}(f^n[a(x)]), {}_\lambda g_{-m}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big)\big\} . \tag{38}$$

But we have by (12)

$$\begin{aligned}&{}_\lambda g_{-m}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big)\\ &\qquad = G^{-1}\big\{f_\lambda^{-m}(f^n[a(x)]), {}_\lambda g_{-m+1}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big)\big\} ,\end{aligned}$$

whence

$$\begin{aligned}&{}_\lambda g_{-m+1}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big)\\ &\qquad = G\big\{f_\lambda^{-m}(f^n[a(x)]), {}_\lambda g_{-m}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big)\big\} .\end{aligned} \tag{39}$$

From (37), (39) and (38) we obtain

$$\varphi[f(x)] = G(x, \varphi(x)) ,$$

which proves that $\varphi(x)$ satisfies equation (1).

If $x \in A$, then $a(x) = x$ and (36) holds for $m = n = 0$. Then (35) becomes $\varphi(x) = \varphi_0(x)$, i.e. condition (34) is fulfilled.

Lastly, let $\varphi(x)$ be any function belonging to the class Φ, satisfying equation (1) in E and fulfilling (34). Let us fix an arbitrary $x \in E$ and n, m, λ such that (36) holds. Then we have by Lemma 11 and condition (34)

$$\varphi[f^n(a(x))] = g_n[a(x), \varphi(a(x))] = g_n[a(x), \varphi_0(a(x))] ,$$

and again by Lemma 11

$$\begin{aligned}\varphi(x) &= \varphi\big(f_\lambda^{-m}[f^n(a(x))]\big) = {}_\lambda g_{-m}\big(f^n[a(x)], \varphi[f^n(a(x))]\big)\\ &= {}_\lambda g_{-m}\big(f^n[a(x)], g_n[a(x), \varphi_0(a(x))]\big) ,\end{aligned}$$

which means that $\varphi(x)$ coincides with function (35). This completes the proof.

Remark. According to Lemma 12, formula (35) gives the general solution of equation (1) when φ_0 ranges over $\Psi[A]$.

The following theorem is an immediate consequence of Theorem 1, and Lemma 12.

THEOREM 2. *Under hypotheses* (i) *and* (ii) *relation* (32) *is a necessary and sufficient condition of the existence of a solution belonging to the class* Φ *of equation* (1) *in* E.

§ 4. We shall illustrate the above results by three examples.

EXAMPLE 1. Let Ξ be an Abelian group (written additively) in which the following condition is fulfilled:

(iii) For any integer $n \neq 0$ and any $y \in \Xi$ the equation $ny = 0$ has the unique solution $y = 0$.

Let $c \neq 0$ be an element of Ξ. We consider the Abel equation (cf. [1])

$$\varphi[f(x)] = \varphi(x) + c\,, \tag{40}$$

where $f(x)$ is a function fulfilling hypothesis (i). Here $G(x, y) = y + c$, and so hypothesis (ii) is fulfilled, since Ξ is a group. The functions ${}_\lambda g_n(x, y)$ are given by

$${}_\lambda g_n(x, y) = y + nc$$

and are independent of x and λ. It follows from (iii) that

$$V[x] = \emptyset \quad \text{for} \quad x \in \bigcup_{k=1}^{\infty} E_k\,,$$

and of course $V[x] \neq \emptyset$ for $x \in E_0$. Thus Theorem 2 implies the following

THEOREM 3. *If* $f(x)$ *fulfils hypothesis* (i) *and* Ξ *is an Abelian group fulfilling condition* (iii), *then equation* (40) *has in* E *a solution assuming values in* Ξ *if and only if* $f^j(x) \neq x$ (4) *for every* $x \in E$ *and every integer* $j > 0$.

In the case where $\Xi = \boldsymbol{R}$ is the group of real numbers, this is a known result of R. Tambs Lyche ([10]).

In order to give the general solution of equation (40) let us fix a set A containing exactly one element of every cycle contained in E. We define the function $a(x)$ by (33) and the function $d(x)$ by

$$d(x) = D\big(a(x), x\big)\,. \tag{41}$$

THEOREM 4. *If* $f(x)$ *fulfils hypothesis* (i), Ξ *is an Abelian group fulfilling condition* (iii) *and* $f^j(x) \neq x$ *for* $x \in E$ *and* $j > 0$, *then to every function* $\varphi_0(x)$ *defined in* A *and taking values from* Ξ *there exists exactly one functions* $\varphi(x)$ *satisfying equation* (40) *in* E *and condition* (34). *This function is given by*

$$\varphi(x) = \varphi_0[a(x)] + d(x)c\,.$$

(4) Let us note that if for an $x \in E$ relation (5) holds, then for $x^* = f^l(x)$ we have $f^k(x^*) = x^*$.

The above theorem results directly from Theorem 1. For $\Xi = \boldsymbol{R}$ Theorem 4 was also found by R. Tambs Lyche ([10]).

EXAMPLE 2. Let Ξ be a vector space over a number field K, and let $s \neq 0$ be a number from the field K which is not a root of unity. We consider the Schröder equation (cf. [9])

$$\varphi[f(x)] = s\varphi(x)\,, \tag{42}$$

where $f(x)$ is a function fulfilling (i). The function $G(x, y) = sy$ fulfils hypothesis (ii), since K is a field and $s \neq 0$. The functions ${}_\lambda g_n(x, y)$ are given by

$${}_\lambda g_n(x, y) = s^n y$$

and are independent of x and λ. It follows from the condition that s is not a root of unity that

$$V[x] = \{\theta\} \quad \text{for} \quad x \in \bigcup_{k=1}^{\infty} E_k\,,$$

where θ is the null element of Ξ.

We fix a set A containing exactly one element of every cycle contained in E_0 and define the functions $a(x)$ and $d(x)$ for $x \in E_0$ by (33) and (41), respectively. From Theorem 1 we obtain

THEOREM 5. *If $f(x)$ fulfils hypothesis* (i), *then to every function $\varphi_0(x)$ defined in A and taking values from Ξ there exists exactly one function $\varphi(x)$ satisfying equation* (42) *in E and condition* (34). *This function is given by*

$$\varphi(x) = \begin{cases} \theta & \textit{for} \quad x \in \bigcup_{k=1}^{\infty} E_k\,, \\ s^{d(x)}\varphi_0[a(x)] & \textit{for} \quad x \in E_0\,. \end{cases}$$

Theorem 5 improves our earlier result ([7], theorem 1.1).

EXAMPLE 3. Let Ξ be an arbitrary non-empty set. We consider the equation of automorphic functions (cf. [4])

$$\varphi[f(x)] = \varphi(x)\,, \tag{43}$$

where $f(x)$ is a function fulfilling (i). The function $G(x, y) = y$ evidently fulfils hypothesis (ii). We have for every n ${}_\lambda g_n(x, y) = y$ and thus

$$V[x] = \Xi \quad \text{for} \quad x \in E\,.$$

Let A be a set containing exactly one element of every cycle contained in E and define the function $a(x)$ by (33). Thus $a(x_1) = a(x_2)$ if and only if $x_1 \iota x_2$. Formula (35) takes the form

$$\varphi(x) = \varphi_0[a(x)]\,,$$

whence it follows that if $x_1 \iota x_2$, then $\varphi(x_1) = \varphi(x_2)$. Hence we obtain ([5])

THEOREM 6. *Suppose that the function $f(x)$ fulfils hypothesis* (i). *A function $\varphi(x)$ satisfies equation* (43) *in E if and only if it is constant on every cycle contained in E.*

This is an improvement of a result of S. B. Prešić ([8]).

Automorphic functions play an important part in the theory of functional equations. There are some results to the effect that the general solution of the linear functional equation of nth order

$$\varphi[f^n(x)] = b_0(x)\varphi[f^{n-1}(x)] + \ldots + b_{n-1}(x)\varphi(x) + b_n(x) \tag{44}$$

can be expressed by some particular solutions of (44) and at most n arbitrary solutions of equation (43) (cf. [2], [3], [4]).

([5]) Theorem 6 can be proved without the use of the axiom of choice. In fact, if $\varphi(x)$ satisfies (43) and $f^n(x_1) = f^m(x_2)$, then $\varphi(x_1) = \varphi[f^n(x_1)] = \varphi[f^m(x_2)] = \varphi(x_2)$. The "if" part of Theorem 6 is trivial.

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