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Some relations between Toeplitz and singular integral operators on odd spheres

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Abstract. The work contains a characterization of various C^* algebras (modulo compact operators) generated by singular integral operators (with a continuous symbol) on odd spheres. These characterizations depend on the analogous results for Toeplitz operators on odd spheres.

Let $S = \{z \in C^n |z| = 1\}$. Denote by $L^2(S)$ the Hilbert space of all square summable functions with respect to the Lebesgue measure on S. Let us denote by $H^2(S)$ the Hardy subspace of $L^2(S)$ of all functions which are radial boundary values of holomorphic functions in $B = \{z \in C^n |z| < 1\}$. See for example [2]. It has been proved by Koranyi and Vagi in [4] that the orthogonal projection operator P of $L^2(S)$ onto $H^2(S)$ is given by a singular integral operator. More precisely, we have for $z \in S$ and $f \in L^2(S)$

$$(1)\qquad Pf(z) = f(z)/2 + \text{P.V.}\int (1-\bar{w}\cdot z)^{-n} f(w)dw, \quad \text{where } \bar{w}\cdot z = \sum_{i=1}^{n} \bar{w}_i z_i$$

and dw is the normalized Lebesgue measure on S. Here the symbol P.V. means a principal value integral. See [4] for the definition. We are going to give some applications of Toeplitz operators to some singular integral operators, depending on relation (1). Namely, we will describe some C^* algebras generated by singular integral operators (with kernels given by the Szego kernel $(1-\bar{w}\cdot z)^{-n}$). It seems that our results can be also obtained by a direct computation, but we have wanted to emphasize the relation of Toeplitz operators to singular integral operators (which is well known for $n = 1$).

In what follows for a given operator T we denote by T^* the adjoint operator and by $[A, B] = AB - BA$ the commutator of operators A and B.

Let $S_1 f(z) = \text{P.V.}\int (1-\bar{w}\cdot z)^{-n} f(w)\, dw$, $f \in L^2(S)$. Then we can rewrite (1) in the form

$$P = \tfrac{1}{2}I + S_1.$$

PROPOSITION 1. *For any* $\varphi \in C(S)$ (*the algebra of continuous functions on*

S) the commutator $[S_1, L_\varphi]$ *is compact in* $L^2(S)$*, where* L_φ *denotes the operator of multiplication by* φ *in* $L^2(S)$.

Proof. Let z_i: $S \ni z = (z_1, \ldots, z_n) \to z_i$ be the coordinate function. Using the approximation by polynomials in z and $\bar{z}$ (the Stone–Weierstrass theorem) and the symmetry of z_i's it is sufficient to prove that $[S_1, L_{z_i}]$ is compact. Since $S_1|_{H^2} = \frac{1}{2}I$, we have for each $f \in L^2(S)$

$$\begin{aligned}(S_1 L_{z_i} - L_{z_i} S_1)f &= S_1(z_i Pf + (I-P)z_i f) - z_i S_1(Pf + (I-P)f) \\ &= \tfrac{1}{2} z_i Pf + S_1 z_i (I-P)f - \tfrac{1}{2} z_i Pf - z_i S_1 (I-P)f \\ &= (P - \tfrac{1}{2}I) z_i (I-P)f - \tfrac{1}{2} z_i (P-I)f = P z_i (I-P)f.\end{aligned}$$

Writing $Pz_i(I-P) = ((I-P)\bar{z}_i P)^*$ we conclude that the operator $Pz_i(I-P)$ is compact by Lemma of [3]. The proof is complete.

From the above result we easily get

PROPOSITION 2. *Let* $\varphi \in C(S)$. *We define the operator*

$$S_\varphi f(\xi) = \mathrm{P.V.}\int (1 - \bar{\xi} \cdot w)^{-n} \varphi(w) f(w)\, dw, \qquad \xi \in S,\ f \in L^2(S).$$

If $\psi \in C(S)$, *then the commutator* $[S_\varphi, L_\psi]$ *is compact.*

Proof. Applying Proposition 1 the result follows immediately by the equality $[S_\varphi, L_\psi] = [S_1, L_{\varphi\psi}] + L_\psi [L_\varphi, S_1]$. The proof is finished.

Note that $S_\varphi = S_1 L_\varphi$.

PROPOSITION 3. *For any* $\varphi, \psi \in C(S)$ *the commutator* $[S_\varphi, S_\psi]$ *is compact.*

Proof. We can write $[S_\varphi, S_\psi] = S_1 L_\varphi [S_1, L_\psi] + S_1 L_\psi [L_\varphi, S_1]$ and the above claim follows by Proposition 1.

The above proposition proves that the operator S_φ is essentially normal. In order to describe the C^* algebra generated by the family $(S_\varphi)_{\varphi \in C(S)}$ we will also need the following proposition.

PROPOSITION 4. *Let* $\mathscr{A}$ *be the* C^* *algebra generated by the family* $(S_\varphi)_{\varphi \in C(S)}$. *The algebra* $\mathscr{A}$ *is irreducible.*

Proof. Let X be a projection and $XS_\varphi = S_\varphi X$ for every $\varphi \in C(S)$. Since $XP = PX$ we can write the following decomposition of X with respect $H^2(S) \oplus H^2(S)^\perp$: $X = X_1 \oplus X_2$. But if $\varphi = z_i$ $(i = 1, \ldots, n)$, then $S_{\varphi|H^2} = \frac{1}{2}T_\varphi$, where T_φ denotes the Toeplitz operator.

We know by [1] that the C^* algebra generated by Toeplitz operators is irreducible. Thus $X_1 = 0$ or $X_1 = I$.

Let us assume first that $X = I \oplus X_2$. Since $XS_\varphi = S_\varphi X$ we have

$$(I-P) L_\varphi Pf = X_2 (I-P) L_\varphi Pf, \qquad f \in L^2(S).$$

For $\varphi \in C(S)$ and $\varphi \perp H^2(S)$ the last equality is reduced (on $f = 1$) to $X_2 L_\varphi = L_\varphi$ and this proves that $X_2 = I$.

If $X = 0 \oplus X_2$, then by a similar reasoning we get $X_2(I-P) L_\varphi Pf = 0$, for every $\varphi \in C(S)$, $\varphi \perp H^2(S)$, $f \in L^2(S)$. Thus as above we get $X_2 = 0$ and the argument is complete.

Applying the above propositions we are ready to prove the following theorems.

THEOREM 1. *The C^* algebra $\mathscr{A}$ generated by the family $(S_\varphi)_{\varphi\in C(S)}$ contains the ideal $\mathscr{K}$ of compact operators in $L^2(S)$ and the sequence*

$$(0)\to\mathscr{K}\to\mathscr{A}\xrightarrow{\tau}C(S)\to(0)\quad \text{is exact,}$$

where $\tau(S_\varphi)=\frac{1}{2}\varphi$.

Proof. By Proposition 4 we know that the algebra $\mathscr{A}$ is irreducible. Thus $\mathscr{A}\supset\mathscr{K}$ since $[S_\varphi, S_\psi]\in\mathscr{K}$, for any $\varphi,\psi\in C(S)$. Therefore by a standard reasoning (see for example [1]) it is sufficient to prove: S_φ compact implies $\varphi=0$. Assume that S_φ is compact. Denote by $\sigma(S_\varphi)$ ($\sigma_e(S_\varphi)$ respectively) the spectrum of S_φ (the essential spectrum of S_φ respectively). For a given set $Z\subset C$ we denote by $Z^2=\{a^2, a\in Z\}$. Then we can write

$$\begin{aligned}\sigma(S_\varphi)^2&=\sigma(S_1L_\varphi)^2\supset\sigma_e(S_1L_\varphi)^2=\sigma_e\big((S_1L_\varphi)^2\big)=\sigma_e(S_1^2L_{\varphi^2}+K)\\&=\sigma_e(\tfrac{1}{4}L_{\varphi^2})=\tfrac{1}{4}\varphi(S)^2,\end{aligned}$$

where $K\in\mathscr{K}$. It follows that $\varphi=0$ and the proof is complete.

THEOREM 2. *The C^* algebra $\mathscr{B}$ generated by the family $\{S_\varphi, L_\psi|_{\varphi,\psi\in C(S)}\}$ is commutative modulo the ideal $\mathscr{K}$, i.e. the sequence*

$$(0)\to\mathscr{K}\to\mathscr{B}\xrightarrow{\varrho}C(S)\to(0)\quad \text{is exact,}$$

where $\varrho(S_\varphi)=\frac{1}{2}\varphi$, $\varrho(L_\psi)=\psi$.

Proof. The proof of this theorem is similar to that of Theorem 1, by using Propositions 2, 3 and 4.

In the following notation the bar denotes the complex conjugation. Now we will give a characterization of the C^* algebra (modulo $\overline{\text{the ideal}}$ of compact operators) generated by the compressions to $H^2(S)\oplus\overline{H_0^2(S)}$ of operators L_φ ($\varphi\in C(S)$) and the projection $\tilde{P}=I\oplus 0$. Here $H_0^2(S)=\{f\in H^2(S), f(0)=0\}$. It seems that this characterization is new for $n>1$.

Let $Q\colon L^2(S)\to\overline{H_0^2(S)}$ be the orthogonal projection. By a direct computation or applying Theorem VIII of [2] we know that for any polynomial $p(z_1,\dots,z_n)$ the operator PL_pQ is compact. If

$$\tilde{L}_p=\begin{pmatrix}T_p, & PL_pQ\\ 0, & QL_pQ\end{pmatrix}$$

is the compression of L_p to $H^2(S)\oplus\overline{H_0^2(S)}$, then the commutator

$$[\tilde{P},\tilde{L}_p]=\begin{pmatrix}0, & PL_pQ\\ 0, & 0\end{pmatrix}\quad\text{is compact.}$$

Now we prove that the commutators $[\tilde{L}_{z_i}, \tilde{L}_{z_j}]$ and $[\tilde{L}^*_{z_i}, \tilde{L}_{z_i}]$ are compact for $i, j = 1, \dots, n$. We have

$$\tilde{L}_{z_i} = \begin{pmatrix} T_{z_i}, & R_i \\ 0, & S_i \end{pmatrix}, \quad \text{where } R_i = PL_{z_i}Q, \ S_i = QL_{z_i}Q.$$

Thus we get

$$[\tilde{L}_{z_i}, \tilde{L}_{z_j}] = \begin{pmatrix} T_{z_i z_j}, & T_{z_i}R_j + R_iS_j \\ 0, & S_iS_j \end{pmatrix} - \begin{pmatrix} T_{z_i z_j}, & T_{z_j}R_i + R_jS_i \\ 0, & S_jS_i \end{pmatrix},$$

$$[\tilde{L}^*_{z_i}, \tilde{L}_{z_i}] = \begin{pmatrix} T^*_{z_i} T_{z_i}, & T_{z_i} R_i \\ R_i T_{z_i}, & R^*_i R_i + S^*_i S_i \end{pmatrix} - \begin{pmatrix} T_{z_i} T^*_{z_i} + R_i R^*_i, & R_i S_i \\ S_i R^*_i, & S_i S^*_i \end{pmatrix}.$$

Since $[T^*_{z_i}, T_{z_i}]$ and R_i are compact, it is enough to prove the compactness of $[S_i, S_j]$ and $[S^*_i, S_i]$. In order to do that it is sufficient to show the compactness of $(I-Q)L_pQ$, where p is an arbitrary polynomial. Note that for any $f \in \overline{H^2_0(S)}$ we have

$$\overline{Qpf} = T_{\bar{p}}\bar{f} - P(\bar{p}\bar{f})(0).$$

Thus denoting $P(\bar{p}\bar{f})(0)$ by $\tilde{f}(0)$ we have

$$\|\overline{(I-Q)pf}\|^2 = \|\bar{p}\bar{f} - T_{\bar{p}}\bar{f} + \tilde{f}(0)\|^2$$
$$= \|\bar{p}\bar{f} - T_{\bar{p}}\bar{f}\|^2 + 2Re(\bar{p}\bar{f} - T_{\bar{p}}\bar{f}, \tilde{f}(0)) + \|\tilde{f}(0)\|^2 = \|(I-P)\bar{p}\bar{f}\|^2 + \|\tilde{f}(0)\|^2.$$

Therefore for $f_k \in \overline{H^2_0(S)}$, $f_k \rightharpoonup 0$ (weakly) we can write

$$\|\overline{(I-Q)pf_k}\|^2 = \|(I-P)\bar{p}\bar{f}_k\|^2 + \|\tilde{f}_k(0)\|^2 \underset{k\to\infty}{\to} 0,$$

by the above mentioned Lemma of [3] and since $\tilde{f}_k(0) \underset{k\to\infty}{\to} 0$. Hence the compactness of $(I-Q)L_pQ$ is proved.

Before we formulate the promised description of C^* algebra generated by $\tilde{P}$ and $\tilde{L}_p$ we also need the following

PROPOSITION 5. *The C^* algebra generated by $\tilde{P}$, $\tilde{L}_p$ (p — runs over the set of all polynomials) is irreducible.*

Proof. Let A be an orthogonal projection in $H^2(S) \oplus \overline{H^2_0(S)} = H$, such that

(2) $$A\tilde{L}_p = \tilde{L}_pA \quad \text{for every polynomial } p.$$

Since also $A\tilde{P} = \tilde{P}A$ we can write $A = X \oplus W$ with respect to the decomposition $H = PH \oplus (I-P)H$. By (2) we know that $XT_{z_i} = T_{z_i}X$, $i = 1, \dots, n$. It follows that $X = T_\varphi$ with some φ in the Banach algebra H^∞ of all bounded and holomorphic functions in the unit ball B. Since X is a projection, $\varphi = 0$ or $\varphi = 1$. Similarily $Wf = \bar{\psi}f$, $f \in \overline{H^2_0(S)}$, $\psi \in H^\infty$. Let us consider first the case

$\varphi = 1$. Then, by (2)

$$PL_b Qf = PL_b Q\psi f \quad \text{for any polynomial } b \text{ in } (z_1, \ldots, z_n)$$

and $f \in L^2(S)$. Thus taking the inner product of the above equality (for $f = 1$) with an arbitrary polynomial g we get

$$\int b\bar{g}\,dw = \int b\bar{g}\bar{\psi}\,dw.$$

Hence $\psi = 1$. If $\varphi = 0$ the repetition of the above reasoning proves that $\psi = 0$. These two cases complete the proof.

Now we are able to prove the last theorem.

THEOREM 3. *Let $\mathscr{D}$ be the C^* algebra generated by $\tilde{P}$ and $\tilde{L}_{z_i}$, $i = 1, \ldots, n$. The following sequence*

$$(0) \to \mathscr{K} \to \mathscr{D} \xrightarrow{\varrho} C(S) \oplus C(S) \to (0)$$

is exact, where $\varrho(\tilde{L}_{z_i}) = z_i \oplus z_i$ and $\varrho(\tilde{P}) = 1 \oplus 0$.

Proof. Let π be the canonical projection of $L(H^2 \oplus \overline{H_0^2})$ onto $L(H^2 \oplus \overline{H_0^2})/\mathscr{K}$. By the above reasoning we know that the elements $\pi(\tilde{P})$ and $\pi(\tilde{L}_{z_i})$ $(i = 1, \ldots, n)$ are normal and commute. Since $\mathscr{D}$ is irreducible and $\mathscr{D} \cap \mathscr{K} \neq \{0\}$ it follows that $\mathscr{D} \supset \mathscr{K}$. The standard reasoning [1] proves that for the commutator ideal $\mathscr{I}$ of $\mathscr{D}$ we have $\ker \varrho = \mathscr{I} = \mathscr{K}$. The proof is complete.

References

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