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On sign-preserving solutions of a linear functional equation

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1. The object of the present paper is the functional equation

$$\varphi[f^n(x)]+A_1\varphi[f^{n-1}(x)]+\ldots+A_n\varphi(x) = F(x)\,, \tag{1}$$

where $\varphi(x)$ is the required function, $f(x)$ and $F(x)$ are known functions and A_i are real constant coefficients, $A_n \neq 0$. $f^k(x)$ denotes the kth iteration of the function $f(x)$, i.e.

$$f^0(x) = x\,, \quad f^{v+1}(x) = f[f^v(x)]\,, \quad v = 0, \pm 1, \pm 2, \ldots$$

We shall seek the solutions satisfying one of the conditions

$$\varphi(x) \geqslant 0\,, \tag{2}$$

$$\varphi(x) \leqslant 0\,, \tag{3}$$

$$\varphi(x) \geqslant G(x)\,, \tag{4}$$

$$\varphi(x) \leqslant G(x)\,, \tag{5}$$

where $G(x)$ is a known function.

There may exist infinitely many such solutions of equation (1). For example, every function $\varphi(x)$ periodic with period 1 is the solution of the equation

$$\varphi(x+1)-\varphi(x) = 0\,.$$

But in the case of the equation, for example,

$$\varphi(x+1)+\varphi(x) = 0$$

the function $\varphi(x) \equiv 0$ is the only solution fulfilling condition (2). In the present paper we shall establish some conditions of the uniqueness of solutions of equation (1) fulfilling one of the conditions (2) through (5). For the general linear equation of the first order

$$\varphi[f(x)]+g(x)\varphi(x) = F(x)$$

the corresponding problem (as well as the problem of existence) has been investigated by J. Burek [1].

2. At first we shall prove the following

LEMMA 1. *Suppose that the function $f(x)$ is defined in an interval I and satisfies the condition $f(I) \subset I$. Then there may exist at most one function $\varphi(x)$ satisfying the equation*

$$\varphi[f(x)] - \lambda\varphi(x) = F(x), \quad \lambda \neq 0 \tag{6}$$

and the condition

$$\lim_{v\to\infty} \frac{\varphi[f^v(x)]}{\lambda^v} = 0 \quad \text{for} \quad x \in I. \tag{7}$$

If it does exist, then $\varphi(x)$ is given by the formula

$$\varphi(x) = -\sum_{v=0}^{\infty} \frac{F[f^v(x)]}{\lambda^{v+1}}. \tag{8}$$

Proof. Let $\varphi(x)$ be any function satisfying equation (6) and fulfilling condition (7). From relation (6) we have

$$\varphi(x) = -\frac{F(x)}{\lambda} + \frac{\varphi[f(x)]}{\lambda}. \tag{9}$$

Next

$$\varphi[f(x)] = -\frac{F[f(x)]}{\lambda} + \frac{\varphi[f^2(x)]}{\lambda}. \tag{10}$$

From (9) and (10) we obtain

$$\varphi(x) = -\frac{F(x)}{\lambda} - \frac{F[f(x)]}{\lambda^2} + \frac{\varphi[f^2(x)]}{\lambda^2}.$$

By induction one can obtain the relation

$$\varphi(x) = -\sum_{v=0}^{n} \frac{F[f^v(x)]}{\lambda^{v+1}} + \frac{\varphi[f^{n+1}(x)]}{\lambda^{n+1}}.$$

Passing to the limit as $n\to\infty$, we obtain, according to (7), formula (8), which completes the proof.

THEOREM 1. *Let the function $f(x)$ fulfil the hypotheses of Lemma 1. If the characteristic polynomial of equation* (1)

$$W(\lambda) = \lambda^n + A_1\lambda^{n-1} + \ldots + A_n \tag{11}$$

has n real roots $\lambda_1, \ldots, \lambda_n$ such that

$$\lambda_i < 0 \quad \text{for} \quad i = 1, 2, \ldots, n \tag{12}$$

and, moreover,

$$\lim_{v\to\infty} \frac{F[f^v(x)]}{\lambda_0^v} = 0 \quad \text{for} \quad x \in I, \tag{13}$$

where $\lambda_0 = \min_i |\lambda_i|$, *then equation* (1) *possesses at most one solution satisfying condition* (2) *resp. condition* (3). *The necessary condition for the existence of such a solution is the relation* $F(x) \geqslant 0$ *resp.* $F(x) \leqslant 0$ *for* $x \in I$. *If the solution does exist, then it is given by the formula*

$$\varphi(x) = (-1)^n \sum_{v_1=0}^{\infty} \dots \sum_{v_n=0}^{\infty} \frac{1}{\lambda_1^{v_1+1}} \dots \frac{1}{\lambda_n^{v_n+1}} F[f^{v_1+\dots+v_n}(x)]. \tag{14}$$

Proof. It has been proved in [3] that under the hypotheses of the present theorem equation (1) is equivalent to the system of equations

$$\begin{aligned} \varphi[f(x)] - \lambda_1\varphi(x) &= \psi_1(x), \\ \psi_1[f(x)] - \lambda_2\psi_1(x) &= \psi_2(x), \\ &\dots\dots\dots \\ \psi_{n-1}[f(x)] - \lambda_n\psi_{n-1}(x) &= F(x). \end{aligned} \tag{15}$$

In the above sentence equivalence is understood in the following manner: If the function $\varphi(x)$ satisfies equation (1), then there exists a stystem of functions

$$\psi_1(x), \dots, \psi_{n-1}(x)$$

such that the system of functions

$$\varphi(x),\ \psi_1(x),\ \dots,\ \psi_{n-1}(x) \tag{*}$$

satisfies the system of equations (15), and conversely, if the system of functions (*) satisfies the system of equations (15), then the function $\varphi(x)$ satisfies equation (1).

It is enough to prove that the system of equations (15) has at most one solution with a constant sign in I. Let the function $\psi_{n-1}(x) \geqslant 0$ satisfy the last equation of system (15). Replacing x by $f^v(x)$ and dividing by λ_i^v we obtain

$$\frac{\psi_{n-1}[f^{v+1}(x)]}{\lambda_i^v} + (-\lambda_n)\frac{\psi_{n-1}[f^v(x)]}{\lambda_i^v} = \frac{F[f^v(x)]}{\lambda_i^v},$$

whence, passing to the limit as $v \to \infty$, we obtain, according to (13),

$$\lim_{v\to\infty}\left[\frac{\psi_{n-1}[f^{v+1}(x)]}{\lambda_i^v} + (-\lambda_n)\frac{\psi_{n-1}[f^v(x)]}{\lambda_i^v}\right] = 0.$$

According to condition $\psi_{n-1}(x) \geqslant 0$ the terms

$$\frac{\psi_{n-1}[f^{v+1}(x)]}{\lambda_i^v}; \quad \frac{\psi_{n-1}[f^v(x)]}{\lambda_i^v}$$

have the same sign. Hence, according to $-\lambda_n > 0$, we obtain the relation

(16) $$\lim_{v\to\infty}\frac{\psi_{n-1}[f^v(x)]}{\lambda_i^v} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n .$$

Similarly, if $\psi_{n-2}(x) \geqslant 0$ satisfies the equation

$$\psi_{n-2}[f(x)] - \lambda_{n-1}\psi_{n-2}(x) = \psi_{n-1}(x) ,$$

then, on account of relation (16), we have

$$\lim_{v\to\infty}\frac{\psi_{n-2}[f^v(x)]}{\lambda_i^v} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n .$$

Analogously we obtain relation

$$\lim_{v\to\infty}\frac{\psi_{n-k}[f^v(x)]}{\lambda_i^v} = 0 \quad \text{for} \quad k, i = 1, 2, \ldots, n ,$$

where $\psi_0(x) = \varphi(x)$. Hence, according to Lemma 1, the uniqueness of the solution of the system of equations (15) follows. If we assume the relation $\psi_{n-1}(x) \leqslant 0$, the proof is analogous.

Since $\lambda_i < 0$ we have on account of Vietá's formulas

$$A_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, n .$$

If $\varphi(x)$ is a solution of equation (1) having a constant sign in the interval I, then the function

$$F(x) = \varphi[f^n(x)] + \ldots + A_n\varphi(x)$$

must also have the same constant sign in I. Consequently the relation $F(x) \geqslant 0$ (resp. $F(x) \leqslant 0$) is a necessary condition for the existence of such a solution of equation (1).

Let us suppose that there exists a function $\varphi(x)$ satisfying equation (1) and preserving a constant sign in the interval I. Hence, according to (15) and Lemma 1, we obtain the formula

$$\psi_k(x) = -\sum_{v=0}^{\infty}\frac{1}{\lambda_{k+1}^{v+1}}\psi_{k+1}[f^v(x)] , \quad k = 0, 1, 2, \ldots, n-1 ,$$

where $\psi_0(x) = \varphi(x)$, $\psi_n(x) = F(x)$, and consequently also formula (14), which completes the proof.

COROLLARY 1. *Let the function $f(x)$ fulfil the hypotheses of Lemma 1 and let $f(x)$ be strictly increasing in the interval $I = \langle a, b\rangle$, and $f(a) = a$, $f(b) = b$ and, moreover,*

(17) $$\lambda_i \leqslant -1 \quad \textit{for} \quad i = 1, 2, \ldots, n .$$

Let us suppose further that

$$(18) \qquad f(x) > x \quad \textit{in } (a, b) \quad \textit{and} \quad \lim_{x\to b} F(x) = 0,$$

respectively

$$(19) \qquad f(x) < x \quad \textit{in } (a, b) \quad \textit{and} \quad \lim_{x\to a} F(x) = 0.$$

Then there may exist at most one solution of equation (1) *satisfying condition* (2) *respectively condition* (3).

The proof follows from Theorem 1 and from the remark that for every $x \in (a, b)$ we have

$$(20) \qquad \begin{aligned} &\lim_{n\to\infty} f^n(x) = b \quad \text{in case (18)},\\ &\lim_{n\to\infty} f^n(x) = a \quad \text{in case (19)}. \end{aligned}$$

A proof of condition (20) is to be found in [4].

Now we shall prove

THEOREM 2. *Let the function $f(x)$ fulfil the hypotheses of Lemma* 1. *If a characteristic polynomial* (11) *has n real roots λ_i satisfying condition* (12) *and, moreover,*

$$(21) \qquad \lim_{v\to\infty} \frac{F^*[f^v(x)]}{\lambda_0^v} = 0 \quad \textit{for} \quad x \in I,$$

where

$$F^*(x) = F(x) - \sum_{i=0}^{n} A_i G[f^{n-i}(x)], \qquad \lambda_0 = \min_i |\lambda_i|, \qquad A_0 = 1,$$

and $G(x)$ is a given function defined in I, then equation (1) *may possess at most one solution satisfying condition* (4) *respectively* (5). *The necessary condition of the existence of such a solution is the relation $F^*(x) \geqslant 0$ in I, respectively $F^*(x) \leqslant 0$ in I. If the solution does exist, then it is given by the formula*

$$(22) \qquad \varphi(x) = G(x) + (-1)^n \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} \frac{1}{\lambda_1^{v_1+1}} \cdots \frac{1}{\lambda_n^{v_n+1}} F^*[f^{v_1+\ldots+v_n}(x)].$$

Proof. We shall assume that the function $\varphi(x)$ fulfils equation (1) and condition (4). Then the function $\psi(x) = \varphi(x) - G(x)$ satisfies equation

$$(23) \qquad \psi[f^n(x)] + \ldots + A_n \psi(x) = F^*(x)$$

and condition $\psi(x) \geqslant 0$. Conversely, if the function $\psi(x) \geqslant 0$ in I satisfies equation (23), then the function $\varphi(x) = G(x) + \psi(x)$ satisfies equation (1) and condition (4). The uniqueness of such solutions and formula (22) follow from Theorem 1 according to (21). If we assume relation (5), the proof is analogous.

For the special case of equation (1)

$$\varphi[f^n(x)]+\ldots+\varphi[f(x)]+\varphi(x) = F(x) \tag{24}$$

we can also give some criteria of the existence of a solution fulfilling one of the conditions (2) through (5).

THEOREM 3. *Let the function* $F(x)$ *fulfil the condition*

$$\lim_{v\to\infty} F[f^v(x)] = 0 \quad \textit{for every } x \in I \tag{25}$$

and

$$f(x) \neq x\,, \quad f(I) \subset I\,.$$

If, moreover,

$$F(x) \geqslant 0 \quad \textit{and} \quad \Delta_{(f)}F[f^v(x)] \leqslant 0 \quad \textit{for } x\in I \textit{ and } v = 0,1,2,\ldots, \tag{26}$$

respectively

$$F(x) \leqslant 0 \quad \textit{and} \quad \Delta_{(f)}F[f^v(x)] \geqslant 0 \quad \textit{for } x\in I \textit{ and } v = 0,1,2,\ldots, \tag{27}$$

where

$$\Delta_{(f)}F(x) = F[f(x)]-F(x)\,,$$

then there exists exactly one solution $\varphi(x)$ *of equation* (24) *satisfying condition* (2), *respectively condition* (3). *This solution is given by the formula*

$$\varphi(x) = -\sum_{v=0}^{\infty} \Delta_{(f)}F[f^{(n+1)v}(x)]\,. \tag{28}$$

This theorem follows immediately from Theorem IV in [2].

Similarly, we have also the following

THEOREM 4. *Let the function*

$$F^*(x) = F(x)-\sum_{v=0}^{n} G[f^v(x)]$$

fulfil conditions (25) *and* (26), *respectively* (27), *and let* $f(I)\subset I$, $f(x)\neq x$ *in* I. *Then there exists exactly one solution* $\varphi(x)$ *of equation* (24) *satisfying condition* (4), *respectively* (5). *This solution is given by the formula*

$$\varphi(x) = G(x)-\sum_{v=0}^{\infty} \Delta_{(f)}F[f^{(n+1)v}(x)]\,. \tag{29}$$

The proof follows easily from Theorem 3.

References

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[2] S. Czerwik, *On a recurrence relation*, Ann. Polon. Math., this volume, pp. 61-71.

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