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ASYMPTOTIC BEHAVIOUR OF ONE-DIMENSIONAL FLOWS THROUGH POROUS MEDIA

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Let $Q = \{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$ and let Γ be the boundary of Q.

Consider the boundary value problem of the first type for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \tag{1}$$

where φ is a given function satisfying certain condition to be specified in the sequel, with the following initial and boundary data:

$$\begin{aligned} u(x, 0) &= u_0(x) \quad \text{for } 0 \leqslant x \leqslant 1, \\ u(0, t) = m, \quad u(1, t) &= M \quad \text{for } t \geqslant 0, \end{aligned} \tag{2}$$

m and M being non-negative constants.

We assume that

(i) u_0 is continuous and non-negative in the interval $\langle 0, 1\rangle$, and $u_0(0) = m$, $u_0(1) = M$;

(ii) $\varphi \in C^5((0, +\infty))$, $\varphi(u) > 0$, $\varphi'(u) > 0$, $\varphi''(u) > 0$ for $u > 0$, $\varphi(0) = \varphi'(0) = 0$, $\varphi^{(5)}$ satisfies the Lipschitz condition on every interval $\langle a, b\rangle$, $0 < a < b < \infty$;

(iii) $\varphi(u_0(x))$ is Lipschitz continuous on $\langle 0, 1\rangle$;

(iv) $[\varphi'(u)]^2/\varphi''(u)$ is bounded for bounded u.

Definition. Let u be a non-negative continuous function defined on Q and satisfying (2). The function u is called a *weak solution* of problem (1), (2) if

(1) $\varphi(u(x, t))$ has a strong derivative with respect to x which is locally square-integrable in Q;

(2) for each function $f \in C_0^1(Q)$ which vanishes at $x = 0$ and $x = 1$ we have

$$\iint_Q \left[\frac{\partial f}{\partial t} u - \frac{\partial f}{\partial x}\frac{\partial \varphi(u)}{\partial x}\right] dx dt + \int_0^1 f(x, 0) u_0(x) dx = 0.$$

Oleĭnik et al. [1] have shown that if conditions (i)-(iv) hold, then problem (1), (2) has a unique weak solution u. Furthermore, in a neighbourhood of each point in $Q \setminus \Gamma$ at which u is positive, all derivatives which appear in the equation are continuous and satisfy this equation in the classical sense (Theorems 3 and 4 in [1]).

Now observe that the function

$$u(x, t) = \varphi^{-1}\big((1-x)\varphi(m) + x\varphi(M)\big)$$

is a solution of (1), independent of t, for which $u(0, t) = m$ and $u(1, t) = M$.

The following problem arises ([1]): do the conditions $u(0, t) = m$ and $u(1, t) = M$ imply

$$(3) \qquad \lim_{t\to\infty} u(x, t) = \varphi^{-1}\big((1-x)\varphi(m) + x\varphi(M)\big)?$$

A positive answer to this question gives the following

THEOREM. *Assume* (i)-(iv) *and let* $u(x, t)$ *be a weak solution of problem* (1), (2). *Then* (3) *holds for* $x \in \langle 0, 1\rangle$.

Proof. We consider separately two cases:

(A) $u_0 > 0$ on $\langle 0, 1\rangle$,

(B) $u_0 \geqslant 0$ on $\langle 0, 1\rangle$.

Case (A). Since $u_0 > 0$, it follows from Theorem 11 in [1] that $u(x, t) > 0$ in Q. Consequently, all derivatives in $Q \setminus \Gamma$ which appear in (1) are continuous and the equation is satisfied in the classical sense.

The substitutions $\bar{u}_0 = \varphi(u_0)$ and $\bar{u} = \varphi(u)$ transform (1), (2) into

$$(4) \qquad \frac{\partial \bar{u}}{\partial t} = \varphi'\big(\varphi^{-1}(\bar{u})\big)\frac{\partial^2 \bar{u}}{\partial x^2},$$

$$(5) \qquad \begin{aligned} &\bar{u}(x, 0) = \bar{u}_0(x) \quad \text{for } x \in \langle 0, 1\rangle, \\ &\bar{u}(0, t) = \varphi(m), \quad \bar{u}(1, t) = \varphi(M) \quad \text{for } t \geqslant 0. \end{aligned}$$

From the inequality $\varphi'(u) > 0$ valid for $u > 0$ we infer that the transformation $u \to \bar{u}$ is one-to-one. Thus, instead of (3) it suffices to prove that

$$(6) \qquad \lim_{t\to\infty} \bar{u}(x, t) = (1-x)\varphi(m) + x\varphi(M) \quad \text{for } x \in \langle 0, 1\rangle.$$

([1]) For the asymptotic behaviour of similar though different problems see [2] and [3].

Take two functions $u_{0,1}, u_{0,2} \in C^3(\langle 0, 1\rangle)$ which satisfy the conditions

(7) $$0 < u_{0,1}(x) \leqslant u_0(x) \leqslant u_{0,2}(x) \quad \text{for } x \in \langle 0, 1\rangle,$$

(8) $$u_{0,i}(0) = m, \quad u_{0,i}(1) = M \quad (i = 1, 2),$$

(9) $$\bar{u}''_{0,i}(0) = \bar{u}''_{0,i}(1) = 0 \quad (\bar{u}_{0,i} = \varphi(u_{0,i}),\ i = 1, 2),$$

(10) $$\bar{u}_{0,1}(x) \leqslant (1-x)\varphi(m) + x\varphi(M) \leqslant \bar{u}_{0,2}(x) \quad \text{for } x \in \langle 0, 1\rangle,$$

(11) $$\bar{u}''_{0,1}(x) \geqslant 0 \quad \text{for } x \in \langle 0, 1\rangle,$$

(12) $$\bar{u}_{0,2}(x) \leqslant A, \quad |\bar{u}'_{0,2}(x)| \leqslant 2B \quad \text{for } x \in \langle 0, 1\rangle,$$

where

$$A = \sup_{x \in \langle 0,1\rangle} \bar{u}_0(x),$$

and B is a Lipschitz constant for $\bar{u}_0$.

Let u_i $(i = 1, 2)$ be the weak solutions of (1) corresponding to the initial data $u_{0,i}$ and having boundary values $u_i(0, t) = m$, $u_i(1, t) = M$, $t \geqslant 0$.

Inequalities (7), (10) and Theorem 17 in [1] imply

$$u_1(x, t) \leqslant \varphi^{-1}\big((1-x)\varphi(m) + x\varphi(M)\big) \leqslant u_2(x, t) \quad \text{for } (x, t) \in Q.$$

Thus, putting $\bar{u}_i = \varphi(u_i)$, $i = 1, 2$, we have

$$\bar{u}_1(x, t) \leqslant (1-x)\varphi(m) + x\varphi(M) \leqslant \bar{u}_2(x, t) \quad \text{for } (x, t) \in Q.$$

Therefore, it is enough to show (6) for $\bar{u}_1$ and $\bar{u}_2$.

By (8), (9) and Lemma 1 in [1], the functions $\bar{u}_i$, $\partial\bar{u}_i/\partial t$, $\partial\bar{u}_i/\partial x$, $\partial^2\bar{u}_i/\partial x^2$ are continuous on Q. Furthermore, in the domain Q all derivatives of $\bar{u}$ which appear in the equations obtained by differentiating (4), with $\bar{u} = \bar{u}_i$, four times with respect to x and once with respect to t, are continuous.

To prove (6) for $\bar{u} = \bar{u}_1$ we need the following

LEMMA 1. *The function $\bar{u}_1$ defined as above has the following properties:*

(13) $$\frac{\partial\bar{u}_1(x, t)}{\partial t} \geqslant 0, \quad \frac{\partial^2\bar{u}_1(x, t)}{\partial x^2} \geqslant 0 \quad \textit{for } (x, t) \in Q.$$

(14) *The functions $\partial\bar{u}_1(0, t)/\partial x$ and $-\partial\bar{u}_1(1, t)/\partial x$ are non-decreasing for $t \geqslant 0$.*

(15) *For $x \in \langle 0, 1\rangle$ and $T > 0$,*

$$\left|\frac{\partial\bar{u}_1(x, T)}{\partial x} + \varphi(m) - \varphi(M)\right| \leqslant \frac{K}{T},$$

$$\textit{where } K = \int_0^1 \varphi^{-1}\big((1-x)\varphi(m) + x\varphi(M)\big)\,dx.$$

Proof of (13). Differentiating both sides of equation (4), with $\bar{u} = \bar{u}_1$, with respect to t and substituting

$$v(x,t) = \frac{\partial \bar{u}_1(x,t)}{\partial t},$$

we obtain

$$\frac{\partial v}{\partial t} = \frac{\varphi''(\varphi^{-1}(\bar{u}_1))}{[\varphi'(\varphi^{-1}(\bar{u}_1))]^2} v^2 + \varphi'(\varphi^{-1}(\bar{u}_1)) \frac{\partial^2 v}{\partial x^2}. \tag{16}$$

It follows from (8), (11) and (4) that $v|_\Gamma \geqslant 0$. Now, applying the minimum principle to (16), we obtain $v(x,t) \geqslant 0$ for $(x,t) \in Q$, which gives

$$\frac{\partial^2 \bar{u}_1(x,t)}{\partial x^2} \geqslant 0.$$

Proof of (14). It follows from (13) that $\bar{u}_1(x,t)$ is non-decreasing with respect to t for each $x \in \langle 0,1\rangle$. Hence, for $t > t' > 0$ and $x \in \langle 0,1\rangle$ we have the inequalities

$$\frac{\bar{u}_1(x,t) - \bar{u}_1(0,t)}{x} \geqslant \frac{\bar{u}_1(x,t') - \bar{u}_1(0,t')}{x}$$

and

$$\frac{\bar{u}_1(1,t) - \bar{u}_1(x,t)}{1-x} \leqslant \frac{\bar{u}_1(1,t') - \bar{u}_1(x,t')}{1-x}$$

which prove (14).

Proof of (15). Let $D = \{(x,t)\colon 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T\}$. Then from Green's formula we get

$$\int_{\partial D} u_1 dx + \frac{\partial \bar{u}_1}{\partial x} dt = \iint_D \left(\frac{\partial u_1}{\partial t} - \frac{\partial^2 \bar{u}_1}{\partial x^2}\right) dx dt = 0.$$

Hence

$$\int_0^T \left[\frac{\partial \bar{u}_1(1,t)}{\partial x} - \frac{\partial \bar{u}_1(0,t)}{\partial x}\right] dt = \int_0^1 [u_1(x,T) - u_{0,1}(x)] dx.$$

Since $u_1(x,T) \leqslant \varphi^{-1}\big((1-x)\varphi(m) + x\varphi(M)\big)$ and $u_{0,1} \geqslant 0$, we have

$$\int_0^T \left[\frac{\partial \bar{u}_1(1,t)}{\partial x} - \frac{\partial \bar{u}_1(0,t)}{\partial x}\right] dt \leqslant K.$$

From (13) and (14) it follows that the integrand in this inequality is non-negative and non-increasing. Thus

$$0 \leqslant \frac{\partial \bar{u}_1(1, T)}{\partial x} - \frac{\partial \bar{u}_1(0, T)}{\partial x} \leqslant \frac{K}{T}.$$

The inequality in (15) follows from the last inequality, from

$$\frac{\partial \bar{u}_1(0, T)}{\partial x} \leqslant \frac{\partial \bar{u}_1(x, T)}{\partial x} \leqslant \frac{\partial \bar{u}_1(1, T)}{\partial x}$$

which is valid for $x \in \langle 0, 1 \rangle$, and from

$$\frac{\partial \bar{u}_1(\xi, T)}{\partial x} = \bar{u}_1(1, T) - \bar{u}_1(0, T) = \varphi(M) - \varphi(m)$$

valid for some $\xi \in (0, 1)$.

Let us write the inequality from (15) in the following way:

$$\varphi(M) - \varphi(m) - \frac{K}{t} \leqslant \frac{\partial \bar{u}_1(x, t)}{\partial x} \leqslant \varphi(M) - \varphi(m) + \frac{K}{t} \qquad (t > 0).$$

Now, integrating these inequalities with respect to x over $\langle 0, x \rangle$ and taking into account the condition $\bar{u}_1(0, t) = \varphi(m)$, we obtain

$$\varphi(m) + x\big(\varphi(M) - \varphi(m)\big) - \frac{K}{t} \leqslant \bar{u}_1(x, t) \leqslant \varphi(m) + x\big(\varphi(M) - \varphi(m)\big) + \frac{K}{t}$$

from which (6) follows immediately.

Now we prove (6) for $\bar{u} = \bar{u}_2$. For this purpose let us put

$$w(x, t) = \bar{u}_2(x, t) - (1 - x)\varphi(m) - x\varphi(M).$$

It follows from (4) that $w(x, t)$ satisfies the equation

$$\frac{\partial w}{\partial t} = p(x, t) \frac{\partial^2 w}{\partial x^2}, \tag{17}$$

where $p(x, t) = \varphi'\big(\varphi^{-1}(w + (1 - x)\varphi(m) + x\varphi(M))\big)$.

LEMMA 2. *If the function $r(x, t)$ is a solution of the equation*

$$\frac{\partial r}{\partial t} = q(x) \frac{\partial^2 r}{\partial x^2} \tag{18}$$

with q satisfying the inequalities $p(x, t) \geqslant q(x)$ for $x \in \langle 0, 1 \rangle$, $t \geqslant 0$, and $r|_\Gamma \geqslant w|_\Gamma$, $\partial^2 r / \partial x^2 \leqslant 0$ in Q, then $w(x, t) \leqslant r(x, t)$ for $(x, t) \in Q$.

Proof. Put $h = r - w$. Subtracting (17) from (18), we obtain

$$\frac{\partial h}{\partial t} = p \frac{\partial^2 h}{\partial x^2} + (q - p) \frac{\partial^2 r}{\partial x^2}.$$

Since

$$h|_\Gamma \geqslant 0, \quad p \geqslant 0, \quad (q-p)\frac{\partial^2 r}{\partial x^2} \geqslant 0,$$

it follows from the minimum principle that

$$h(x, t) = r(x, t) - w(x, t) \geqslant 0 \quad \text{for } (x, t) \in Q.$$

LEMMA 3. *The inequality*

$$p(x, t) \geqslant C(x - x^2)$$

holds, where a positive constant C depends only on A,

$$A = \sup_{(x,t)\in Q} \bar{u}_2(x, t).$$

Proof. Due to our assumptions on φ, we can write

$$\frac{z}{\varphi^{-1}(z)} = \frac{\varphi(\varphi^{-1}(z))}{\varphi^{-1}(z)} = \varphi'(\xi) \leqslant \varphi'(\varphi^{-1}(z))$$

with some ξ, $0 < \xi < \varphi^{-1}(z)$. Consequently, we have

$$\varphi'(\varphi^{-1}(z)) \geqslant \frac{z}{\varphi^{-1}(z)}.$$

Hence

$$\begin{aligned} p(x, t) &= \varphi'\big(\varphi^{-1}(w + (1-x)\varphi(m) + x\varphi(M))\big) \\ &\geqslant \frac{w + (1-x)\varphi(m) + x\varphi(M)}{\varphi^{-1}\big(w + (1-x)\varphi(m) + x\varphi(M)\big)} \geqslant \frac{x\varphi(M)}{\varphi^{-1}(A)} \geqslant \frac{\varphi(M)}{\varphi^{-1}(A)}(x - x^2). \end{aligned}$$

Let us consider a special case of Lemmas 2 and 3 in which

$$r(x, t) = E(x - x^2)e^{-2Ct}.$$

The function r satisfies the equation

$$\frac{\partial r}{\partial t} = C(x - x^2)\frac{\partial^2 r}{\partial x^2}.$$

Let E be a real number so large that for $x \in \langle 0, 1\rangle$ the inequality

$$w(x, 0) \leqslant E(x - x^2)$$

holds. The existence of E is guaranteed by (8), (12) and by the definition of w.

It follows from Lemmas 2 and 3 that

$$0 \leqslant w(x, t) \leqslant \frac{E}{4} e^{-2Ct},$$

whence

$$|\bar{u}_2(x,t)-(1-x)\varphi(m)-x\varphi(M)| \leqslant \frac{E}{4}e^{-2Ct}$$

and, consequently, (6) holds for $\bar{u}_2$.

Case (B). Let $\{\bar{u}_0^n(x)\}$ be any sequence of positive functions convergent on $\langle 0,1\rangle$ to $\bar{u}_0(x)$.

We assume that for any n $(n = 1, 2, \ldots)$

$$\bar{u}_0^n \geqslant \bar{u}_0^{n+1}, \quad \bar{u}_0^n \in C^\infty(\langle 0,1\rangle), \quad |(\bar{u}_0^n)'| \leqslant 2B, \quad (\bar{u}_0^n)''(0) = (\bar{u}_0^n)''(1) = 0,$$

B being the Lipschitz constant for the function $\bar{u}_0$.

Let $\bar{u}^n(x,t)$ $(n = 1, 2, \ldots)$ be a solution of equation (4) for which

$$\bar{u}^n(x,0) = \bar{u}_0^n(x), \quad \bar{u}^n(0,t) = \bar{u}_0^n(0) = \varphi(m_n),$$
$$\bar{u}^n(1,t) = \bar{u}_0^n(1) = \varphi(M_n) \quad \text{for } x \in \langle 0,1\rangle,\ t \geqslant 0.$$

In a way similar to that in the proof of Theorem 4 in [1] we may show that the limit

$$u(x,t) = \lim_{n\to\infty} \varphi^{-1}(\bar{u}^n(x,t))$$

exists and that it is a weak solution of problem (1), (2).

From the proof of our theorem for case (A) it follows immediately that the convergence

$$\lim_{t\to\infty} u^n(x,t) = \varphi^{-1}\big((1-x)\varphi(m_n)+x\varphi(M_n)\big) \qquad (u^n = \varphi^{-1}(\bar{u}^n))$$

is uniform with respect to n, which means that for any $\varepsilon > 0$ there exists a $T > 0$ such that

$$\big|u^n(x,t)-\varphi^{-1}\big((1-x)\varphi(m_n)-x\varphi(M_n)\big)\big| < \varepsilon$$

for $x \in \langle 0,1\rangle$, $t > T$ and $n = 1, 2, \ldots$ This inequality implies

$$\lim_{t\to\infty} u(x,t) = \varphi^{-1}\big((1-x)\varphi(m)+x\varphi(M)\big) \quad \text{for } x \in \langle 0,1\rangle,$$

which completes the proof of the Theorem.

REFERENCES

[1] О. А. Олейник, А. С. Калашников и Чжоу Юй-Линь, *Задача Коши и краевые задачи для уравнений типа нестационарной фильтрации*, Известия Академии наук СССР, серия математическая, 22 (5) (1958), p. 667-704.

[2] Н. В. Хуснутдинова, *О поведении решений краевых задач и задачи Коши для уравнения типа нестационарной фильтрации при неограниченном возрастании времени*, Труды Казанского авиационного института 64 (1961), p. 47-63.

[3] — *О проведении решений задачи Коши и краевых задач для некоторых квазилинейных уравнений при неограниченном возрастании времени*, in: Материалы 1-й конференции молодых научных работников, г. Казань, Физико-техническая и математическая секция, Казань 1959.

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