COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 1

HEREDITARILY σ-CONNECTED CONTINUA

BY

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1. Preliminaries and definitions. A topological space is said to be *σ-connected* (respectively, *weakly σ-connected*) if it is not the union of countably many mutually disjoint closed (respectively, closed connected) non-empty sets. A topological space is said to be *hereditarily σ-connected* (respectively, *hereditarily weakly σ-connected*) if every connected subset is σ-connected (respectively, weakly σ-connected). Sierpiński has proved that every compact, connected Hausdorff space is σ-connected. By a *continuum* we mean a compact connected metric space. A continuum X is said to be *hereditarily locally connected* if every subcontinuum of X is locally connected. A continuum X is said to be *finitely Suslinian* if every sequence of pairwise disjoint subcontinua of X forms a null sequence. It has been proved in [3], Theorem 2.2, that for hereditarily locally connected continua the three concepts — hereditarily σ-connected, hereditarily weakly σ-connected and finitely Suslinian — coincide. A continuum X is said to be *Suslinian* if every collection of pairwise disjoint subcontinua of X is countable. We let ω denote the first infinite ordinal. We let $\mathrm{Cl}(U)$ denote the closure of the set U and let $\mathrm{Bd}(U)$ denote the boundary of U. If (X, ϱ) is a metric space, $U \subset X$ and $\varepsilon > 0$, we let $S(U, \varepsilon)$ be the ε-neighborhood of U.

In Section 2, it is proved that a hereditarily weakly σ-connected continuum is hereditarily decomposable and a locally connected hereditarily weakly σ-connected continuum is Suslinian. An example also is provided to show that hereditarily σ-connected non-locally connected continua are not necessarily Suslinian.

In Section 3, it is proved that an arc-like continuum is hereditarily weakly σ-connected if and only if it is hereditarily decomposable. An arc-like continuum that is hereditarily σ-connected is Suslinian.

Finally, some questions are raised.

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* This research was supported in part by National Research Council (Canada), grant No. A5616.

2. General continua.

2.1. PROPOSITION. *Every hereditarily weakly σ-connected continuum is hereditarily decomposable.*

Proof. Let X be an indecomposable metric continuum. Let $C_1, C_2, \ldots$ be a sequence of disjoint composants of X. For each $i = 1, 2, \ldots$ let A_i be a finite set in C_i such that $X \subset S(A_i, i^{-1})$. For each i let B_i be a continuum in C_i which contains A_i. Let

$$K = \bigcup_{i=1}^{\infty} B_i.$$

Then we claim that K is connected. To prove this just suppose that $K = A \cup B$, where A and B are separated sets. Without loss of generality, we can assume that A contains infinitely many of the continua B_i. Since $A_i \subset B_i$ and $X \subset S(A_i, i^{-1})$, it follows that A is dense in X. Hence, B is empty and K is connected. However, K is not weakly σ-connected. This completes the proof of Proposition 2.1.

In Section 3 we shall prove that the converse of Proposition 2.1 is true for arc-like continua.

The proof of the next theorem is based on a technique used by the first-named author and A. Lelek to show that the cone over the Cantor set is not hereditarily weakly σ-connected (unpublished).

2.2. THEOREM. *A locally connected, hereditarily weakly σ-connected continuum is Suslinian.*

Proof. Let X be a locally connected continuum which is not Suslinian. Let $\mathbf{C}$ be an uncountable family of pairwise disjoint non-degenerate continua in X. Since the hyperspace of subcontinua of X is a separable metric space, we may assume that $\mathbf{C}$ is a continuous family (see [1], 2.1). Let d be a metric on X.

Let $C \in \mathbf{C}$, $x \in C$ and let U be a connected neighborhood of x such that $C \not\subset \mathrm{Cl}(U)$. We may suppose, without loss of generality, that $U \cap D \neq \emptyset$ for each $D \in \mathbf{C}$. Let $y \in C \setminus \mathrm{Cl}(U)$. Without loss of generality, we may suppose that $d(y, U) = 1$. Let $(C_n)_{n\in\omega}$ be a sequence of distinct elements of $\mathbf{C} \setminus \{C\}$ which converges to C. For each $n \in \omega$ let $y_n \in C_n \setminus S(U, 2^{-1})$ be such that

$$\lim_{n\to\infty} y_n = y.$$

For each $n \in \omega$ let K_n be the component of $C_n \setminus S(U, 2^{-1})$ which contains y_n.

Let r be a positive integer and suppose that $C_{n_1,\ldots,n_k}$, $y_{n_1,\ldots,n}$, and $K_{n_1,\ldots,n_k}$ have been defined for each $k = 1, \ldots, r$ and $(n_1, \ldots, n_k) \in \omega^k$ so that

(i) for each $k = 1, \ldots, r-1$ and each $(n_1, \ldots, n_k) \in \omega^k$, $(C_{n_1,\ldots,n_k,i})_{i\in\omega}$ is a sequence in C which converges to $C_{n_1,\ldots,n_k}$;

(ii) $C_{n_1,\ldots,n_k} \neq C_{m_1,\ldots,m_j}$ for $(n_1, \ldots, n_k) \neq (m_1, \ldots, m_j)$;

(iii) $y_{n_1,\ldots,n_k,i} \in C_{n_1,\ldots,n_k,i} \setminus S(U, 2^{-k-1})$ is such that

$$\lim_{i\to\infty} y_{n_1,\ldots,n_k,i} = y_{n_1,\ldots,n_k};$$

(iv) $K_{n_1,\ldots,n_k}$ is the component of $C_{n_1,\ldots,n_k} \setminus S(U, 2^{-k})$ which contains $y_{n_1,\ldots,n_k}$.

For each $(n_1, \ldots, n_r) \in \omega^r$ let $(C_{n_1,\ldots,n_r,i})_{i\in\omega}$ be a sequence of distinct elements of C such that

$$\lim_{i\to\infty} C_{n_1,\ldots,n_r,i} = C_{n_1,\ldots,n_r}$$

and $C_{n_1,\ldots,n_p} \neq C_{m_1,\ldots,m_q}$ for $(n_1, \ldots, n_p) \neq (m_1, \ldots, m_q)$ and $1 \leqslant p \leqslant n+1$, $1 \leqslant q \leqslant n+1$.

Let $y_{n_1,\ldots,n_r,i} \in C_{n_1,\ldots,n_r,i} \setminus S(U, 2^{-n-1})$ be such that

$$\lim_{i\to\infty} y_{n_1,\ldots,n_r,i} = y_{n_1,\ldots,n_r}.$$

Let $K_{n_1,\ldots,n_r,i}$ be the component of $C_{n_1,\ldots,n_r,i} \setminus S(U, 2^{-r-1})$ which contains $y_{n_1,\ldots,n_r,i}$. By induction, $K_{n_1,\ldots,n_r}$ is defined for each $r = 1, 2, \ldots$ and for each $(n_1, \ldots, n_r) \in \omega^r$. Let

$$P = \mathrm{Cl}(U) \cup \bigcup \{K_{n_1,\ldots,n_r} \mid r = 1, 2, \ldots \text{ and } (n_1, \ldots, n_r) \in \omega^r\}.$$

Then P is easily seen to be a connected F_σ-subset which is not weakly σ-connected. This completes the proof of Theorem 2.2.

2.3. Remarks. The proof of Theorem 2.2 is valid for semi-aposyndetic continua.

One may ask whether the conclusion in 2.2 can be strengthened. We say that a continuum is *rational* provided it has a basis consisting of open sets with countable boundaries. It is known that hereditarily locally connected continua are rational and rational continua are Suslinian. The following example shows that locally connected hereditarily σ-connected continua are not necessarily hereditarily locally connected.

2.4. Example. By (x, y) we denote the point of the Euclidean space $\boldsymbol{R}^2$ with Cartesian coordinates x and y. Let

$$A_n = \left\{(x, y):\ 0 \leqslant x \leqslant \frac{1}{2^n},\ y = -2^n x + 1\right\}$$

for each $n = 1, 2, \ldots$ and

$$A_0 = \{(0, y):\ 0 \leqslant y \leqslant 1\}.$$

For each dyadic rational $m/2^n$ with $0 < m < 2^n$, let

$$B_{m,n} = \left\{(x, y)\colon\ 0 \leqslant x \leqslant \frac{2^n - m}{2^{2n}},\ y = \frac{m}{2^n}\right\}.$$

Consider now the set

$$X = \bigcup_{n=0}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^n-1} B_{m,n}.$$

Then X, as a subspace of $\boldsymbol{R}^2$, is a locally connected hereditarily σ-connected continuum which is not hereditarily locally connected.

One could ask the following question:

PROBLEM I (**P 1067**). Is every locally connected hereditarily weakly σ-connected (respectively, hereditarily σ-connected) continuum rational?

The following example shows that the local connectedness in the hypothesis of Theorem 2.2 is essential.

2.5. Example. This is an example of a non-locally connected hereditarily σ-connected continuum X which is not Suslinian.

We first construct a dendrite D in the xy-hyperplane of $\boldsymbol{R}^3$ as follows. Let

$$C_0 = [0, 1] \times \{0\} \times \{0\}.$$

Let A_0 be the union of two line segments from the endpoints of C_0 to the point $(\frac{1}{2}, \frac{1}{2}, 0)$. Let C_1 be obtained from C_0 by removing from C_0 the open middle third. Then

$$C_1 = \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) \times \{0\} \times \{0\}.$$

Let A_1 consist of A_0 together with the line segment with slope -1 and length $1/3\sqrt{2}$ from $(\frac{1}{3}, 0, 0)$ to A_0 and with the line segment with slope 1 and length $1/3\sqrt{2}$ from $(\frac{2}{3}, 0, 0)$ to A_0.

Let n be a natural number and suppose that $A_0 \subset A_1 \subset \ldots \subset A_n$ have been constructed to be dendrites. Suppose that $C_n \subset C_{n-1} \subset \ldots \subset C_0$, where C_n consists of 2^n disjoint closed intervals each of length $1/3^n$. Let C_{n+1} be obtained from C_n by removing from each component of C_n the open middle third. Then C_{n+1} consists of 2^{n+1} disjoint closed intervals each of length $1/3^{n+1}$. Let A_{n+1} consist of A_n together with 2^{n+1} intervals each of length $1/3^{n+1}\sqrt{2}$ which are obtained as follows:

If x is the left-hand (respectively, right-hand) endpoint of a maximal interval in $[0, 1]$ which was removed from C_n to form C_{n+1}, join x to A_n by an irreducible line segment with slope -1 (respectively, slope 1).

By induction, C_n and A_n are defined for each natural number n. Then

$$C = \bigcap_{n=0}^{\infty} C_n$$

is the Cantor ternary set. Let

$$D = C \cup \bigcup_{n=0}^{\infty} A_n.$$

Then D is obviously a dendrite.

To construct the continuum X consider the set $D \setminus C$ as an open subset of D, and the compact set

$$K = \bigcup \{(x, 0, z)\colon \ -1 \leqslant z \leqslant 1, (x, 0, 0) \in C\}.$$

Let (X, h) be a compactification of $D \setminus C$ such that the remainder $X \setminus h(D \setminus C)$ is exactly the set K and X admits a map f onto D such that $f | X \setminus K$ is h^{-1} and $f(x, 0, z) = (x, 0, 0)$ for $z \in [-1, 1]$. Then X is a non-locally connected, hereditarily decomposable but not Suslinian continuum. We show that X is hereditarily σ-connected. Suppose that F is a connected subset of X. Since D is a dendrite, it is also a finitely Suslinian continuum. Therefore, by [3], Theorem 2.2, D is hereditarily σ-connected. Thus,

$$F \cap K \neq \emptyset \neq F \cap h(D \setminus C).$$

Let

$$F = \bigcup_{i=1}^{\infty} F_i$$

be a decomposition of F into mutually disjoint closed subsets of F. Note that $F \setminus K$ is connected. Since $D \setminus C$ is hereditarily σ-connected,

$$F \cap h(D \setminus C) \subset F_n \quad \text{for some } n \in \omega.$$

By the construction of X, we also have

$$F \setminus F_n \subset \mathrm{Cl}\big(F \cap h(D \setminus C)\big) \subset \mathrm{Cl}(F_n).$$

Since F_n is closed in F, $F_i = \emptyset$ for $i \neq n$. This proves that X is hereditarily σ-connected.

3. Irreducible continua. A continuum is said to be *arc-like* if it has open coverings of sets of arbitrarily small diameters such that the nerves of these coverings are arcs. An irreducible continuum is said to be of *type* λ if it admits a continuous mapping onto $[0, 1]$ such that point inverses under this mapping are continua with void interiors.

3.1. THEOREM. *A connected dense subset of an irreducible continuum of type λ is weakly σ-connected.*

Proof. Let X be an irreducible continuum of type λ. Let C be a connected dense subset of X. Let $\pi\colon X \to [0,1]$ be a continuous mapping of X onto $[0,1]$ such that the point inverses under π are nowhere dense continua in X.

Suppose that

$$C = \bigcup_{i=1}^{\infty} C_i,$$

where the C_i are pairwise disjoint, connected, closed subsets of C. Since C is connected, $\pi(C_i)$ is connected for each $i = 1, 2, \ldots$ Suppose that $\pi(C_i)$ contains (a, b), where $a < b$. Since $\mathrm{Cl}(C_i)$ meets both $\pi^{-1}(a)$ and $\pi^{-1}(b)$ and X is irreducible,

$$\mathrm{Cl}\big(\pi^{-1}((a, b))\big) \subset \mathrm{Cl}(C_i).$$

Since $\pi^{-1}\big((a, b)\big)$ is open in X,

$$\pi^{-1}\big((a, b)\big) \cap C \subset C_i$$

and

$$(*) \qquad \mathrm{Cl}\big(\pi^{-1}((a, b)) \cap C_i\big) = \mathrm{Cl}\big(\pi^{-1}((a, b))\big).$$

If $0 < a < 1$, then $\pi^{-1}(a) \cap C \neq \emptyset$, for otherwise C could be written as the union of the separated sets $C \cap \pi^{-1}([0, a])$ and $C \cap \pi^{-1}\big((a, 1]\big)$. Thus $\pi(C_i) \cap (0, 1)$ is closed in $(0, 1)$.

Just suppose that there exist $b < a < c$ and $i, j \in \{1, 2, \ldots\}$ such that $(b, a] \subset \pi(C_i)$ and $[a, c) \subset \pi(C_j)$. Since $\pi^{-1}(a)$ is nowhere dense,

$$X = \mathrm{Cl}\big(\pi^{-1}((0, a))\big) \cup \mathrm{Cl}\big(\pi^{-1}((a, 1))\big).$$

Therefore, there exists

$$x \in \mathrm{Cl}\big(\pi^{-1}((b, a))\big) \cap \mathrm{Cl}\big(\pi^{-1}((a, c))\big) \cap C.$$

So $x \in \mathrm{Cl}(C_i) \cap \mathrm{Cl}(C_j) \cap C$. Hence $x \in C_i \cap C_j$ and $C_i = C_j$.

Let

$$K = \{x \in (0, 1) \mid x \text{ is not an interior point of } \pi(C_j) \text{ for any } j\}.$$

Then K is closed in $(0, 1)$. Hence K has no isolated points. Since K is contained in the set of endpoints of the countable family of intervals $\pi(C_i)$, K is countable. Hence K is empty. By $(*)$, if $i \neq j$, then $\pi(C_i) \cap \pi(C_j)$ contains at most one point and that is a common endpoint of the intervals $\pi(C_i)$ and $\pi(C_j)$. Thus all but one of the sets C_i are empty and C is weakly σ-connected.

The following corollary provides a converse to Proposition 2.1 in case where the continuum X is arc-like.

3.2. Corollary. *An arc-like continuum is hereditarily decomposable if and only if it is hereditarily weakly σ-connected.*

Proof. The sufficiency was proved in 2.1. The necessity follows from 3.1 and from the fact that every subcontinuum of an arc-like continuum is arc-like and every hereditarily decomposable arc-like continuum is an irreducible continuum of type λ (see [4], p. 216).

3.3. Remark. We give an example to show that the hypothesis in Theorem 3.1 that X be irreducible is essential. Let X be the closure of the subset of E^2 which is described on p. 175 of [4]. Then X is a rational dendroid that is not hereditarily weakly σ-connected.

Let X be an arc-like hereditarily decomposable continuum and let $x \in X$. Then there exists a finest monotone mapping of X onto $[0, 1]$ (see [4], p. 199). The inverse images of points under this mapping are called the *tranches* of X. We denote by $T(X, x)$ the tranche of X containing x. We write $T^0(X, x) = X$, and we use a transfinite induction to define $T^a(X, x)$ for each ordinal a, namely

$$T^{a+1}(X, x) = T\big(T^a(X, x)\big)$$

and

$$T^{\lambda}(X, x) = \bigcap_{a<\lambda} T^a(X, x)$$

for each limit ordinal λ. The set $T^a(X, x)$ is called the *tranche of order* a of X at the point x (see [2], p. 172).

3.4. THEOREM. *Arc-like hereditarily σ-connected continua are Suslinian.*

Proof. Let X be an arc-like hereditarily σ-connected continuum. By 2.1, X is hereditarily decomposable. Suppose, on the contrary, that X is not Suslinian. By [4], p. 216, X is an irreducible continuum of type λ. Let $\varphi: X \to [0, 1]$ be a finest monotone mapping of X onto $[0, 1]$ (see [4], p. 199).

It is known (see [2], p. 172) that if $x \in X$, then there exists a countable ordinal a such that $T^a(X, x) = \{x\}$. Consequently, there exists an ordinal $\beta < a$ such that $T^{\beta}(X, x)$ contains uncountably many non-degenerate tranches for some $x \in X$, since X is not Suslinian. Therefore, we may assume, without loss of generality, that X has uncountably many non-degenerate tranches. Let $(K_c)_{c\in C}$ be an uncountable collection of non-degenerate tranches of X with diameters greater than or equal to some $\varepsilon > 0$. Without loss of generality, assume that this collection is continuous and the set C indexing the collection $K_c)_{c\in C}$ is the Cantor ternary set on the interval $[0, 1]$ and $\varphi^{-1}(c) = K_c$ for $c \in C$ (see [1], 2.1).

Consider now for each positive integer i a subinterval $[a_i, b_i]$ of $[0, 1]$ with the following properties:

(i) $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for $i \neq j$ $(i, j = 1, 2, \ldots)$;

(ii) the set $\{a_1, b_1, a_2, b_2, \ldots\}$ is a dense subset of C;

(iii) for each $i = 1, 2, \ldots,$

$$\mathrm{Bd}\big(\varphi^{-1}([0, a_i))\big) = \varphi^{-1}(a_i) = \mathrm{Bd}\big(\varphi^{-1}((a_i, 1])\big),$$
$$\mathrm{Bd}\big(\varphi^{-1}([0, b_i))\big) = \varphi^{-1}(b_i) = \mathrm{Bd}\big(\varphi^{-1}((b_i, 1])\big).$$

This is possible by [5], III.2.1. Since each K_c is a frontier set of X (see [4], p. 201), there exists a perfect set $P \subset X$ such that

$$\emptyset \neq P \cap K_i \underset{\neq}{\subset} K_c \quad \text{for each } c \in C.$$

Let $p_i \in \varphi^{-1}(a_i) \setminus P$ and $q_i \in \varphi^{-1}(b_i) \setminus P$. Now consider the set

$$C = \big(P \setminus \bigcup_{i=1}^{\infty} \varphi^{-1}([a_i, b_i])\big) \cup \bigcup_{i=1}^{\infty} \big(\varphi^{-1}((a_i, b_i)) \cup \{p_i, q_i\}\big).$$

Then C is a connected but not σ-connected subset of X. To show that C is connected, just suppose that $C = A \cup B$, where A and B are non-empty separated subsets of C. It is clear that C is dense in $\varphi^{-1}([0,1])$, $\varphi(C) = [0,1]$, and that for each i either $\varphi^{-1}\big((a_i, b_i)\big) \subset A$ or $\varphi^{-1}\big((a_i, b_i)\big) \subset B$.

Let

$$t \in [0,1] \setminus \bigcup_{i=1}^{\infty} [a_i, b_i],$$

and let $a_{i_1}, a_{i_2}, \ldots$ be a subsequence of a_i converging to t. Without loss of generality, assume that $\varphi^{-1}\big((a_{i_j}, b_{i_j})\big) \subset A$ for each $j \in \omega$. Then, since A is closed in C, and since

$$\varphi^{-1}(t) \subset \operatorname*{Lim\,sup}_{j \to \infty} \varphi^{-1}\big((a_{i_j}, b_{i_j})\big),$$

it is clear that $\varphi^{-1}(t) \cap C \subset A$. Hence, $\varphi(A)$ and $\varphi(B)$ are separated subsets of $[0,1]$ such that $\varphi(A) \cup \varphi(B) = [0,1]$. By the connectedness of $[0,1]$, we deduce that either $A = \emptyset$ or $B = \emptyset$. This contradiction completes the proof of Theorem 3.4.

3.5. Remark. Theorem 3.4 can be generalized to atriodic hereditarily σ-connected continua, since Theorem 3 on p. 216 of [4] has been generalized to the case of hereditarily decomposable atriodic continua (see [2], Theorem 2.1).

It is reasonable to ask whether Theorem 3.4 can be strengthened to give a characterization of arc-like hereditarily σ-connected continua analogous to that given in 3.1 for hereditarily weakly σ-connected continua. In particular, we have the following questions:

PROBLEM II (**P 1068**). Are arc-like hereditarily σ-connected continua always rational?

PROBLEM III (**P 1069**). Are Suslinian (respectively, rational) arc-like continua always hereditarily σ-connected?

A positive answer to Problem III would imply a negative solution to Problem II.

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Reçu par la Rédaction le 26. 2. 1977