Nodes of eigenfunctions of a certain class of ordinary differential equations of the fourth order

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Abstract. In this paper we shall consider the problem of eigenvalues and eigenfunctions for a certain class of ordinary differential equations of the fourth order with boundary conditions of the first kind. We prove some theorems about multiplicity of zero-points of eigenfunctions and about multiplicity of eigenvalues. Finally we prove the main theorem: The n-th eigenfunction of this problem has in interval \((a, b)\) exactly \(n - 1\) zero-points.

Introduction. Let \(M\) be the set of functions of class \(C^4([a, b])\) satisfying the following boundary conditions:

\[(1) \quad u(a) = u'(a) = u(b) = u'(b) = 0.\]

Consider in \(M\) a differential operator of the fourth order of the form

\[(2) \quad Lu = L_1 L_2 u,\]

where \(L_k \varphi (k = 1, 2)\) denotes the following differential operators of the Sturm–Liouville type, i.e.

\[(3) \quad L_k \varphi = -[p_k(x)\varphi']' + q_k(x)\varphi \quad (k = 1, 2).\]

Let us assume that \(p_k(x) \in C^{k-1}([a, b]), \ q_k(x) \in C^{k-2}([a, b]), \ p_k(x) > 0, \ q_k(x) \geq 0 \) for \(x \in [a, b]\).

We shall consider the eigenvalues and eigenfunctions for the differential equation

\[(4) \quad Lu - \mu \varphi(x)u = 0\]

with boundary conditions (1), where \(Lu\) is defined by (2), \(\mu\) is a real parameter, and \(\varphi(x) > 0\) is a continuous function in \([a, b]\).

Definition. We say that a real number \(\lambda\) is an eigenvalue of problem (4), (1) if there exists a function \(u(x) \neq 0\) on \(M\) such that (4) holds for \(\mu = \lambda\), \(u(x)\) is called an eigenfunction corresponding to the eigenvalue \(\lambda\).

The present paper deals with some properties of eigenvalues and eigenfunctions of problem (4), (1).

Our main theorem reads as follows:
Under some assumptions about the operator $L$ in equation (4), the $n$-th eigenfunction of problem (4), (1) has in interval $(a, b)$ exactly $n-1$ zero-points (see Theorem 6).

1. The auxiliary lemmas and theorems. Let $f(x)$ be a continuous function in $[a, b]$. Put

$$A = \{ x : x \in (a, b); f(x) = 0 \} \quad \text{(cf. [5])}.$$ 

$A$ is the sum of separate intervals closed in $(a, b)$. Let $M$ denote the set of all subintervals in $A$. $M = M_1 + M_2$, where $M_1$ contains subintervals reduced to a single point or such, which are not closed in $[a, b]$, $M_2 = M - M_1$. Let us put

$$Z(f) = \begin{cases} l + 2k & \text{if } M_1 \text{ and } M_2 \text{ are finite sets,} \\ +\infty & \text{if } M_1 \text{ or } M_2 \text{ is an infinite set,} \end{cases}$$

where $l$ and $k$ represent the powers of $M_1$ and $M_2$ respectively. If $f(x)$ has a finite number of isolated zero-points in interval $(a, b)$, then $Z(f)$ denotes this number.

**Lemma 1.** If $f(x) \in \mathbb{R}$ has only a finite number of isolated zero-points in $(a, b)$ equal to $s$ (with multiplicity at most 4), then the number of zero-points of function $h(x) = Lf(x)$ is equal at least to $(s+r)$, where $r$ denotes the number of zero-points of the function $f(x)$ in $(a, b)$ with multiplicity greater than one.

**Proof.** Let $a = x_0 < x_1 < \ldots < x_s < x_{s+1} = b$ denote zero-points of $f(x)$ in $[a, b]$. Put $g(x) = L_2f(x)$. Since the multiplicity of $x_0 = a$ and $x_{s+1} = b$ is at least 2 then, by Lemma 2.3 and Theorem 3.1 of [5], we have $Z(g) \geq s + r + 2$. Let $t_1, t_2, \ldots, t_{s+r+2}$ be zero-points of $g(x)$ in $(a, b)$, where $a < t_1 < t_2 < \ldots < t_{s+r+2} < b$. By Lemma 2.4 of [5] applied to $g(x)$ in the interval $[t_1, t_{s+r+2}]$, we have that $h(x) = L_1g(x)$ has at least $s + r$ zero-points in $(t_1, t_2)$. On the other hand, we have $h(x) = L_1g(x) = L_1L_2f(x) = Lf(x)$ and $(t_1, t_{s+r+2}) \subset (a, b)$, thus $h(x) = Lf(x)$ has at least $s + r$ zero-points in $(a, b)$, which completes the proof.

**Lemma 2.** If $f(x)$ satisfies the assumptions of Lemma 1 and the multiplicity of $a$ or $b$ is 3 or 4, then $Z(Lf) \geq s + r + 1$ ($s, r$ as in Lemma 1).

**Proof.** Let $a = x_0 < x_1 < \ldots < x_s < x_{s+1} = b$ be zero-points of $f(x)$ in $[a, b]$. Write $g(x) = L_2f(x)$. Suppose that the multiplicity of $a$ is 3 or 4. Of course $a$ is also a zero-point of $g(x)$. By Lemma 1 $g(x)$ has at least $s + r + 2$ zero-points in $(a, b)$, and $g(a) = 0$. Applying Lemma 2.4 of [5] to $g(x)$ in the interval $[a, t_{s+r+2}]$, we have that $h(x) = Lf(x) = L_1g(x)$ has at least $s + r + 1$ zero-point in $(a, t_{s+r+2})$. By reasoning as in Lemma 1, we have $Z(h) \geq s + r + 1$. The proof is analogous for $b$ of multiplicity of 3 or 4.
Lemma 3. If \( f(x) \in \mathcal{M} \), then \( Z(Lf) \geq Z(f) \).

The proof of this lemma is quite similar to that of an analogous lemma in [5], and is omitted.

2. Multiplicity of zero-points of eigenfunctions and multiplicity of eigenvalues of problem (4), (1). Let \( u(x) \) be an eigenfunction of problem (4), (1). By the well-known theorem on the uniqueness of solution of Cauchy's problem for equation (4), we have that \( u(x) \) has no zero-point of multiplicity greater than 3 in \([a, b]\), and that the number of zero-points of \( u(x) \) in \([a, b]\) is finite.

We shall prove the following:

Theorem 1. If \( u(x) \) is an eigenfunction of problem (4), (1) corresponding to an eigenvalue \( \lambda \neq 0 \), then all zero-points of \( u(x) \) in \((a, b)\) are single.

Proof. Suppose \( x_0 \in (a, b) \) is a zero-point of \( u(x) \) with multiplicity greater than one. By Lemma 1 we get \( Z(Lu) \geq Z(u) + 1 \). On the other hand, according to (4), we have \( Lu(x) = \lambda \varphi(x)u(x) \). Since \( \varphi(x) > 0 \) in \([a, b]\) and \( \lambda \neq 0 \), \( Z(Lu) = Z(u) - a \) contradiction.

Theorem 2. Under the assumptions of Theorem 1, the points \( a \) and \( b \) are the zero-points of \( u(x) \) with multiplicity equal to 2.

The proof of the above theorem follows from Lemma 2, the same as Theorem 1 follows from Lemma 1.

Theorem 3. Any two eigenfunctions of problem (4), (1) corresponding to the same eigenvalue \( \lambda \neq 0 \), are linearly dependent.

Proof. Let \( u(x) \) and \( v(x) \) be two eigenfunctions of problem (4), (1) corresponding to an eigenvalue \( \lambda \neq 0 \). Suppose \( u(x) \) and \( v(x) \) are linearly independent in \([a, b]\). Put \( w(x) = u(x) - au(x) \), where \( a = u''(a)/v''(a) \). Suppose \( w(x) \neq 0 \) in \([a, b]\), and \( w(a) = w'(a) = w''(a) = 0 \). Hence \( w(x) \in \mathcal{M} \), and satisfies (4) for \( \mu = \lambda \). This means that \( w(x) \) is an eigenfunction of problem (4), (1) corresponding to a eigenvalue \( \lambda \neq 0 \), which has at the point \( a \) a zero-point with multiplicity at least equal 3. We obtain a contradiction with Theorem 2.

Theorem 3 implies

Corollary 1. Every non-zero eigenvalue of problem (4), (1) is a single eigenvalue.

In the sequel we shall need the following lemmas.

Lemma 4. Suppose \( \{f_n(x)\} \) is a sequence of functions such that:

1° \( f_n(x) \in \mathcal{M} \);
2° \( \{f_n(x)\}, \{f'_n(x)\} \) and \( \{f''_n(x)\} \) tend uniformly to \( f(x), f'(x) \) and \( f''(x) \) respectively;
3° \( f(x) \in \mathcal{M} \) and \( f(x) \) has \( p \) zero-points in \((a, b)\), \( p < \infty \), which are not zero-points of \( f'(x) \) and \( f''(a)f''(b) \neq 0 \).
Then the number of zero-point in \((a, b)\) for every \(f_n(x)\), for sufficiently large \(n\), is equal to \(p\).

The proof is similar to the analogous lemma in [1], and is omitted.

**Lemma 5** (cf. [5]). Let \(\{f_n(x)\}\) be a sequence of continuous functions in \([a, b]\) tending uniformly in \([a, b]\) to the function \(f(x)\) of class \(C^1([a, b])\), and let \(f(x)\) have a finite number of single zero-points in \((a, b)\). Then for sufficiently large \(n\) we have \(Z(f_n) \geq Z(f)\).

Let \(u_m(x), u_{m+1}(x), \ldots, u_n(x)\) \((n \geq m)\) be eigenfunctions of problem (4), (1) corresponding to eigenvalues \(\lambda_m, \lambda_{m+1}, \ldots, \lambda_n\), where \(0 < |\lambda_m| < |\lambda_{m+1}| < \ldots < |\lambda_n|\). Put

\[
f(x) = c_m u_m(x) + c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x),
\]

\(c_m, c_{m+1}, \ldots, c_n\) being real constants such that \(c_m^2 + c_{m+1}^2 + \ldots + c_n^2 > 0\) and \(c_m c_n \neq 0\).

Analogously as in [1] we shall prove the following:

**Theorem 4.** If \(f(x)\) denotes the function defined by (5), then we have

\[
Z(u_m) \leq Z(f) \leq Z(u_n).
\]

**Proof.** It follows from (5) that \(f(x)\) satisfies the assumptions of Lemma 3. Let us put

\[
f_1(x) = \frac{1}{\lambda_n \varepsilon(x)} \int f(x) = \frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x).
\]

By Lemma 3, we get \(Z(f_1) \geq Z(f)\). \(f_1(x)\) has the same form as \(f(x)\) and satisfies the assumptions of Lemma 3. Applying again Lemma 3 to \(f_1(x)\), we have for

\[
f_2(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^2 c_m u_m(x) + \ldots + c_n u_n(x),
\]

the inequality \(Z(f_2) \geq Z(f_1)\).

Proceeding thus further we get an infinite sequence of functions

\[
f(x), f_1(x), f_2(x), \ldots,
\]

for which the number of zero-points does not decrease with the increase of the index.

Put

\[
f_r(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^r c_m u_m(x) + \left(\frac{\lambda_{m+1}}{\lambda_n}\right)^r c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x).
\]

Since \(|\lambda_s/\lambda_n| < 1\) \((s = m, m+1, \ldots, n-1)\) and \(u_s(x), u'_s(x)\) and \(u''_s(x)\) \((s = m, m+1, \ldots, n-1)\) are bounded in \([a, b]\), we find that (7) and its
limit $c_n u_n(x)$ (for $n \to +\infty$) satisfy the assumptions of Lemma 4. Hence

\begin{equation}
Z(f) \leq Z(u_n).
\end{equation}

The inequality

\begin{equation}
Z(u_m) \leq Z(f)
\end{equation}
is proved in a similar way by putting

\[ g_1(x) = c_m u_m(x) + \frac{\lambda_m}{c_{m+1}} c_{m+1} u_{m+1}(x) + \ldots + \frac{\lambda_m}{\lambda_n} c_n u_n(x). \]

We verify that $Lg_1(x) = \lambda_m g(x) f(x)$. Because $g_1(x)$ satisfies the assumptions of Lemma 3, $Z(g_1) \leq Z(f)$. Similarly we construct the functions

\begin{equation}
g_\nu(x) = c_m u_m(x) + \left( \frac{\lambda_m}{\lambda_{m+1}} \right)^\nu c_{m+1} u_{m+1}(x) + \ldots + \left( \frac{\lambda_m}{\lambda_n} \right)^\nu c_n u_n(x),
\end{equation}

such that

\[ Lg_\nu(x) = \lambda_m g(x) g_{\nu-1}(x), \quad \nu = 2, 3, \ldots \]

By reasoning similarly as in the proof of (9) we get inequality (10), considering that for the functions of sequence (11) the number of each of their zero-points does not increase with the increase of the index. But inequalities (9) and (10) are just the thesis of Theorem 4.

**Theorem 5.** Let $u(x)$ and $v(x)$ be eigenfunctions of problem (4), (1) corresponding to the eigenvalues $\lambda$ and $\mu$ respectively, where $|\lambda| < |\mu|$. Then we have the inequality

\begin{equation}
Z(u) < Z(v).
\end{equation}

**Proof.** By Theorem 4 we have $Z(u) \leq Z(v)$. Suppose that $Z(u) = Z(v)$, and let $w(x) = u(x) - \gamma v(x)$, where $\gamma = u''(a)/v''(a)$. By definition of $w(x)$ we get $w(a) = w'(a) = w''(a) = 0$. By Theorem 4 we have that $Z(w) = Z(u) = Z(v)$. Since $a$ is a zero-point of $w(x)$ of multiplicity of at least 3, by Lemma 2 we get $Z(Lw) \geq Z(w) + 1$.

Let us put

\[ w_1(x) = \frac{1}{\mu \nu} L u(x) = \frac{\lambda}{\mu} u(x) - \gamma v(x), \]

whence $Z(w_1) \geq Z(w) + 1 = Z(v) + 1$. Applying Lemma 3 to $w_1(x)$ we get the inequality $Z(w_2) \geq Z(w_1)$, where

\[ w_2(x) = \left( \frac{\lambda}{\mu} \right)^2 u(x) - \gamma v(x). \]

Proceeding thus further we get an infinite sequence of functions

\begin{equation}
w(x), w_1(x), w_2(x), \ldots,
\end{equation}

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for which

$$Z(w_*) \geq Z(w_{r-1}) \geq Z(w) + 1 = Z(v) + 1, \quad r = 2, 3, \ldots,$$

where

$$w_*(x) = \left( \frac{\lambda}{\mu} \right)^r u(x) - \gamma v(x), \quad r = 2, 3, \ldots$$

On the other hand, since the sequence (13) and its limit $v(x)$, for $r \rightarrow +\infty$, satisfy the assumptions of Lemma 4, we find that $Z(w_*) = Z(v)$ for sufficiently large $r$, which contradicts (14).

4. Nodes of the n-th eigenfunction of problem (4), (1). Let us make some additional assumptions:

**Assumption Z.** The operator $L$ in equation (4) is symmetric and positive definite in $\mathfrak{M}$, i.e.

$$(Lu, u) = (u, Lu) \geq \beta(u, u) \quad \text{for } u \in \mathfrak{M}, \quad \beta > 0.$$

**Remark 1.** Assumption Z is satisfied if for instance we have $p_1(x) = p_2(x) > 0$ and $q_1(x) = q_2(x) \geq 0$ for $x \in [a, b]$.

It is known (cf. [4]) that if Assumption Z is satisfied, then there exists an infinite sequence of eigenvalues of problem (4), (1)

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \quad \lim \lambda_n = +\infty$$

and a corresponding sequence of eigenfunctions

$$u_1(x), u_2(x), u_3(x), \ldots,$$

which form a complete system in $L_2([a, b])$.

By Theorem 3 we have that (15) is a strongly increasing sequence. According to Theorem 5 we get the inequality

$$Z(u_n) \geq n - 1.$$

Our main purpose now is to prove that $Z(u_n) = n - 1$.

We shall prove the following:

**Theorem 6.** Let Assumption Z hold. Then the n-th eigenfunction $u_n(x)$ of problem (4), (1) has exactly $n - 1$ zero-points in $(a, b)$.

**Proof.** By (17) it is sufficient to prove that $Z(u_n) \leq n - 1$ for each $n$. Suppose that there exists $n = m$ such that $Z(u_m) > m$. Let us denote zero-points of $u_m(x)$ in $(a, b)$ by $a = x_0 < x_1 < \ldots < x_{m+1} = b$. Put

$$U_j(x) = \begin{cases} u_m(x) & \text{in } [x_{j-1}, x_j], \\ 0 & \text{in } [a, b] - [x_{j-1}, x_j], \end{cases} \quad j = 1, \ldots, m + 1.$$
It is evident that \( U_1(x), \ldots, U_{m+1}(x) \) are linearly independent in \([a, b]\). Put

\[
\varphi(x) = c_1 U_1(x) + \ldots + c_{m+1} U_{m+1}(x),
\]

where \( c_1, \ldots, c_{m+1} \) are real numbers such that \( c_1^2 + \ldots + c_{m+1}^2 > 0 \), and \( \varphi(x) \) is orthogonal to \( u_1(x), \ldots, u_m(x) \) in \( L_2([a, b]) \), i.e.

\[
\langle \varphi, u_i \rangle = 0, \quad i = 1, \ldots, m.
\]

Let us note that \( \varphi(x) \) defined by (19) is continuous in \([a, b]\), and

\[
Z(\varphi) \leq Z(u_m).
\]

Hence it can be expanded into Fourier's series of functions (17) convergent in the norm of \( L_2([a, b]) \).

By (20) we get

\[
\varphi(x) = \sum_{k=m+1}^{\infty} a_k u_k(x), \quad a_k = \langle \varphi, u_k \rangle.
\]

We can assume that \( a_{m+1} \neq 0 \).

By [5] we find that the function

\[
\varphi_1(x) = \sum_{k=m+1}^{\infty} \left( \frac{\lambda_{m+1}}{\lambda_k} \right)^{\frac{\nu}{2}} a_k u_k(x)
\]

is an element of \( \mathcal{M}_l \), since the series in (23) is uniformly and absolutely convergent in \([a, b]\), and that

\[
L\varphi_1(x) = \lambda_{m+1} \varphi(x) \varphi(x).
\]

Hence by Lemma 3 we get \( Z(\varphi_1) \leq Z(\varphi) \). Let us put

\[
\varphi_\nu(x) = \sum_{k=m+1}^{\infty} \left( \frac{\lambda_{m+1}}{\lambda_k} \right)^\nu a_k u_k(x), \quad \nu = 2, 3, \ldots
\]

The following holds

\[
L\varphi_\nu(x) = \lambda_{m+1} \varphi(x) \varphi_{\nu-1}(x), \quad \nu = 2, 3, \ldots
\]

Hence

\[
Z(\varphi) \geq Z(\varphi_1) \geq Z(\varphi_2) \geq \ldots
\]

It follows from the definition of the sequence (24) and from previous remarks, that \( \{\varphi_\nu(x)\} \) tends uniformly to \( a_{m+1} u_{m+1}(x) \) in \([a, b]\). Since \( a_{m+1} \neq 0 \), by Lemma 5, from (21) and (25) we find that

\[
Z(u_m) \geq Z(\varphi) \geq Z(u_{m+1}),
\]

which contradicts Theorem 5.
COROLLARY 2. When Assumption 2 holds, the first eigenfunction of problem (4), (1) does not vanish at any point of the interval $(a, b)$.


References


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