COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1975 FASC. 1

THE TOPOLOGY OF THE FREE TOPOLOGICAL SEMIGROUP

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1. Introduction and preliminaries. Relying more often than not on existential arguments rather than explicit constructions, investigators have had difficulty in determining the topological properties of objects such as the free topological semigroup, the free topological group and the free product of topological groups. For example, the space of the free topological semigroup generated by a metric space is itself metrizable; however, this result is by no means obvious from the construction found in [1], p. 344. For certain classes of generating spaces the topology of the free topological group or the free product of topological groups has been explicitly described and from the description a number of new and interesting results have been obtained (see [7], [9], [13], [15] and [16]). In Section 2 of this paper we provide an explicit construction of the free topological semigroup $F(X)$ generated by a space X, a notion first defined by Christoph in [1], p. 344.

A number of internal characterizations of $F(X)$ are also given in Section 2 along with conditions sufficient for a topological subsemigroup of $F(X)$ to be free. The topological properties of $F(X)$ are investigated in Section 3, and in Section 4 we discuss the uniformizability of the free topological semigroup.

An associative binary operation on a set will be called a *multiplication*. A *topological semigroup* is a topological space together with a continuous multiplication defined on it.

For a set X and a subset A of a semigroup S, we denote by X^n the Cartesian product of n copies of X, and by $A^{(n)}$ the algebraic product, in S, of n copies of A.

A mapping between topological semigroups will be called a *homomorphism* if it is a continuous semigroup homomorphism. It will be called an *isomorphism* if it is both a homeomorphism and a semigroup isomorphism.

For details concerning the theory of free algebraic semigroups, the reader is referred to [2], Chapter 9.

2. Construction and characterizations.

Definition 2.1. Let X be a topological space. A pair $(F(X), \theta)$ is called a *free topological semigroup* generated by the space X if the following are satisfied:

(F1) $F(X)$ is a topological semigroup;

(F2) $\theta\colon X \to F(X)$ is an embedding;

(F3) $\theta[X]$ generates $F(X)$ algebraically; and

(F4) for each topological semigroup T and each continuous mapping $\omega\colon X \to T$, there exists a unique homomorphism $\Omega\colon F(X) \to T$ such that $(\Omega \circ \theta)(x) = \omega(x)$ for every $x \in X$.

If a pair $(F(X), \theta)$ satisfies (F1)-(F4), we say that $F(X)$ is *freely generated* by the space X.

The following existence and uniqueness theorems follow from the paper of Christoph [1]:

THEOREM 2.1 (existence). *For each topological space X, there exists a pair $(F(X), \theta)$ satisfying* (F1)-(F4).

THEOREM 2.2 (uniqueness). *Let (T_1, γ_1) satisfy* (F1)-(F4). *Then a pair (T_2, γ_2) satisfies* (F1)-(F4) *if and only if there exists an isomorphism $\Gamma\colon T_1 \to T_2$ such that $\Gamma \circ \gamma_1 = \gamma_2$.*

We proceed now with our construction from which the main result of this section, Theorem 2.5, will follow.

For a non-empty set X let $S(X)$ be the disjoint union of the family $\{X^n\colon n \in N\}$ together with the multiplication μ defined by

$$\mu\big((x_1, \ldots, x_n), (y_1, \ldots, y_m)\big) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Then $S(X)$ is the free algebraic semigroup generated by the set X. Now suppose X carries a topology. Providing each X^n with the usual product topology and $S(X)$ with the summation of these topologies, $S(X)$ becomes a topological semigroup. In fact, with respect to this topology, we have the following

THEOREM 2.3. *Multiplication on $S(X)$ is a continuous, open, and closed mapping.*

Proof. The continuity and openness of μ are easily verified; we will establish that μ is a closed mapping. Let A be closed in $S(X)^2$ and

$$x = (x_1, \ldots, x_p) \in S(X) \setminus \mu[A].$$

For each n, $1 \leqslant n < p$, select p open sets $U(n, 1), \ldots, U(n, p)$ in X such that $x_i \in U(n, i)$, $1 \leqslant i \leqslant p$, and

$$\prod_{i=1}^{n} U(n, i) \times \prod_{i=n+1}^{p} U(n, i) \subset S(X)^2 \setminus A.$$

For each $i \leqslant p$, set

$$V(i) = \bigcap_{n=1}^{p-1} U(n, i).$$

If $y = (y_1, \ldots, y_p) \in \mu[A]$, then $((y_1, \ldots, y_n), (y_{n+1}, \ldots, y_p)) \in A$ for some $n < p$. It follows that $y_m \notin U(n, m)$ for some $m \leqslant p$, whence we have $y \notin V(1) \times \ldots \times V(p)$. We conclude that

$$x \in V(1) \times \ldots \times V(p) \subset S(X) \setminus \mu[A].$$

Let β denote the inclusion mapping from X into $S(X)$.

THEOREM 2.4. *For a topological space X the pair $(S(X), \beta)$ satisfies* (F1)-(F4).

Proof. Suppose T is a topological semigroup and ω: $X \to T$ is continuous. The mapping Ω, defined by

$$\Omega(x_1, x_2, \ldots, x_n) = \omega(x_1)\omega(x_2)\ldots\omega(x_n),$$

is, of course, the unique extension of ω to a semigroup homomorphism on $S(X)$. Moreover, Ω is continuous since, for each n, the restriction of Ω to X^n is a product of the continuous mapping ω.

Theorems 2.3 and 2.4 provide the following important result concerning the topological structure of the free topological semigroup:

THEOREM 2.5. *If $(F(X), \theta)$ is the free topological semigroup generated by X, then $F(X)$ is homeomorphic to the topological sum of all the finite Cartesian products of X. Moreover, θ: $X \to F(X)$ is a homeomorphism onto an open-closed subspace of $F(X)$.*

Next we consider the analog of the important theorem in discrete semigroup theory which states that every semigroup is a quotient of a free semigroup. Let I: $F(T) \to T$ be the extension to $F(T)$ of the identity mapping on a topological semigroup T. Then T is a retract of $F(T)$ and necessarily has the quotient topology induced by I; whence we have the following theorem:

THEOREM 2.6. *Each topological semigroup is a quotient topological semigroup of a free topological semigroup.*

Remark. It should be mentioned that a somewhat weaker version of Lemma 2.6 appears in [1], Proposition 1.5, p. 345, and there the observation is made that T will have the quotient topology if I is an open mapping (such a condition is necessary and sufficient in the setting of topological groups (see [6], 8.23, p. 82). As the following example indicates, the mapping I need not be open:

Example 2.1. Providing $T = [0, 1]$ with the usual topology and the multiplication defined by $(x, y) \to 0$, we see that T^2 is open in $F(T)$ while $I(T^2) = \{0\}$ is not open in T.

We now offer several alternative characterizations of free topological semigroups (Theorem 2.8); each of these is expressed in terms of the internal structure of the topological semigroup rather than an "external" mapping property.

A set X is called a *free algebraic base* for a semigroup T if T is the free algebraic semigroup on the set X.

THEOREM 2.7. *A topological semigroup T is freely generated by a space if and only if it is freely generated by $T \setminus T^{(2)}$.*

Proof. Let Y be a space and let ω be a mapping from Y to T such that (T, ω) is a free topological semigroup generated by Y. Setting $X = T \setminus T^{(2)}$, $\omega[Y] = X$ according to 9.2 of [2], p. 117. Consequently, there exist isomorphisms $\Omega_1\colon F(Y) \to T$ and $\Omega_2\colon F(X) \to F(Y)$ extending ω and ω^{-1}, respectively. Therefore, $\Omega_1 \circ \Omega_2\colon F(X) \to T$ is an isomorphism satisfying $(\Omega_1 \circ \Omega_2) \circ \theta = \omega$.

THEOREM 2.8. *Let T be a topological semigroup and let $X \subset T$ be a free algebraic base for T. Then the following are equivalent:*

(a) *the space X freely generates T;*

(b) *X is open in T and multiplication is an open mapping;*

(c) *X is closed in T and multiplication is a closed mapping; and*

(d) *multiplication is an open-closed mapping.*

Proof. That (a) implies each of the remaining conditions is a consequence of Theorems 2.4 and 2.2, while implications (d) $\to$ (c) and (d) $\to$ (b) follow easily from the fact that $X = T \setminus T^{(2)}$.

Define the mapping $\lambda\colon S(X) \to T$ by

$$\lambda(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n.$$

Then λ is a continuous, algebraic isomorphism.

(b) $\to$ (a). In assuming that (b) holds, we infer that X^n is open in T^n for each $n \in N$, and that λ maps each open set in X^n onto an open set in T. Recalling that $S(X)$ is the topological sum of $\{X^n\colon n \in N\}$, we conclude the openness of λ.

(c) $\to$ (a). (i) We will first show that the restriction of λ to X^n is closed for each n. Let $\mathscr{F}$ be the collection of all sets of the form $F_1 \times \ldots \times F_n$, where each F_i is a closed set in X. Condition (c) implies that λ maps each member of $\mathscr{F}$ to a closed set in T. Since $\mathscr{F}$ is a subbase for the closed sets of X^n and λ is a bijection, λ is a closed mapping on X^n.

(ii) Consider the family $\mathscr{P} = \{X^{(n)}\colon n \in N\}$ which partitions the set T. If (c) is satisfied by T, then $X^{(n)}$ and $T^{(n)}$ are closed in T for every n. Since

$$X^{(1)} = T \setminus T^{(2)} \quad \text{and} \quad X^{(i+1)} = T \setminus [T^{(i+2)} \cup X^{(1)} \cup \ldots \cup X^{(i)}],$$

we conclude that $X^{(n)}$ is open in T for each $n \in N$. Therefore, $\mathscr{P}$ is an open partition of T.

We are now prepared to establish that λ is a closed mapping (provided (c) is satisfied). It follows from (i) and (ii) that, for each closed set A in $S(X)$,

$$\mathscr{B}(A) = \{\lambda[A \cap X^n]: n \in N\}$$

is a locally finite family of closed subsets of T; whence $\lambda[A] = \bigcup \mathscr{B}(A)$ is a closed set.

Let T be a free topological semigroup and let S be a subsemigroup of T that is algebraically free. In general, S is not a free topological semigroup. Even with the added condition that S be open (respectively, closed) in T, which guarantees that multiplication on S is open (respectively, closed), S need not be freely generated by a topological space (see Example 2.2 below). Conditions sufficient for S to be a free topological semigroup are given in Theorems 2.9 and 2.10.

Example 2.2. Let R denote the reals with the usual topology and let S be the subsemigroup of $F(R)$ generated by

$$X = (0,1) \cup \big([1,2) \times (0,1)\big) \subset F(R).$$

The set X is a free algebraic base for S. If, for $n \in N$ and $1 \leqslant i \leqslant n$, A_i is $(0,1)$ and B_i is either $(0,1)$ or $[1,2)$, then

$$\prod A_i \cup \prod B_i = \prod (A_i \cup B_i)$$

is open in R^n. Since $S \cap R^n$ is the union of sets of this form, $S \cap R^n$ must be open in R^n; whence S is open in $F(R)$ and multiplication on S is continuous and open. However, S is not a free topological semigroup since X is not open in S.

THEOREM 2.9. *Let Y be a topological space and $X \subset Y$. The topological subsemigroup of $F(Y)$ generated by X is a free topological semigroup.*

THEOREM 2.10. *Let Y be a topological space and let S be a topological subsemigroup of $F(Y)$ that is algebraically free. If $S \setminus S^{(2)}$ is either open or closed in $F(Y)$, then S is a free topological semigroup.*

Proof. Set $X = S \setminus S^{(2)}$. If X is open in $F(Y)$, then $S = \bigcup X^{(n)}$ is also open and, therefore, condition (b) of Theorem 2.8 is satisfied. If X is closed in $F(Y)$, $\{X^{(n)}: n \in N\}$ is a locally finite family of closed sets in $F(Y)$. Thus $S = \bigcup X^{(n)}$ is closed and condition (c) of Theorem 2.8 is satisfied.

We conclude this section with the analog of another important result, for discrete semigroups, which states that two free semigroups are isomorphic if and only if there exists a bijection between their bases (see [2], 9.3, p. 117). It has been shown that this result does not extend to the setting of free topological groups (see [4], Section 5).

THEOREM 2.11. *A pair of topological spaces generate isomorphic free topological semigroups if and only if the spaces are homeomorphic.*

Proof. If X and X_0 are spaces and

$$\Omega:\ \big(F(X),\theta\big) \to \big(F(X_0),\theta_0\big)$$

is an isomorphism, then $(\Omega\circ\theta)[X] = \theta_0[X_0]$ by Corollary 9.3 of [2], p. 117.

3. Topological properties of $F(X)$. In this section we determine a number of topological properties which are transmitted from the space X to its free topological semigroup. We also introduce the notions of P-semigroup and free P-semigroup and discuss the existence and structure of the latter.

The following result is immediate from Theorem 2.5:

THEOREM 3.1. (a) *Let P be a finitely productive, countably summable topological property. Then P is a property of $F(X)$ whenever P is a property of X.*

(b) *Let P be either an open-hereditary or a closed-hereditary topological property. Then P is a property of X whenever P is a property of $F(X)$.*

It follows from Theorem 3.1 that $F(X)$ is completely regular (respectively, T_0, T_1, T_2, regular) if and only if the same is true for X. The following corollary lists additional properties which satisfy the hypothesis of Theorem 3.1 (a) and (b):

COROLLARY. *Let P be any one of the following topological properties:*

(a) *local compactness,*
(b) *σ-compactness,*
(c) *metrizability,*
(d) *separability,*
(e) *1-st countability,*
(f) *2-nd countability,*
(g) *real compactness,*
(h) *topological completeness,*
(i) *local connectedness.*

Then P is a property of the space $F(X)$ if and only if P is a property of X.

The additive semigroup N of positive integers becomes a free topological semigroup when given the discrete topology. The semigroup or, more precisely, the space N plays an important role in determining the properties of $F(X)$.

THEOREM 3.2. *For each topological space X, N is the continuous image of $F(X)$ under an open-closed homomorphism.*

Proof. The continuous homomorphism $\Omega: F(X)\to N$, satisfying $(\Omega\circ\theta)(x) = 1$ for all $x\in X$, is an open and closed mapping.

Remark 3.1. It follows from Theorem 3.2, as well as from Theorem 2.5, that $F(x)$ can never have such topological properties as compactness, pseudocompactness, countable compactness, connectedness and indiscreteness.

THEOREM 3.3. *Let P be a countably productive, closed-hereditary topological property and let X be a space having a closed, singleton subset. Then $F(X)$ has property P if and only if both X and N have P.*

Proof. Necessity. The space X has property P by Theorem 2.5. Furthermore, the subsemigroup of $F(X)$ generated by a closed singleton subset is necessarily closed and homeomorphic to N.

Sufficiency. For each $n \in N$ let (q_n) be that point in X^n whose every coordinate is the fixed point q, where $\{q\}$ is a closed subset of X. Then the mapping h defined by

$$h(x) = \{n\} \times \{x\} \times \prod_{m \neq n} \{(q_m)\}, \qquad x \in X^n, n \in N,$$

is an embedding of $\sum_{n \in N} X^n$ onto a closed subspace of the countable product $N \times \prod_{n \in N} X^n$.

The next theorem can be proved in much the same manner as Theorem 3.3.

THEOREM 3.4. *Let X be a topological space and let P be a countably productive, hereditary topological property. Then $F(X)$ has property P if and only if both X and N have property P.*

Definition 3.1. Let P be a topological property. A topological semigroup T is called a *P-semigroup* if the topological space T has property P.

Remark 3.2. As an exception to this definition, we define a topological semigroup T to be *space-metrizable* if T is metrizable as a topological space. The term "metrizable semigroup" will retain its usual meaning (see Section 4).

The definition of the *free P-semigroup* generated by X is obtained by replacing "topological semigroup" by "P-semigroup" everywhere in Definition 2.1. The free P-semigroup will be denoted by $(FP(X), \theta_P)$. We note that if the free P-semigroup exists, it is unique in the sense of Theorem 2.2.

LEMMA. *If Y is any topological space, then either the countable product Y^N contains a copy of N or Y is indiscrete.*

Proof. It is well known that N is embeddable in W^N, where W is either the two-point discrete space or the Sierpiński space (see [14], 3.11 and 3.12, p. 170). For a non-indiscrete space Y, there exist points $a, b \in X$ such that $a \notin \mathrm{cl}_Y\{b\}$. Thus the subspace $\{a, b\}$ is either discrete or Sierpiński.

THEOREM 3.5. *Let P be a topological property of a space X. Then $(FP(X), \theta_P)$ exists and is isomorphic to $(F(X), \theta)$ if any one of the following is satisfied:*

(a) *P is finitely productive, countably summable;*

(b) *X has a closed, singleton subset and P is a countably productive, closed hereditary property of N;*

(c) *P is a countably productive, hereditary property of N;*

(d) *P is a countably productive, hereditary property of some non-indiscrete space.*

Proof. The sufficiency of conditions (a), (b) and (c) follows from Theorems 3.1, 3.3 and 3.4, respectively, and, according to the Lemma, (d) implies (c).

It is possible for the free P-semigroup to exist and be distinct from the free topological semigroup; such is the case for the property of indiscreteness (Theorem 3.6 and Remark 3.1).

THEOREM 3.6. *Let P be a productive, hereditary property of X. Then $(FP(X), \theta_P)$ exists. Moreover, if P is a property of some two-point space Y, then $(FP(X), \theta_P)$ is the free algebraic semigroup on the set X.*

Proof. The existence of $(FP(X), \theta_P)$ follows from the construction in [1], p. 344. If there exists a non-indiscrete space having P, then $(FP(X), \theta_P)$ is isomorphic to $(F(X), \theta)$ (Theorem 3.5 (d)).

Now suppose P implies indiscreteness. The two properties are then equivalent; for if W is an indiscrete space, it is embeddable in Y^W and, therefore, has P. Now let ω be a function from X to a semigroup S. When given the indiscrete topology, S is a P-semigroup and ω is a continuous mapping and, therefore, can be extended to $FP(X)$.

4. Uniformizability of $F(X)$. A topological semigroup is said to be *uniformizable* if there exists a uniformity on the semigroup that induces the given topology and with respect to which multiplication is uniformly continuous (see [11], Section 3). In this section it will be shown that the free topological semigroup generated by a completely regular space is uniformizable. We also obtain a very useful representation (stated in terms of metrizable, free topological semigroups) of $(F(X), \theta)$, where X is a completely regular T_1-space.

Throughout this section, X and X^n will be written in place of $\theta[X]$ and $\theta[X]^n$, respectively.

A topological semigroup is said to be *metrizable* (respectively, *pseudometrizable*) if there exists a subinvariant metric (respectively, pseudometric) on the semigroup that induces the given topology. Metrizability of a topological semigroup implies space-metrizability (Remark 3.2), however, the converse is not true (see [12], 4.4).

THEOREM 4.1. *The topological semigroup $F(X)$ is metrizable if and only if X is metrizable.*

Proof. Let d be a compatible metric on X with $d \leqslant 1$. For points

$x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ in $F(X)$, write

$$d_0(x, y) = \min\left\{1, \sum_{i=1}^{n} d(x_i, y_i)\right\} \quad \text{if } n = m,$$

otherwise let $d_0(x, y) = 1$. The metric d_0 is invariant and is compatible with the topology of $F(X)$.

For a completely regular space X let $\mathscr{D}(X)$ denote the collection of all continuous pseudometrics on X that are bounded by 1, let $\mathscr{D}_0(X)$ denote the collection $\{d_0\colon d \in \mathscr{D}(X)\}$, where each d_0 is defined as in the proof of Theorem 4.1, and let $\mathscr{U}_0(X)$ denote the uniformity on $F(X)$ generated by $\mathscr{D}_0(X)$.

THEOREM 4.2. *The topological semigroup $F(X)$ is uniformizable if and only if X is a completely regular space.*

Proof. The uniformity $\mathscr{U}_0(X)$ is compatible with both the multiplication and the topology of $F(X)$ (see [11], Theorem 2).

THEOREM 4.3. *The free topological semigroup generated by a completely regular T_1-space is dense in the inverse limit of an inverse system of metrizable, free topological semigroups.*

Proof. According to Theorem 13 of [11], there exists a dense, uniform embedding of the uniform semigroup $(F(X), \mathscr{U}_0(X))$ into the inverse limit of an inverse system whose objects consist of the metric semigroups $(F(X)^*, d_0^*)$ associated with the pseudometric semigroups $(F(X), d_0)$. For $d \in \mathscr{D}(X)$ let X_d be the metric space associated with the pseudometric space (X, d), and let q denote the corresponding quotient mapping. Since q is open, $q_n[X^n] = X_d^n$, where q_n is the product of n-copies of q. Furthermore, words in $F(X)$ that are of different lengths are not identified in $(F(X)^*, d_0^*)$; whence $F(X)^*$ is the topological summation of $\{X_d^n\colon n \in N\}$.

The category of completely uniformizable T_1-spaces and continuous functions is productive and closed hereditary and, by 15.24 of [3], p. 232, contains all metrizable spaces. Therefore, the complete uniformizability of a T_1-space X is a condition necessary for $F(X)$ to be an inverse limit of metrizable, free topological semigroups. Moreover, this condition is sufficient (see Theorem 4.4). Examples of completely regular T_1-spaces failing to admit complete uniformities are found in [3], Chapter 5.

THEOREM 4.4. *Let X be a completely regular T_1-space. Then $F(X)$ is the inverse limit of metrizable, free topological semigroups if and only if X admits a complete uniformity.*

Proof. The uniformity $\mathscr{U}_0(X)$ induces the largest admissible uniformity on X and the corresponding product uniformity on each X^n, $n > 1$. Therefore, if X is completely uniformizable, X^n for each $n \in N$ is complete with respect to the uniformity inherited from $\mathscr{U}_0(X)$. Given a Cauchy filter $\mathscr{H}$ on $F(X)$, there is a unique k for which $\{H \cap X^k\colon H \in \mathscr{H}\}$ is a Cauchy, hence convergent, filter on X^k. Thus $\mathscr{H}$ itself converges. The completeness of $\mathscr{U}_0(X)$, together with Theorem 16 of [11] and Theorem 4.3, effects the sufficiency.

The author wishes to thank John Mack for his many helpful suggestions relating to the preparation of this paper.

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Reçu par la Rédaction le 17. 5. 1974;
en version modifiée le 23. 7. 1974