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ON CURVATURE COLLINEATIONS ON SIMPLE CONFORMALLY RECURRENT MANIFOLDS

BY

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1. Introduction. An n-dimensional ($n \geqslant 4$) Riemannian manifold M (whose metric g need not be definite) is said to be *conformally recurrent* [1] if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{ik}g_{hj}) \tag{1}$$

satisfies the condition

$$C_{hijk,l}C_{pqtr} = C_{hijk}C_{pqtr,l},$$

where the comma denotes covariant differentiation with respect to g. The above relation states that at any point $x \in M$ such that $C(x) \neq 0$ there exists a unique covariant vector φ (called the *recurrence vector* of C) which satisfies the condition

$$C_{hijk,l} = \varphi_l C_{hijk}. \tag{2}$$

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric ($C_{hijk,l} = 0$) as well as all recurrent manifolds of dimension $n \geqslant 4$.

A conformally recurrent manifold (M, g) is said to be *simple* [6] (s.c.r. for short) if its metric is locally conformal to a non-conformally flat conformally symmetric one, i.e., if for each point $x \in M$ there exist a neighbourhood U of x and a function p on U such that $\bar{g} = (\exp 2p)g$ is a non-conformally flat conformally symmetric metric.

The following theorem gives a characterization of s.c.r. manifolds:

THEOREM A ([6], Theorem 1). *A Riemannian manifold M* ($\dim M = n \geqslant 4$) *is s.c.r. if and only if*

(i) $C_{hijk} \neq 0$ *(everywhere on M),*

(ii) $C_{hijk,l} = \varphi_l C_{hijk}$,

(iii) *the recurrence vector* φ *is locally a gradient,*

(iv) *the Ricci tensor is a Codazzi one, i.e.,*

$$R_{ij,k} = R_{ik,j}. \tag{3}$$

Obviously, every non-conformally flat conformally symmetric manifold is necessarily s.c.r. The existence of s.c.r. manifolds which are neither conformally symmetric nor recurrent has been established in [6]. Thus the class of s.c.r. manifolds is a natural extension of the class of conformally symmetric ones.

In this paper we are concerned with a symmetry property of a manifold which we call a curvature collineation. A vector field v on M is said to be a *curvature collineation* [5] (CC for short) if

$$L_v R^h_{ijk} = 0, \tag{4}$$

where L_v denotes the Lie derivative with respect to v. It is worth noticing that the investigation of this symmetry is strongly motivated by the all-important role of the Riemann curvature tensor in the general theory of relativity.

Section 2 contains some necessary conditions for an s.c.r. manifold to admit a CC. Section 3 deals with CC's in s.c.r. manifolds such that rank $R_{ij} = 2$. Note that the Ricci tensor of every non-locally symmetric s.c.r. manifold satisfies rank $R_{ij} \leqslant 2$ ([7], Theorem 2). Moreover, it is known that any manifold admitting a parallel vector field admits also a CC generated by this field [4]. On the other hand, non-locally symmetric s.c.r. manifolds whose Ricci tensor satisfies rank $R_{ij} = 2$ do not admit parallel vector fields [7]; so there arises a natural question whether there exist CC in such manifolds. It will be shown (Section 4) that this problem has an affirmative answer.

All manifolds under consideration are assumed to be connected and of class C^∞. The Riemannian metrics are not assumed to be positive definite.

2. Preliminaries. In the sequel we shall need the following lemmas:

LEMMA 1 ([3], Lemma 1). *Let an (algebraic) tensor* $A_{lmhs_1\ldots s_p}$ *of type* $(0, p+3)$ *be symmetric in* (l, m) *and skew-symmetric in* (m, h). *Then*

$$A_{lmhs_1\ldots s_p} = 0.$$

LEMMA 2. *The Weyl conformal curvature tensor satisfies the relations*

$$\begin{gathered} C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \\ C_{hijk} + C_{hjki} + C_{hkij} = 0, \qquad C^r_{ijr} = C^r_{irj} = C^r_{rij} = 0. \end{gathered} \tag{5}$$

LEMMA 3. *Let* a *be a symmetric tensor of type* $(0, 2)$ *on a Riemannian manifold* M *which satisfies the condition* $a_{ij,lm} - a_{ij,ml} = 0$. *Then*

$$a_{rj} R^r_{ilm} + a_{ir} R^r_{jlm} = 0, \tag{6}$$

$$a_{ir} R^r_{jlm} + a_{lr} R^r_{jmi} + a_{mr} R^r_{jil} = 0, \tag{7}$$

$$a_{lr} R^r_{m} = a_{mr} R^r_{l}, \tag{8}$$

$$a_{ir} C^r_{jlm} + a_{lr} C^r_{jmi} + a_{mr} C^r_{jil} = 0. \tag{9}$$

Proof. The first equation follows at once from the Ricci identity. Permuting in (6) the indices i, l, m cyclically and adding the resulting equations to (6), we obtain (7). Now, contracting (7) with g^{ij}, we get (8). Finally, using (7) and (8), we obtain easily (9).

Moreover, in the sequel we shall use the following properties of s.c.r. manifolds:

LEMMA 4 ([6]). *The curvature tensor and the Weyl conformal curvature tensor of every s.c.r. manifold satisfy the conditions*

$$R_{hijk,lm}-R_{hijk,ml}=0, \tag{10}$$

$$\varphi_l C_{hijk}+\varphi_h C_{iljk}+\varphi_i C_{lhjk}=0, \tag{11}$$

where φ is the recurrence vector of C.

LEMMA 5 ([6], [7]). *Every non-locally symmetric s.c.r. manifold satisfies the relations*

$$R=0, \tag{12}$$

$$R_{ir}R^{r}{}_{j}=0, \tag{13}$$

$$R_{ir}R^{r}{}_{j,k}=0, \tag{14}$$

$$R_{ir}C^{r}{}_{jkl}=0, \tag{15}$$

$$R_{il}C_{hmjk}+R_{ij}C_{hmkl}+R_{ik}C_{hmlj}=0. \tag{16}$$

LEMMA 6 ([6], Theorem 4). *Let M be a non-locally symmetric s.c.r. manifold. Then M admits a unique function F such that*

$$FC_{hijk}=R_{hk}R_{ij}-R_{hj}R_{ik}. \tag{17}$$

F is said to be the *fundamental function* of M. It is clear that $F(x)=0$ if and only if $\operatorname{rank} R_{ij}(x)\leqslant 1$.

Using (4) and the well-known formulas [8]

$$L_v R^{h}{}_{ijk}=(L_v\Gamma^{h}_{ij})_{,k}-(L_v\Gamma^{h}_{ik})_{,j},$$

$$L_v\Gamma^{h}_{ij}=\tfrac{1}{2}g^{hs}(a_{si,j}+a_{sj,i}-a_{ij,s}), \tag{18}$$

where

$$a_{ij}=L_v g_{ij}=v_{i,j}+v_{j,i}, \tag{19}$$

we have

LEMMA 7 (cf. [4]). (i) *A vector field v on a manifold M is a* CC *if and only if*

$$(a_{hi,j}+a_{hj,i}-a_{ij,h})_{,k}-(a_{hi,k}+a_{hk,i}-a_{ik,h})_{,j}=0, \tag{20}$$

where a is given by (19).

(ii) *Every* CC *vector v satisfies the relations*

$$a_{ij,lm}-a_{ij,ml}=0, \tag{21}$$

$$L_v R_{ij}=0. \tag{22}$$

LEMMA 8. *Let v denote a* CC *on a non-locally symmetric s.c.r. manifold M. Then the following equations hold:*

$$a^{rs}R_{rs} = 0,$$

$$a^{rs}R_{ri}R_{sk} = 0, \tag{23}$$

$$a_{ir}R^{r}{}_{k} = \omega R_{ik}, \tag{24}$$

$$a_{pr}C^{r}{}_{ijk} + a_{ir}C^{r}{}_{pjk} = \frac{1}{n-2}[R_{pk}(a_{ij}-\omega g_{ij}) - R_{pj}(a_{ik}-\omega g_{ik}) + R_{ik}(a_{pj}-\omega g_{pj}) - R_{ij}(a_{pk}-\omega g_{pk})], \tag{25}$$

$$T_{ij} = a^{rs}C_{rijs} = \frac{1}{n-2}(n\omega-\alpha)R_{ij}, \quad \alpha = a_{rs}g^{rs}, \tag{26}$$

$$R_{ij}T_{hl} = R_{il}T_{hj}, \tag{27}$$

where ω is a function on the open subset $U = \{x \in M\colon R_{ij}(x) \neq 0\}$.

Proof. Using (12) and (22), we obtain easily the first equation. Similarly, in virtue of (13) and (22), we get (23). By (21), Lemma 3 implies (6) and (8). Now, using (1), (12) and (6), we have

$$a_{pr}C^{r}{}_{ijk} + a_{ir}C^{r}{}_{pjk} = -\frac{1}{n-2}(g_{ij}a_{pr}R^{r}{}_{k} - g_{ik}a_{pr}R^{r}{}_{j} + g_{pj}a_{ir}R^{r}{}_{k} - g_{pk}a_{ir}R^{r}{}_{j} + a_{pk}R_{ij} - a_{pj}R_{ik} + a_{ik}R_{pj} - a_{ij}R_{pk}). \tag{28}$$

Transvecting (28) with $R^{p}{}_{l}$ and using (15), (13), (8) and (23), we obtain

$$R_{jl}B_{ik} - R_{kl}B_{ij} + R_{ij}B_{kl} - R_{ik}B_{jl} = 0,$$

where $B_{ij} = a_{ir}R^{r}{}_{j} = B_{ji}$. Alternating this relation in (k, l) and (i, j), we get

$$R_{jl}B_{ik} = R_{ik}B_{jl},$$

which implies immediately

$$B_{ik} = \omega R_{ik}$$

for some function ω on U. So we have (24). Substituting (24) into (28), we get (25). Now, contraction of (25) with g^{pk}, by (5), (12) and (24), leads to (26). Finally, transvecting (16) with a^{mk} and taking (5), (8) and (15) into account, we obtain (27). This completes the proof.

LEMMA 9. *Let M be a non-locally symmetric s.c.r. manifold admitting a* CC *v. Then the relations*

$$L_{v}C^{h}{}_{ijk} = \frac{1}{n-2}[R^{h}{}_{j}(a_{ik}-\omega g_{ik}) - R^{h}{}_{k}(a_{ij}-\omega g_{ij})], \tag{29}$$

$$(30)\quad 2(L_v\varphi_l)C_{hijk}-\frac{2}{n-2}\varphi_l[R_{hk}(a_{ij}-\omega g_{ij})-R_{hj}(a_{ik}-\omega g_{ik})]$$

$$+\frac{2}{n-2}[R_{hk,l}(a_{ij}-\omega g_{ij})-R_{hj,l}(a_{ik}-\omega g_{ik})+R_{hk}(a_{ij,l}-\omega_{,l}g_{ij})-R_{hj}(a_{ik,l}-\omega_{,l}g_{ik})]$$

$$=(a_{hl,r}+a_{hr,l}-a_{lr,h})C^r{}_{ijk}+(a_{ri,l}+a_{rl,i}-a_{il,r})C^r{}_{hjk}$$

$$-(a_{rj,l}+a_{rl,j}-a_{jl,r})C^r{}_{khi}+(a_{rk,l}+a_{rl,k}-a_{kl,r})C^r{}_{jhi},$$

$$(31)\quad R_{hj}(a_{sk,l}R^s{}_m-\omega_{,l}R_{km})=R_{hk}(a_{sj,l}R^s{}_m-\omega_{,l}R_{jm})$$

hold on U.

Proof. Using (1), (12), (4), (22) and (24), we obtain (29). Differentiating (29) covariantly and applying (18) and the well-known formula ([8], p. 16)

$$L_v(C^h{}_{ijk,l})-(L_vC^h{}_{ijk})_{,l}=(L_v\Gamma^h_{lr})C^r{}_{ijk}-(L_v\Gamma^r_{li})C^h{}_{rjk}-(L_v\Gamma^r_{lj})C^h{}_{irk}-(L_v\Gamma^r_{lk})C^h{}_{ijr},$$

in virtue of (2) we have (30). Transvecting (30) with $R^h{}_m$ and using (13)–(15), we obtain

$$(32)\quad R_m{}^s(a_{sl,r}+a_{sr,l}-a_{lr,s})C^r{}_{ijk}=0.$$

Similarly, transvection of (30) with $R^i{}_m$, implies

$$(33)\quad \frac{2}{n-2}[R_{hk}(R_m{}^sa_{sj,l}-\omega_{,l}R_{mj})-R_{hj}(R_m{}^sa_{sk,l}-\omega_{,l}R_{mk})]$$

$$=R_m{}^s(a_{rs,l}+a_{rl,s}-a_{sl,r})C^r{}_{hjk}=2R_m{}^sa_{sr,l}C^r{}_{hjk}$$

in virtue of (24) and (32). Differentiating (9) covariantly, using (2), (9) and transvecting the resulting relation with $R_m{}^i$, by (15) we get

$$R_m{}^sa_{sr,p}C^r{}_{jhk}=0.$$

This turns (33) into our assertion, which completes the proof.

LEMMA 10. *Let v denote a* CC *on a non-locally symmetric s.c.r. manifold M. Then*

$$(34)\quad F(b_{ml}C_{hijk}+b_{mj}C_{hikl}+b_{mk}C_{hilj})=0,$$

where $b_{ml}=a_{ml}-\omega g_{ml}$ *and F is the fundamental function of M.*

Proof. Raising the index h in relation (16) and applying the Lie derivative, by (22) and (29) we obtain

$$R_{il}(R^h{}_jb_{mk}-R^h{}_kb_{mj})+R_{ij}(R^h{}_kb_{ml}-R^h{}_lb_{mk})+R_{ik}(R^h{}_lb_{mj}-R^h{}_jb_{ml})=0.$$

But the last equation, in virtue of (17), is equivalent to our assertion.

3. Main results.

PROPOSITION 1. *Let M be a non-locally symmetric s.c.r. manifold* ($\dim M = n \geqslant 4$) *such that* $\operatorname{rank} R_{ij} = 2$ *on M. Then the equations*

$$a_{ir}R^{r}{}_{j} = \frac{\alpha}{n}R_{ij}, \tag{35}$$

$$a_{ir}C^{r}{}_{jkl} = \frac{\alpha}{n}C_{ijkl} \tag{36}$$

hold for every CC v *on M, where $a_{ij} = L_v g_{ij}$, $\alpha = a_{rs}g^{rs}$.*

Proof. We assert that $T_{ij} = a^{rs}C_{rijs} = 0$. This is a consequence of the assumption $\operatorname{rank} R_{ij} = 2$, because if T did not vanish identically, then by (27) we would have $\operatorname{rank} R_{ij} \leqslant 1$ at some point $x \in M$. Now, (26) implies

$$\omega = \alpha/n, \tag{37}$$

which turns (24) into (35). By (34) and $F \neq 0$, we have

$$b_{ml}C_{hijk} + b_{mj}C_{hikl} + b_{mk}C_{hilj} = 0.$$

Contracting this equation with g^{lh}, by (5) we obtain $b_{mr}C^{r}{}_{ijk} = 0$, which in virtue of (37) is equivalent to (36).

THEOREM 1. *Let M be a non-locally symmetric s.c.r. manifold whose Ricci tensor satisfies* $\operatorname{rank} R_{ij} = 2$. *If v is a* CC *on M, then*

$$a_{ij} - \frac{\alpha}{n}g_{ij} = \Phi R_{ij} \tag{38}$$

for some function Φ on M.

Proof. Substituting (36) into (25) and using (5) and (37), we have

$$R_{pk}\left(a_{ij} - \frac{\alpha}{n}g_{ij}\right) - R_{pj}\left(a_{ik} - \frac{\alpha}{n}g_{ik}\right) + R_{ik}\left(a_{pj} - \frac{\alpha}{n}g_{pj}\right) - R_{ij}\left(a_{pk} - \frac{\alpha}{n}g_{pk}\right) = 0.$$

This, analogously as in the proof of Lemma 8, implies

$$\left(a_{ij} - \frac{\alpha}{n}g_{ij}\right)R_{pk} = \left(a_{pk} - \frac{\alpha}{n}g_{pk}\right)R_{ij},$$

which leads immediately to our assertion.

Substituting (38) into (29), by (37), (17) and (36) we obtain

COROLLARY 1. *For every* CC v *on a non-locally symmetric s.c.r. manifold whose Ricci tensor satisfies* $\operatorname{rank} R_{ij} = 2$, *the relations*

$$L_v C^{h}{}_{ijk} = -\frac{1}{n-2}F\Phi C^{h}{}_{ijk},$$

$$L_v C_{hijk} = \left(\frac{\alpha}{n} - \frac{F\Phi}{n-2}\right)C_{hijk} \tag{39}$$

hold.

PROPOSITION 2. *Let M be a non-locally symmetric s.c.r. manifold such that* $\operatorname{rank} R_{ij} = 2$. *If v is a* CC *on M, then it satisfies the following equations:*

$$\frac{F\Phi}{n-2} - \frac{\alpha}{n} = (L_v F)/F, \tag{40}$$

$$\alpha_{,l} C_{hijk} + \alpha_{,h} C_{iljk} + \alpha_{,i} C_{lhjk} = 0, \tag{41}$$

$$L_v \varphi_l + ((L_v F)/F)_{,l} + \frac{3}{n}\alpha_{,l} = 0. \tag{42}$$

Proof. Applying the Lie derivative to (17), in virtue of (22) and (39), we obtain (40). Differentiating (38) and (17) covariantly, we have

$$a_{ij,l} - \frac{1}{n}\alpha_{,l} g_{ij} = \Phi_{,l} R_{ij} + \Phi R_{ij,l}, \tag{43}$$

$$(F_{,l} + F\varphi_l) C_{hijk} = R_{hk,l} R_{ij} + R_{hk} R_{ij,l} - R_{hj,l} R_{ik} - R_{hj} R_{ik,l}.$$

Substituting these relations into (30), in virtue of (37), (38) and (17) we can write the left-hand side of (30) in the form

$$2\left(L_v \varphi_l + \frac{1}{n-2}(F\Phi)_{,l}\right) C_{hijk}.$$

Using (40), we see that (30) can now be written as

$$\begin{aligned} 2\left[L_v \varphi_l + \frac{1}{n}\alpha_{,l} + ((L_v F)/F)_{,l}\right] C_{hijk} \\ = (a_{hl,r} + a_{hr,l} - a_{lr,h}) C^r{}_{ijk} + (a_{ri,l} + a_{rl,i} - a_{il,r}) C^r{}_{hjk} \\ - (a_{rj,l} + a_{rl,j} - a_{jl,r}) C^r{}_{khi} + (a_{rk,l} + a_{rl,k} - a_{kl,r}) C^r{}_{jhi}. \end{aligned} \tag{44}$$

Transvecting (43) with $C^i{}_{hkm}$, in virtue of (15) and $R_{rj,l} C^r{}_{hkm} = 0$, which is an obvious consequence of (15) and (2), we obtain

$$a_{rj,l} C^r{}_{hkm} = \frac{1}{n}\alpha_{,l} C_{jhkm}.$$

Analogously, transvecting (43) with $C^l{}_{hkm}$ and using also (3), we have

$$a_{ij,r} C^r{}_{hkm} = \frac{1}{n}\alpha_{,r} C^r{}_{hkm} g_{ij} + R_{ij} \Phi_{,r} C^r{}_{hkm}.$$

Substituting the two last equations into (44), we get

$$\begin{aligned} 2n(L_v \varphi_l + ((L_v F)/F)_{,l}) C_{hijk} + 4\alpha_{,l} C_{hijk} + \alpha_{,h} C_{lijk} + \alpha_{,i} C_{hljk} - \alpha_{,j} C_{klhi} + \alpha_{,k} C_{jlhi} \\ = g_{hl}\alpha_{,r} C^r{}_{ijk} - g_{il}\alpha_{,r} C^r{}_{hjk} + g_{jl}\alpha_{,r} C^r{}_{khi} - g_{kl}\alpha_{,r} C^r{}_{jhi} \\ + n(R_{hl}\Phi_{,r} C^r{}_{ijk} - R_{il}\Phi_{,r} C^r{}_{hjk} + R_{jl}\Phi_{,r} C^r{}_{khi} - R_{kl}\Phi_{,r} C^r{}_{jhi}), \end{aligned} \tag{45}$$

which, by transvection with $R^h{}_m$, in view of (15) and (13) implies

$$\alpha_{,r}R^r{}_m C_{lijk} = R_{ml}\alpha_{,r}C^r{}_{ijk}.$$

Now, putting $A_{mlijk} = R_{ml}\alpha_{,r}C^r{}_{ijk}$ and applying Lemma 1, we obtain

$$\alpha_{,r}C^r{}_{ijk} = 0. \tag{46}$$

Differentiating (17) covariantly and alternating the resulting equality in l, h, i, by (3) and (11) we obtain

$$F_{,l}C_{hijk} + F_{,h}C_{iljk} + F_{,i}C_{lhjk} = 0. \tag{47}$$

This implies

$$F_{,r}C^r{}_{ijk} = 0. \tag{48}$$

Applying the Lie derivative to (47), in view of (39) we have

$$\begin{aligned} 0 &= (L_v F_{,l})C_{hijk} + (L_v F_{,h})C_{iljk} + (L_v F_{,i})C_{lhjk} \\ &= (L_v F)_{,l}C_{hijk} + (L_v F)_{,h}C_{iljk} + (L_v F)_{,i}C_{lhjk}, \end{aligned}$$

which, together with (47), implies

$$(L_v F/F)_{,l}C_{hijk} + (L_v F/F)_{,h}C_{iljk} + (L_v F/F)_{,i}C_{lhjk} = 0. \tag{49}$$

Analogously, (11) leads to

$$(L_v \varphi_l)C_{hijk} + (L_v \varphi_h)C_{iljk} + (L_v \varphi_i)C_{lhjk} = 0. \tag{50}$$

Now, (40), (46), (48) and $(L_v F/F)_{,r}C^r{}_{ijk} = 0$, which is an immediate consequence of (49), imply

$$\Phi_{,r}C^r{}_{ijk} = 0.$$

Substituting this equality and (46) into (45), we obtain

$$\begin{aligned} 2n\big(L_v \varphi_l + (L_v F/F)_{,l}\big)C_{hijk} + 4\alpha_{,l}C_{hijk} + \alpha_{,h}C_{lijk} \\ + \alpha_{,i}C_{hljk} - \alpha_{,j}C_{klhi} + \alpha_{,k}C_{jlhi} = 0. \end{aligned} \tag{51}$$

Permuting in (51) the indices l, h, i cyclically, adding the resulting equations to (51) and making use of (49), (50) and (5), we get (41). Finally, (41) together with (5), turns (51) into

$$2n\big(L_v \varphi_l + (L_v F/F)_{,l}\big)C_{hijk} + 6\alpha_{,l}C_{hijk} = 0,$$

which, in view of $C \neq 0$ everywhere, yields (42). This completes the proof.

Remark 1. It is obvious that every essentially conformally symmetric manifold, i.e., such a conformally symmetric manifold which is neither conformally flat nor locally symmetric, is a non-locally symmetric s.c.r. manifold. Thus all the above results remain true for essentially conformally symmetric manifolds.

4. Example. We are now in a position to show the existence of CC's in non-locally symmetric s.c.r. manifolds. Let M be a non-locally symmetric s.c.r. manifold and assume that the fundamental function F of M and a CC v on M satisfy the relations

$$F = \text{const} \neq 0, \qquad L_v \varphi_l = 0.$$

Thus (42) implies $\alpha = \text{const}$, and (40) leads to

$$\Phi = \frac{\alpha(n-2)}{nF} = \text{const}.$$

Substituting this equation into (38), we obtain

$$a_{ij} = \frac{\alpha}{n}\left(g_{ij} + \frac{n-2}{F} R_{ij}\right).$$

On the other hand, if the above equation is satisfied and $\alpha = \text{const}$, then

$$a_{ij,k} = \frac{\alpha(n-2)}{nF} R_{ij,k},$$

which, together with (3), turns (20) into

$$\frac{\alpha(n-2)}{nF}(R_{ij,kl} - R_{ij,lk}) = 0.$$

Thus, taking (10) into account, we have

COROLLARY 2. *Let M be a non-locally symmetric s.c.r. manifold whose fundamental function is a non-zero constant. If v is a vector field on M such that $L_v \varphi_l = 0$, then v is a* CC *if and only if*

$$a_{ij} = \frac{\alpha}{n}\left(g_{ij} + \frac{n-2}{F} R_{ij}\right) \quad \text{and} \quad a_{rs} g^{rs} = \alpha = \text{const}. \tag{52}$$

Now, we need the following result:

LEMMA 11 ([7], Example 3). *Let M denote the Euclidean n-space ($n \geqslant 4$) endowed with metric g defined by*

$$g_{ij} = \begin{cases} -2e & \text{if } i = j = 1, \\ \exp F_i & \text{if } i + j = n+1, \\ 0 & \text{otherwise}, \end{cases}$$

where the functions $F_i = F_{n+1-i}$ are given by

$$F_1(x, y, \ldots, x^{n-1}, x^n) = G(x, y) + A(x),$$

$$F_2(x, y, \ldots) = G(x, y) + B(y), \qquad F_\lambda(x, y, \ldots) = G(x, y)$$

for $\lambda \in \{3, \ldots, n-2\}$ (empty for $n = 4$), and $e = \text{const} \neq 0$. Define functions A, B

and G by

$$G(x, y) = x + \int (\exp H) dy, \quad B = H - 2\int (\exp H) dy,$$

$$A = -\frac{2}{3}\left(x - \frac{4}{(n-2)^2}\int F dx\right),$$

where $H = H(y)$ is an arbitrary function of y only, and F is a given constant or a non-constant function of x only. Moreover, let $e = 1$. Then M is an s.c.r. manifold which is neither conformally flat nor recurrent and its fundamental function is F.

To simplify calculations we put

$$H = 0, \quad F = \frac{(n-2)^2}{4}.$$

This implies $G(x, y) = x + y$, $B = -2y$, $A = 0$ and $F_1 = x + y = F_\lambda$, $F_2 = x - y$. Thus the only non-zero components of g, the reciprocal of g, Christoffel symbols, Ricci tensor and Weyl conformal curvature tensor are those related to ([2], [7])

$$g_{11} = -2, \quad g_{1n} = g_{\lambda,n+1-\lambda} = \exp(x+y), \quad g_{2,n-1} = \exp(x-y),$$

$$g^{nn} = 2\exp(-2x-2y), \quad g^{1n} = g^{\lambda,n+1-\lambda} = \exp(-x-y),$$

$$g^{2,n-1} = \exp(-x+y),$$

$$\Gamma^1_{11} = 1, \quad \Gamma^1_{12} = \Gamma^\lambda_{2\lambda} = \Gamma^n_{2n} = \Gamma^2_{12} = \Gamma^\lambda_{1\lambda} = \Gamma^{n-1}_{1,n-1} = \tfrac{1}{2},$$

$$\Gamma^2_{22} = -1, \quad \Gamma^{n-1}_{1n}, \quad \Gamma^n_{12}, \quad \Gamma^{n-1}_{\lambda,n+1-\lambda}, \quad \Gamma^n_{11}, \quad \Gamma^n_{2,n-1}, \quad \Gamma^n_{\lambda,n+1-\lambda},$$

$$R_{12} = \frac{n-2}{4} = R_{11}, \quad R_{22} = \frac{3(2-n)}{4}, \quad C_{1212} = 1.$$

Moreover, the components of the recurrence vector φ of C are the following:

$$\varphi_1 = -3, \quad \varphi_2 = 1, \quad \varphi_3 = \ldots = \varphi_n = 0.$$

Define a vector field v by the formulas

$$v_1 = \frac{\alpha}{n} x^n \exp(x+y) + \frac{\alpha}{2n},$$

$$v_2 = \frac{\alpha}{n} x^{n-1} \exp(x-y) - \frac{3\alpha}{2n},$$

$$v_\lambda = \frac{\alpha}{2n} x^{n+1-\lambda} \exp(x+y), \quad \lambda = 3, \ldots, n-2,$$

$$v_{n-1} = v_n = 0,$$

where $\alpha = \text{const} \neq 0$. It is easy to see that $v^r \varphi_r = 0$. Moreover, using the formula

$$L_v \varphi_l = v^r \varphi_{l,r} + v^r{}_{,l} \varphi_r = (v^r \varphi_r)_{,l}$$

(since φ is a gradient), we obtain $L_v \varphi_l = 0$. By an elementary but somewhat lengthy calculation we can easily show that the only non-zero components of $a\,(a_{ij} = v_{i,j} + v_{j,i})$ are those related to

$$a_{11} = -\frac{\alpha}{n}, \quad a_{12} = \frac{\alpha}{n}, \quad a_{1n} = \frac{\alpha}{n}\exp(x+y),$$

$$a_{2,n-1} = \frac{\alpha}{n}\exp(x-y), \quad a_{\lambda,n+1-\lambda} = \frac{\alpha}{n}\exp(x+y) \text{ if } \lambda \in \{3, \ldots, n-2\}.$$

It follows now easily that

$$a_{rs} g^{rs} = \alpha \quad \text{and} \quad a_{ij} = \frac{\alpha}{n}\left(g_{ij} + \frac{4}{n-2} R_{ij}\right) = \frac{\alpha}{n}\left(g_{ij} + \frac{n-2}{F} R_{ij}\right).$$

Thus, in virtue of Corollary 2, v is CC.

Remark 2. It is worth noticing that the above CC v is an almost isometry. A vector field v on M is called an *almost isometry* [9] if

$$g^{ij}(L_v \Gamma^h_{ij}) = 0.$$

The above relation is equivalent to $a^r{}_{h,r} = \frac{1}{2}\alpha_{,h}$. Thus, taking (52), (3) and (12) into account, we see that v is an almost isometry but is not an infinitesimal isometry.

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