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Observations on quasi-linear partial differential equations

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Abstract. Some observations are given, allowing an immediate extension of certain results for quasi-linear partial differential equations to more general equations containing functionals. The procedure is exemplified with quasi-linear first order hyperbolic systems in the Schauder canonical form. However, it can also be applied to other types of quasi-linear differential equations and systems (e.g. parabolic, elliptic). The extensions stated here include in particular all the results presented in [8]–[11], [17]–[20] (see the references).

1. Introduction. We consider a differential-functional system in the Schauder canonical form:

$$(1)\qquad \sum_{j=1}^{m} A_{ij}(x, y, z(\cdot))\left[\frac{\partial z_j(x, y)}{\partial x}+\sum_{k=1}^{r} \varrho_{ik}(x, y, z(\cdot))\frac{\partial z_j(x, y)}{\partial y_k}\right] = f_i(x, y, z(\cdot)), \quad i = 1, \ldots, m,$$

with the unknowns $z(x, y):=(z_1(x, y), \ldots, z_m(x, y))$, where $y = (y_1, \ldots, y_r)$, $(x, y)\in D_a:= I_a\times \mathbf{R}^r$, $I_a:= [0, a]$, $0 < a < \infty$.

In (1) and below, the symbol $z(\cdot) = (z_1(\cdot), \ldots, z_m(\cdot))$ stands for an argument varying in a function space; thus A_{ij}, ϱ_{ik}, f_i are (given) functionals with respect to this argument (cf. e.g. [16]).

System (1) is more general than the quasi-linear systems widely investigated in the literature, e.g. in [1], [2], [4], [5], [15] (see also the references in [2], [4]), where $z(x, y)$ appears in place of $z(\cdot)$ (see system (20) below).

We shall consider (1) together with the initial conditions

$$(2)\qquad z_i(0, y) = \varphi_i(y)$$

and with the general boundary conditions

$$(3)\qquad \sum_{j=1}^{m}\sum_{k=1}^{N} b_{ijk}(y)z_j(a_k, y) = \psi_i(y),$$

$i = 1, \ldots, m$, $y\in\mathbf{R}^r$, where φ_i, ψ_i, b_{ijk} are given functions and $a_1, \ldots, a_N\in I_a$, $m \leqslant N < \infty$, are arbitrarily given numbers.

The boundary conditions (3), introduced by P. Pucci in [15], include as particular cases all the well-known conditions: Cesari's conditions ($N = m$, $b_{ijk}(y) = b_{ij}(y)\,\delta_{ik}$, δ_{ik} being the Kronecker symbol, [4], [5]), generalized Cauchy conditions (of the form $z_i(a_i, y) = \psi_i(y)$, $i = 1, \ldots, m$, [14], [16], also called the Niccoletti conditions) and, obviously, the usual Cauchy conditions (2).

Papers [8]–[11] concern differential systems with retarded arguments, while [17]–[20] generalize the results of former papers, respectively, to systems containing some operators. Further, [8], [9], [17], [18] deal with Cauchy conditions (2), whereas [10], [11], [19], [20] discuss Pucci's conditions (3) for the systems of equations treated in [8], [9], [17], [18], respectively.

We state two theorems on the existence and uniqueness of solutions and their continuous dependence on some initial or boundary data. The first theorem deals with the Cauchy problem (1), (2). It is a generalization of Cesari's result ([4], Theorem I), and includes all the theorems of [8], [9], [17], [18]. The second theorem concerns the problem (1), (3). It is in turn a generalization of a theorem by P. Bassanini ([2], Theorem 1), and includes all the results of [10], [11], [19], [20]. The second theorem can be deduced from Bassanini's in the same way as the first one from Cesari's; therefore, we omit its proof.

Our theorems also include some cases not covered in [8]–[11], [17]–[20] (see Sections 5, 7, 8). On the other hand, the theorems are only examples illustrating the pattern we give; the same pattern can be followed in many other cases.

2. Statement of the existence theorem for the Cauchy problem (1), (2). For any vector $v = (v_1, \ldots, v_d)$ we write $|v| = \max_{1 \leqslant k \leqslant d} |v_k|$. Given constants $a, a_0, \Omega > 0$, $0 < a \leqslant a_0$, we define K to be the class of all continuous vector-functions $z = (z_1, \ldots, z_m)\colon D_a \to \mathbf{R}^m$ such that $|z| \leqslant \Omega$.

ASSUMPTION H_1. 1° The real functions $\varrho_{ik}(x, y, z(\cdot))$, $f_i(x, y, z(\cdot))$, $A_{ij}(x, y, z(\cdot))$, $i, j = 1, \ldots, m$; $k = 1, \ldots, r$, are defined for $(x, y, z) \in D_{a_0} \times K$; for any fixed $(y, z) \in \mathbf{R}^r \times K$, ϱ_{ik} and f_i, considered as functions of x, are measurable;

2° there are summable functions $m_0, m_1\colon I_{a_0} \to [0, \infty)$ and constants μ, $H, H' > 0$ such that

$$|\varrho_{ik}(x, y, z(\cdot))| \leqslant m_1(x), \qquad |f_i(x, y, z(\cdot))| \leqslant m_0(x), \tag{4}$$

$$\det[A_{ij}] \geqslant \mu, \qquad |A_{ij}(x, y, z(\cdot))| \leqslant H, \qquad |\alpha_{ij}(x, y, z(\cdot))| \leqslant H' \tag{5}$$

for $(x, y, z) \in D_{a_0} \times K$, where $[\alpha_{ij}]$ is the inverse matrix to $[A_{ij}]$.

Note that the first two inequalities in (5) imply the third with some H'.

Let J be the set of all functions $\varphi = (\varphi_1, \ldots, \varphi_m)$ defined on $\mathbf{R}^r$ such that for some constants ω, $0 < \omega < \Omega$, $\Lambda \geqslant 0$, we have

$$|\varphi(y)| \leqslant \omega, \qquad |\varphi(y) - \varphi(\bar{y})| \leqslant \Lambda |y - \bar{y}| \tag{6}$$

for $y, \bar{y} \in \mathbf{R}^r$.

Choose arbitrary constants p, Q, k, R_0, R_1 with

(7) $$0 < p < 1, \quad Q > \Lambda\big(1+m^2H'H(2+p)\big), \quad 0 < k < 1,$$

(8) $$R_0 > mH', \quad R_1 > m^2H'H\Lambda(1-k)^{-1},$$

and an arbitrary summable function $\chi\colon I_{a_0} \to [0, \infty)$, such that

(9) $$\chi(x) \geqslant R_0 m_0(x) + R_1 m_1(x).$$

We denote by K_1 the class of all $z \in K$ satisfying (2) and the inequalities

(10) $$|z(x, y) - z(x, \bar{y})| \leqslant Q|y - \bar{y}|,$$

(11) $$|z(x, y) - z(\bar{x}, y)| \leqslant \Big|\int_{\bar{x}}^{x} \chi(t)\,dt\Big|,$$

for all (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in D_a$. Let

(12) $$\|z\|_x = \sup_{D_x} |z(t, y)|, \quad D_x = [0, x] \times \mathbf{R}^r.$$

ASSUMPTION H_2. 1° There are summable functions l, $l_1\colon I_{a_0} \to [0, \infty)$, such that

(13) $$\big|\varrho_{ik}(x, y, z(\cdot)) - \varrho_{ik}(x, \bar{y}, \bar{z}(\cdot))\big| \leqslant l(x)\big[|y - \bar{y}| + \|z - \bar{z}\|_x\big],$$

(14) $$\big|f_i(x, y, z(\cdot)) - f_i(x, \bar{y}, \bar{z}(\cdot))\big| \leqslant l_1(x)\big[|y - \bar{y}| + \|z - \bar{z}\|_x\big]$$

for all (x, y, z), $(x, \bar{y}, \bar{z}) \in D_{a_0} \times K_1$, $i = 1, \ldots, m$; $k = 1, \ldots, r$;

2° there exist constants $C, C' \geqslant 0$ and summable functions $m_2, m_3\colon I_{a_0} \to [0, \infty)$ such that $m_2 \geqslant C\chi$, $m_3 \geqslant C'\chi$,

(15) $$\big|A_{ij}(x, y, z(\cdot)) - A_{ij}(x, \bar{y}, \bar{z}(\cdot))\big| \leqslant C\big[|y - \bar{y}| + \|z - \bar{z}\|_x\big],$$

(16) $$\big|A_{ij}(x, y, z(\cdot)) - A_{ij}(\bar{x}, y, z(\cdot))\big| \leqslant \Big|\int_{\bar{x}}^{x} m_2(t)\,dt\Big|,$$

(17) $$\big|\alpha_{ij}(x, y, z(\cdot)) - \alpha_{ij}(x, \bar{y}, \bar{z}(\cdot))\big| \leqslant C'\big[|y - \bar{y}| + \|z - \bar{z}\|_x\big],$$

(18) $$\big|\alpha_{ij}(x, y, z(\cdot)) - \alpha_{ij}(\bar{x}, y, z(\cdot))\big| \leqslant \Big|\int_{\bar{x}}^{x} m_3(t)\,dt\Big|$$

for (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z) \in D_{a_0} \times K_1$; $i, j = 1, \ldots, m$;

3° there are constants R_2, $R_3 > 0$ such that

(19) $$\chi(x) \geqslant R_0 m_0(x) + R_1 m_1(x) + R_2\big(m_2(x) - C\chi(x)\big) + R_3\big(m_3(x) - C'\chi(x)\big).$$

In fact, (17), (18) with some C', m_3 are consequences of (5), (15), (16).

Note that if (15)–(18) hold for some $C, C' \geqslant 0$ and $m_2 = C\chi$, $m_3 = C'\chi$, then (19) is satisfied.

The following theorem is a generalization of one by Cesari ([4], Theorem I).

THEOREM 1. *If Assumptions* H_1, H_2 *are satisfied and a is sufficiently small,* $0 < a \leqslant a_0$, *then there exists in* K_1 *a unique function z which satisfies* (1) *almost everywhere in* D_a *and* (2) *everywhere in* $\mathbf{R}^r$. *Moreover, z depends continuously on* $\varphi \in J$ *(in the same sense as in* [4]).

3. Cesari's theorem. For convenience of the reader we quote Cesari's theorem. Consider the quasi-linear differential system

$$\sum_{j=1}^{m} \bar{A}_{ij}(x, y, z(x, y)) \left[\frac{\partial z_j(x, y)}{\partial x} + \sum_{k=1}^{r} \bar{\varrho}_{ik}(x, y, z(x, y)) \frac{\partial z_j(x, y)}{\partial y_k} \right] = \bar{f}_i(x, y, z(x, y)), \quad i = 1, \ldots, m, \ (x, y) \in D_a, \tag{20}$$

where

$$\bar{A}_{ij}(x, y, w), \quad \bar{\varrho}_{ik}(x, y, w), \quad \bar{f}_i(x, y, w) \tag{21}$$

(with $w \in \mathbf{R}^m$) are given functions from $D_{a_0} \times [-\bar{\Omega}, \bar{\Omega}]^m$ into $\mathbf{R}$; $\bar{\Omega} = \text{const} > 0$.

ASSUMPTIONS A. For any fixed $(y, w) \in \mathbf{R}^r \times [-\bar{\Omega}, \bar{\Omega}]^m$, $\bar{\varrho}_{ik}$, $\bar{f}_i$ viewed as functions of x, are measurable and there exist summable functions $\bar{m}_0, \ldots, \bar{m}_3$, $\bar{l}, \bar{l}_1$: $I_{a_0} \to [0, \infty)$ and constants $\bar{\mu}, \bar{H}, \bar{H}' > 0$, $\bar{C}, \bar{C}' \geqslant 0$ such that the inequalities

$$|\bar{\varrho}_{ik}(x, y, w)| \leqslant \bar{m}_1(x), \qquad |\bar{f}_i(x, y, w)| \leqslant \bar{m}_0(x), \tag{22}$$

$$|\bar{\varrho}_{ik}(x, y, w) - \bar{\varrho}_{ik}(x, \bar{y}, \bar{w})| \leqslant \bar{l}(x)[|y - \bar{y}| + |w - \bar{w}|], \tag{23}$$

$$|\bar{f}_i(x, y, w) - \bar{f}_i(x, \bar{y}, \bar{w})| \leqslant \bar{l}_1(x)[|y - \bar{y}| + |w - \bar{w}|], \tag{24}$$

$$\det[\bar{A}_{ij}(x, y, w)] \geqslant \bar{\mu}, \qquad |\bar{A}_{ij}(x, y, w)| \leqslant \bar{H}, \tag{25}$$

$$|\bar{A}_{ij}(x, y, w) - \bar{A}_{ij}(x, \bar{y}, \bar{w})| \leqslant \bar{C}[|y - \bar{y}| + |w - \bar{w}|], \tag{26}$$

$$|\bar{A}_{ij}(x, y, w) - \bar{A}_{ij}(\bar{x}, y, w)| \leqslant \left| \int_{\bar{x}}^{x} \bar{m}_2(t)\, dt \right| \tag{27}$$

hold together with the analogues of (25)–(27) for the entries $\bar{\alpha}_{ij}$ of the inverse matrix to $[\bar{A}_{ij}]$, with $\bar{H}$, $\bar{C}$, $\bar{m}_2$ replaced by $\bar{H}'$, $\bar{C}'$, $\bar{m}_3$ (the latter inequalities are in fact consequences of (25)–(27)).

Let $\bar{p}$, $\bar{Q}$, $\bar{k}$, $\bar{R}_0$, $\bar{R}_1$, $\bar{R}_2$, $\bar{R}_3$, $\bar{\Omega}$ be any constants such that

$$0 < \bar{p} < 1, \quad \bar{Q} > \Lambda(1 + m^2 \bar{H}' \bar{H}(2 + \bar{p})), \quad 0 < \bar{k} < 1,$$

$$\bar{R}_0 > m\bar{H}', \quad \bar{R}_1 > m^2 \bar{H}' \bar{H} \Lambda (1 - \bar{k})^{-1}, \quad \bar{R}_2, \bar{R}_3 > 0, \quad \bar{\Omega} > \omega.$$

Further, let $\bar{\chi}$ be any summable function such that

$$\bar{\chi}(x) \geqslant \sum_{s=0}^{3} \bar{R}_s \bar{m}_s(x), \qquad 0 \leqslant x \leqslant a_0. \tag{28}$$

(In [4], $\bar{\chi}$ is defined by equality in (28); however, the inequality is less restrictive.)

We shall denote by $\bar{K}_1$ the class of all functions $z\colon D_a \to \mathbf{R}^m$ satisfying (2) and

$$|z(x, y)| \leqslant \bar{\Omega}, \tag{29}$$

$$|z(x, y) - z(x, \bar{y})| \leqslant \bar{Q}|y - \bar{y}|, \tag{30}$$

$$|z(x, y) - z(\bar{x}, y)| \leqslant \Big|\int_{\bar{x}}^{x} \bar{\chi}(t)\, dt\Big|. \tag{31}$$

Cesari's Theorem ([4], Theorem I). *If Assumptions* A *are satisfied and a is sufficiently small,* $0 < a \leqslant a_0$, *then there exists in $\bar{K}_1$ a unique function z which satisfies* (20) *almost everywhere in D_a and* (2) *everywhere in* $\mathbf{R}^r$. *Furthermore, z depends continuously on φ.*

The proof is lengthy and consists in formal integration of the given equations along the bicharacteristics to obtain a system of integral equations. The latter system is then solved by means of the Banach fixed point theorem. Finally, it is shown that the solution of the integral equations system satisfies the given differential system almost everywhere ([1], [4]).

4. Proof of Theorem 1. Theorem 1 can be proved directly, following the lines of Cesari's proof. At the same time, the functional form of the coefficients and free terms of (1) turns out to be very convenient in notation and calculations. This makes the proof shorter than Cesari's. We show, however, that Theorem 1 follows from Cesari's theorem, and some observations concerning its proof.

Now, starting from Cesari's theorem, we first consider the composite function

$$\bar{\varrho}_{ik}(x, y, z(x, y)), \quad (x, y) \in D_a, \tag{32}$$

with $z \in \bar{K}_1$. As a function of x, it is measurable for every $y \in \mathbf{R}^r$, and the first inequality in (22) implies

$$|\bar{\varrho}_{ik}(x, y, z(x, y))| \leqslant \bar{m}_1(x) \quad \text{in } D_a \tag{33}$$

for every $z \in \bar{K}_1$. Further, (23), (30) imply

$$|\bar{\varrho}_{ik}(x, y, z(x, y)) - \bar{\varrho}_{ik}(x, \bar{y}, \bar{z}(x, \bar{y}))| \leqslant \bar{l}(x)[(1+\bar{Q})|y-\bar{y}| + \|z - \bar{z}\|_x] \tag{34}$$

for all $(x, y), (x, \bar{y}) \in D_a$ and all $z, \bar{z} \in \bar{K}_1$. (Note that from (34), by taking for $z, \bar{z}$ constant functions, we get (23) with $|y - \bar{y}|$ replaced by $(1+\bar{Q})|y-\bar{y}|$.)

Now, from Cesari's proof ([4], see also [1]) it is immediately seen that instead of the two separate conditions (23), (30) applied to the function (32), the condition (34) alone (or its particular forms) can be used, which is even more convenient and does not influence the result at all. Moreover, any other property of the function (32) (besides (33), (34) and the measurability in x) is irrelevant to the proof of Cesari's theorem. One can write, e.g.,

$$\varrho_{ik}(x, y, z(\cdot)) := \bar{\varrho}_{ik}(x, y, z(x, y)) \tag{35}$$

and consider (32) as a function from $D_a \times \bar{K}_1$ into $\mathbf{R}$. It follows that Cesari's theorem remains valid if in (20) the coefficient (32) is replaced by $\varrho_{ik}(x, y, z(\cdot))$, where ϱ_{ik} denotes any real function defined on $D_{a_0} \times \bar{K}_1$, measurable in x and satisfying, for (x, y, z), $(x, \bar{y}, \bar{z}) \in D_a \times \bar{K}_1$, the conditions

$$|\bar{\varrho}_{ik}(x, y, z(\cdot))| \leqslant \bar{m}_1(x), \tag{36}$$

$$|\varrho_{ik}(x, y, z(\cdot)) - \varrho_{ik}(x, \bar{y}, \bar{z}(\cdot))| \leqslant \bar{l}(x)[(1+\bar{Q})|y-\bar{y}| + \|z-\bar{z}\|_x] \tag{37}$$

which correspond to (33), (34).

Exactly the same is true of the terms $\bar{f}_i(x, y, z(x, y))$ appearing in (20). By the same argument as above, we deduce that Cesari's theorem remains valid if in (20) these terms are replaced by $f_i(x, y, z(\cdot))$, where each f_i is measurable in x and satisfies

$$|f_i(x, y, z(\cdot))| \leqslant \bar{m}_0(x), \tag{38}$$

$$|f_i(x, y, z(\cdot)) - f_i(x, \bar{y}, \bar{z}(\cdot))| \leqslant \bar{l}_1(x)[(1+\bar{Q})|y-\bar{y}| + \|z-\bar{z}\|_x] \tag{39}$$

for all (x, y, z), $(x, \bar{y}, \bar{z}) \in D_a \times \bar{K}_1$.

For $\bar{A}_{ij}$ the situation is similar. First of all we note that if $z, \bar{z} \in \bar{K}_1$, then by (25)–(27), (29)–(31) we have

$$\det[\bar{A}_{ij}(x, y, z(x, y))] \geqslant \bar{\mu}, \quad |\bar{A}_{ij}(x, y, z(x, y))| \leqslant \bar{H}, \tag{40}$$

$$|\bar{A}_{ij}(x, y, z(x, y)) - \bar{A}_{ij}(x, \bar{y}, \bar{z}(x, \bar{y}))| \leqslant \bar{C}[(1+\bar{Q})|y-\bar{y}| + \|z-\bar{z}\|_x], \tag{41}$$

$$|\bar{A}_{ij}(x, y, z(x, y)) - \bar{A}_{ij}(\bar{x}, y, z(\bar{x}, y))| \leqslant \left|\int_{\bar{x}}^{x} [\bar{m}_2(t) + \bar{C}\bar{\chi}(t)]\, dt\right|. \tag{42}$$

Similar inequalities hold for the entries $\bar{\alpha}_{ij}$ of the inverse matrix to $[\bar{A}_{ij}]$, with $\bar{H}$, $\bar{C}$, $\bar{m}_2$ replaced by $\bar{H}'$, $\bar{C}'$, $\bar{m}_3$, respectively.

Further, one can easily verify that in Cesari's proof the set of conditions (26), (30), (27), (31) on the composite functions

$$\bar{A}_{ij}(x, y, z(x, y)) \tag{43}$$

can be replaced by the two conditions (41), (42) ([1]). We also observe that no other properties of the functions (43) (besides conditions (40)–(42) and their counterparts for $\bar{\alpha}_{ij}(x, y, z(x, y))$) are needed for Cesari's proof.

Hence, we deduce that Cesari's theorem remains true if in (20) the coefficients (43) are replaced by $A_{ij}(x, y, z(\cdot))$, under the assumption that

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([1]) Note that making a direct use of (41) and (42) enables us to improve the estimate of the derivative $(d/d\xi)\bar{A}_{sh}$ on page 324 of [4]; viz., we obtain

$$\left|\frac{d}{d\xi}\bar{A}_{sh}\right| \leqslant \bar{m}_2(\xi) + \bar{C}\bar{\chi}(\xi) + r\bar{C}(1+\bar{Q})\bar{m}_1(\xi). \tag{44}$$

A_{ij} are real functions on $D_a \times \bar{K}_1$ which satisfy the conditions

$$(45)\qquad \det[A_{ij}(x, y, z(\cdot))] \geqslant \bar{\mu}, \quad |A_{ij}(x, y, z(\cdot))| \leqslant \bar{H},$$

$$(46)\qquad |A_{ij}(x, y, z(\cdot)) - A_{ij}(x, \bar{y}, \bar{z}(\cdot))| \leqslant \bar{C}[(1+\bar{Q})|y-\bar{y}| + \|z-\bar{z}\|_x],$$

$$(47)\qquad |A_{ij}(x, y, z(\cdot)) - A_{ij}(\bar{x}, y, z(\cdot))| \leqslant \Big|\int_{\bar{x}}^{x} [\bar{m}_2(t) + \bar{C}\bar{\chi}(t)]\, dt\Big|$$

corresponding to (40)–(42), and similarly

$$(48)\qquad |\alpha_{ij}(x, y, z(\cdot))| \leqslant \bar{H}',$$

$$(49)\qquad |\alpha_{ij}(x, y, z(\cdot)) - \alpha_{ij}(x, \bar{y}, \bar{z}(\cdot))| \leqslant \bar{C}'[(1+\bar{Q})|y-\bar{y}| + \|z-\bar{z}\|_x],$$

$$(50)\qquad |\alpha_{ij}(x, y, z(\cdot)) - \alpha_{ij}(\bar{x}, y, z(\cdot))| \leqslant \Big|\int_{\bar{x}}^{x} [\bar{m}_3(t) + \bar{C}'\bar{\chi}(t)]\, dt\Big|$$

for (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z) \in D_a \times \bar{K}_1$.

(Note that replacing in (13)–(15), (17) the term $|y-\bar{y}|$ by $(1+Q)|y-\bar{y}|$, that is, making weaker assumptions, we would get conditions precisely adjusted to (37), (39), (46), (49), respectively.) Note finally that the set of conditions (16), (18), (19) corresponds exactly to the set of conditions (47), (50), (28).

Thus, by manipulations with Cesari's theorem, we arrive at Theorem 1. The magnitude of a in Theorem 1 can also be determined by the Cesari's theorem.

Remark 1. From assumption (13) ((14), (15)) it follows that the functions ϱ_{ik} (f_i, A_{ij} respectively) satisfy the following condition (which is sometimes called a Volterra condition): if $z, \bar{z} \in K_1$ and $z(t, y) = \bar{z}(t, y)$ for all $(t, y) \in D_x$, then $\varrho_{ik}(t, y, z(\cdot)) = \varrho_{ik}(t, y, \bar{z}(\cdot))$ for all $(t, y) \in D_x$, x being any number in $[0, a]$.

5. Corollaries to Theorem 1. We consider the system (cf. [8]–[11], [17]–[20])

$$(51)\qquad \sum_{j=1}^{m} \hat{A}_{ij}(x, y, z(x, y), (V_{ij}^{(1)}z)(x, y)) \Big[\frac{\partial z_j(x, y)}{\partial x} + \sum_{k=1}^{r} \hat{\varrho}_{ik}(x, y, z(x, y), (V_{ik}^{(2)}z)(x, y)) \frac{\partial z_j(x, y)}{\partial y_k}\Big] = \hat{f}_i(x, y, z(x, y), (V_i^{(3)}z)(x, y)), \quad i = 1, \ldots, m, \ (x, y) \in D_a,$$

where $\hat{A}_{ij}$, $\hat{\varrho}_{ik}$, $\hat{f}_i$ are given functions from $D_{a_0} \times \mathbf{R}^m \times \mathbf{R}^n$ into $\mathbf{R}$, and $V_{ij}^{(1)}$, $V_{ik}^{(2)}$, $V_i^{(3)}$: $S^m \to S^n$ are given operators, S^l being the set of all l-vector-valued functions defined on D_a.

Considering the coefficients and free terms of (51) as the corresponding functionals appearing in (1), we make assumptions on $\hat{A}_{ij}$, $\hat{\varrho}_{ik}$, $\hat{f}_i$ and the operators in order to make use of Theorem 1.

ASSUMPTION H_3. The functions

(52) $$\hat{A}_{ij}(x, y, w, v), \quad \hat{\varrho}_{ik}(x, y, w, v), \quad \hat{f}_i(x, y, w, v)$$

are defined on $D_{a_0} \times [-\Omega, \Omega]^m \times [-\Omega, \Omega]^n$, $\Omega > 0$. For any fixed y, w, v, $\hat{\varrho}_{ik}$, $\hat{f}_i$, treated as functions of x, are measurable. Furthermore, there exist summable functions $m_0, m_1, \hat{m}_2, \hat{m}_3, \hat{l}, \hat{l}_1\colon I_{a_0} \to [0, \infty)$ and constants μ, $H, H' > 0$, $\hat{C}, \hat{C}' \geq 0$ such that, in the domain of the functions (52), the inequalities

(53) $$|\hat{\varrho}_{ik}(x, y, w, v)| \leq m_1(x), \quad |\hat{f}_i(x, y, w, v)| \leq m_0(x),$$

(54) $$|\hat{\varrho}_{ik}(x, y, w, v) - \hat{\varrho}_{ik}(x, \bar{y}, \bar{w}, \bar{v})| \leq \hat{l}(x)[|y-\bar{y}| + |w-\bar{w}| + |v-\bar{v}|],$$

(55) $$|\hat{f}_i(x, y, w, v) - \hat{f}_i(x, \bar{y}, \bar{w}, \bar{v})| \leq \hat{l}_1(x)[|y-\bar{y}| + |w-\bar{w}| + |v-\bar{v}|],$$

(56) $$\det[\hat{A}_{ij}(x, y, w, v)] \geq \mu, \quad |\hat{A}_{ij}(x, y, w, v)| \leq H,$$

(57) $$|\hat{A}_{ij}(x, y, w, v) - \hat{A}_{ij}(x, \bar{y}, \bar{w}, \bar{v})| \leq \hat{C}[|y-\bar{y}| + |w-\bar{w}| + |v-\bar{v}|],$$

(58) $$|\hat{A}_{ij}(x, y, w, v) - \hat{A}_{ij}(\bar{x}, y, w, v)| \leq \left|\int_{\bar{x}}^{x} \hat{m}_2(t)\, dt\right|$$

hold together with the analogous of (56)–(58) for the entries $\hat{\alpha}_{ij}$ of the inverse matrix to $[\hat{A}_{ij}]$, with H, $\hat{C}$, $\hat{m}_2$ replaced by H', $\hat{C}'$, $\hat{m}_3$.

We define

(59) $$\hat{\chi}(x) := R_0 m_0(x) + R_1 m_1(x) + R_2 \hat{m}_2(x) + R_3 \hat{m}_3(x),$$

where $R_0, \ldots, R_3$ are any numbers such that

$$R_0 > mH', \quad R_1 > m^2 H' H \Lambda (1-k)^{-1}, \quad 0 < k < 1,$$

$R_2, R_3 > 0$, and Λ is taken from (6).

Let $\chi\colon I_{a_0} \to [0, \infty)$ be any summable function satisfying

(60) $$\chi(x) \geq b\hat{\chi}(x), \quad x \in I_{a_0},$$

with some $b = \text{const} > 1$. Let Q be any constant such that

$$Q > \Lambda\big(1 + m^2 H' H (2+p)\big), \quad 0 < p < 1.$$

We denote by $\hat{K}$ the class of all functions $z\colon D_a \to \mathbf{R}^m$, satisfying (2), the inequality $|z(x, y)| \leq \Omega$, and inequalities (10), (11) with Q, χ defined above.

ASSUMPTION H_4. There exist constants r_k, Q_k, $q_1 \geq 0$ ($k = 1, 2, 3$) such that for all $z, \bar{z} \in \hat{K}$ and (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in D_a$ we have

(61) $$|(V^{(k)} z)(x, y) - (V^{(k)} \bar{z})(x, y)| \leq r_k \|z - \bar{z}\|_x,$$

(62) $$|(V^{(k)}z)(x, y)| \leqslant \Omega,$$

(63) $$|(V^{(k)}z)(x, y)-(V^{(k)}z)(x, \bar{y})| \leqslant Q_k|y-\bar{y}| \quad (k = 1, 2, 3),$$

(64) $$|(V^{(1)}z)(x, y)-(V^{(1)}z)(\bar{x}, y)| \leqslant q_1\Big|\int_{\bar{x}}^{x} \chi(t)\,dt\Big|,$$

(65) $$(V^{(k)}z)(x, y) \quad (k = 2, 3) \quad \text{are measurable in } x;$$

here and below, $V^{(1)}$ ($V^{(2)}, V^{(3)}$) denotes each of the operators $V_{ij}^{(1)}$, $i, j = 1, \ldots, m$ ($V_{ik}^{(2)}$, $V_i^{(3)}$, respectively, $i = 1, \ldots, m$; $k = 1, \ldots, r$).

Corollary 1. *Under Assumptions* H_3, H_4, *there exist a number* $a \in (0, a_0]$ *and a function* $z \in \hat{K}$ *which satisfies* (51) *almost everywhere in* D_a *and* (2) *everywhere in* $\mathbf{R}^r$. *Furthermore, z is unique in* $\hat{K}$ *and depends continuously on* φ.

Proof. Set

$$\varrho_{ik}(x, y, z(\cdot)) = \hat{\varrho}_{ik}(x, y, z(x, y), (V_{ik}^{(2)}z)(x, y)).$$

Owing to (54), (10), (63) and (61) we have

$$\big|\varrho_{ik}(x, y, z(\cdot))-\varrho_{ik}(x, \bar{y}, \bar{z}(\cdot))\big| \leqslant \hat{l}(x)[(1+Q+Q_2)|y-\bar{y}|+(1+r_2)\|z-\bar{z}\|_x].$$

Therefore, inequality (13) is satisfied with

$$l(x) = \hat{l}(x)C_2, \quad \text{where } C_k := \max(1+Q+Q_k,\ 1+r_k).$$

Similarly, it can be shown that, under suitable notation of the functionals, inequalities (14)–(18) hold for

$$l_1(x) = \hat{l}_1(x)C_3, \quad C = \hat{C}C_1, \quad C' = \hat{C}'C_1,$$

(66) $$m_2(x) = \hat{m}_2(x)+C(1+q_1)\chi(x), \quad m_3(x) = \hat{m}_3(x)+C'(1+q_1)\chi(x).$$

In order to show that condition (19) is satisfied, note first that in view of (66) this condition takes the form

(67) $$\chi(x) \geqslant \hat{\chi}(x)+(R_2C+R_3C')q_1\chi(x).$$

Decreasing R_2, R_3 if necessary, we have

(68) $$(R_2C+R_3C')q_1 \leqslant 1-b^{-1}.$$

Thus inequality (60) remains valid implying (67), that is, (19). Hence Theorem 1 yields Corollary 1.

Now, we deal with a system with retarded argument, i.e. system (51) with (for simplicity)

(69) $$(V^{(k)}z)(x, y) = z(\alpha_k(x, y), \beta_k(x, y)) \quad (k = 1, 2, 3),$$

where $\alpha_k\colon D_a \to I_a$, $\beta_k = (\beta_{k1}, \ldots, \beta_{kr})\colon D_a \to \mathbf{R}^r$ are given functions.

Let us retain Assumption H_3 and the definition of the class $\hat{K}$. We make

ASSUMPTION H_5. 1° There are constants b_k, $\bar{b}_1$ such that

(70) $$|\beta_k(x, y) - \beta_k(x, \bar{y})| \leqslant b_k|y-\bar{y}| \quad (k = 1, 2, 3),$$

(71) $$|\beta_1(x, y) - \beta_1(\bar{x}, y)| \leqslant \bar{b}_1 \Big|\int_{\bar{x}}^{x} \chi(t)\,dt\Big|$$

for (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in D_a$. The functions $\beta_2(x, y)$, $\beta_3(x, y)$ are measurable in x for every $y \in \mathbf{R}^r$;

2° the functions $\alpha_k(x, y)$ $(k = 1, 2, 3)$ are continuous in y for every $x \in I_a$, absolutely continuous in each of the variables $y_1, \ldots, y_r$ separately, $\alpha_k(x, y) \leqslant x$ and

(72) $$\chi(\alpha_k(x, y))\left|\frac{\partial \alpha_k(x, y)}{\partial y_s}\right| \leqslant N_k \quad (s = 1, \ldots, r;\ k = 1, 2, 3)$$

for all $x \in I_a$ and almost all $y \in \mathbf{R}^r$, where $N_k \geqslant 0$ are some constants. The function $\alpha_1(x, y)$ is absolutely continuous in x for any $y \in \mathbf{R}^r$, and

(73) $$\chi(\alpha_1(x, y))\left|\frac{\partial \alpha_1(x, y)}{\partial x}\right| \leqslant \bar{N}\chi(x)$$

for every $y \in \mathbf{R}^r$ and almost every $x \in I_a$, where $\bar{N} \geqslant 0$ is a constant. The functions $\alpha_2(x, y)$, $\alpha_3(x, y)$ are measurable in x for any $y \in \mathbf{R}^r$.

COROLLARY 2. *If a is sufficiently small and Assumptions H_3, H_5 are satisfied, then there exists in $\hat{K}$ a unique function z which satisfies* (51) (*with $V^{(k)}$ given by* (69)) *almost everywhere in D_a and* (2) *everywhere in $\mathbf{R}^r$. Moreover, z depends continuously on φ.*

Proof. By Corollary 1 it is sufficient to verify that operators (69) satisfy Assumption H_4. Evidently, inequalities (61) hold for $r_k = 1$. Since $z \in \hat{K}$, (62) holds as well. Let

$$\tilde{y}_s = (\bar{y}_1, \ldots, \bar{y}_{s-1}, \bar{y}_s, y_{s+1}, \ldots, y_r), \quad \tilde{y}_0 = y$$

$(s = 0, \ldots, r)$. Applying (11), (10), (70), (72) and a theorem on the change of the integration variable [7], we deduce that

$$\begin{aligned} &|z(\alpha_k(x, y), \beta_k(x, y)) - z(\alpha_k(x, \bar{y}), \beta_k(x, \bar{y}))| \\ &\qquad \leqslant \sum_{s=1}^{r} \Big|\int_{\alpha_k(x,\tilde{y}_s)}^{\alpha_k(x,\tilde{y}_{s-1})} \chi(t)\,dt\Big| + Qb_k|y-\bar{y}| \\ &\qquad \leqslant \sum_{s=1}^{r} N_k|y_s - \bar{y}_s| + Qb_k|y-\bar{y}| \leqslant (rN_k + Qb_k)|y-\bar{y}|. \end{aligned}$$

Therefore (63) are satisfied with $Q_k = rN_k + Qb_k$. Similarly, (11), (10), (71), (73) and the theorem on the change of the integration variable imply

$$|z(\alpha_1(x, y), \beta_1(x, y)) - z(\alpha_1(\bar{x}, y), \beta_1(\bar{x}, y))|$$
$$\leqslant \Big|\int_{\alpha_1(\bar{x},y)}^{\alpha_1(x,y)} \chi(t)\,dt\Big| + Q\bar{b}_1 \Big|\int_{\bar{x}}^{x} \chi(t)\,dt\Big| \leqslant (\bar{N} + Q\bar{b}_1)\Big|\int_{\bar{x}}^{x} \chi(t)\,dt\Big|.$$

This means that (64) holds for $q_1 = \bar{N} + Q\bar{b}_1$. By known facts on the measurability of composite functions, (65) is true as well, which completes the proof.

6. Boundary value problem (1), (3). As in [2] we write

$$A_{ij}(x, y, z(\cdot)) = \delta_{ij} + \tilde{A}_{ij}(x, y, z(\cdot)),$$
$$\alpha_{ij}(x, y, z(\cdot)) = \delta_{ij} + \tilde{\alpha}_{ij}(x, y, z(\cdot)),$$
$$b_{ijk}(y) = \delta_{ij} \cdot \delta_{ik} + \tilde{b}_{ijk}(y)$$

and set

$$\sigma_1 = \max_i \Big[\sup_{D_{a_0} \times K} \sum_{j=1}^{m} |\tilde{A}_{ij}(x, y, z(\cdot))| \Big],$$

$$\sigma_2 = \max_i \Big[\sup_{D_{a_0} \times K} \sum_{j=1}^{m} |\tilde{\alpha}_{ij}(x, y, z(\cdot))| \Big],$$

$$\sigma_0 = \max_i \Big[\sup_{y \in \mathbf{R}^r} \sum_{j=1}^{m} \sum_{k=1}^{N} |\tilde{b}_{ijk}(y)| \Big],$$

K being the class defined in Sec. 2.

ASSUMPTION H_6. 1° $\zeta := (\sigma_0 + \sigma_1)(1 + \sigma_2) < 1$;

2° there are constants $\Lambda_0 \geqslant 0$, $\tau_0 \geqslant 0$ such that for all $y, \bar{y} \in \mathbf{R}^r$ and $i = 1, \ldots, m$, we have

$$|\psi_i(y) - \psi_i(\bar{y})| \leqslant \Lambda_0 |y - \bar{y}|, \qquad \sum_{j=1}^{m} \sum_{k=1}^{N} |b_{ijk}(y) - b_{ijk}(\bar{y})| \leqslant \tau_0 |y - \bar{y}|.$$

Now we define a basic function class K_1. Let $k \in (\zeta, 1)$ and let C, C', ω_0 be positive constants, $\omega_0 < \Omega(1-\zeta)(1+\sigma_2)^{-1}$. The products $C\omega_0$, $C'\omega_0$ are assumed to be so small that

(74) $$\delta_0(\varrho_0) := m[C'\omega_0 + C'\varrho_0(\sigma_0 + \sigma_1) + C\varrho_0(1 + \sigma_2)] < k - \zeta,$$

where $\varrho_0 := (1 + \sigma_2)(1 - \zeta)^{-1}\omega_0$ $(< \Omega)$ (²).

(²) We make use of the improved estimate (44). Then assumption (a_3) of [2] takes the form $\delta_0(\varrho_0) < 1$ and becomes superfluous because of assumption (a_2) (having the same form as (74)).

Let $\tilde{\varrho}$ be any constant such that $\varrho_0 < \tilde{\varrho} \leqslant \Omega$ and $\delta_0(\tilde{\varrho}) < k - \zeta$ (3). There is $p \in (0, 1)$ such that

$$\tilde{\alpha}_0(\tilde{\varrho}) := (1+p)\zeta + \delta_0(\tilde{\varrho}) < 1. \tag{75}$$

We choose an arbitrary constant

$$Q > \tilde{\beta}_0 (1 - \tilde{\alpha}_0(\tilde{\varrho}))^{-1} \tag{76}$$

with

$$\tilde{\beta}_0 := (1+p)(1+\sigma_2)(A_0 + \tilde{\varrho}\tau_0) + \delta_0(\tilde{\varrho}). \tag{77}$$

Further, let $\chi\colon I_{a_0} \to [0, \infty)$ be an arbitrary summable function such that

$$\chi(x) \geqslant R_0 m_0(x) + R_1 m_1(x),$$

where m_0, m_1 are the functions appearing in Assumption H_1, which we adopt here, while R_0, R_1 are any constants satisfying the inequalities

$$\begin{gathered} R_0 > (1+\sigma_2)(1-\delta_0(\tilde{\varrho}))^{-1}, \\ R_1 > (1+\sigma_2)(1-\delta_0(\tilde{\varrho}))^{-1}[A_0 + \tilde{\varrho}\tau_0 + (\sigma_0+\sigma_1)Q + \tilde{\varrho} m C(r+1)(1+Q)]. \end{gathered} \tag{78}$$

Now we denote by K_1 the class of all functions $z\colon D_a \to \mathbf{R}^m$, such that $|z| \leqslant \tilde{\varrho}$ and inequalities (10), (11) (with Q, χ chosen above) are satisfied.

THEOREM 2. *Suppose that Assumptions* H_1 *and* H_6 *hold. We assume that* $|\psi_i(y)| \leqslant \omega_0$ $(y \in \mathbf{R}^r,\ i = 1, \ldots, m)$ *and that Assumption* H_2 *is satisfied for* C, C', ω_0, K_1, χ, R_0, R_1 *defined in this section and for some constants* R_2, R_3 *such that*

$$R_2 > m\tilde{\varrho}(1+\sigma_2)(1-\delta_0(\tilde{\varrho}))^{-1}, \quad R_3 > m[\omega_0 + \tilde{\varrho}(\sigma_0+\sigma_1)](1-\delta_0(\tilde{\varrho}))^{-1}. \tag{79}$$

Then, for sufficiently small $a \in (0, a_0]$ *and for any* $a_1, \ldots, a_N \in [0, a]$ *there exists in* K_1 *a unique function* z *satisfying* (1) *almost everywhere in* D_a *and* (3) *everywhere in* $\mathbf{R}^r$. *This function depends continuously on* ψ_i *in the same sense as in* [2], [4].

Theorem 2 can be deduced from Theorem 1 of [2] in the same way as our Theorem 1 from Theorem I of [4]; there is no need to repeat the procedure.

7. Corollaries to Theorem 2. First we consider the problem (51), (3). Let

$$\hat{A}_{ij}(x, y, w, v) = \delta_{ij} + A'_{ij}(x, y, w, v),$$

$$\hat{\alpha}_{ij}(x, y, w, v) = \delta_{ij} + \alpha'_{ij}(x, y, w, v).$$

---

(3) We introduce $\tilde{\varrho}$ which replaces ϱ appearing in [2]. This change is inessential for [2]. In our case it simplifies the formulation of the results below.

Set

$$(80) \qquad \sigma_1 = \max_i \Big[\max_{D_{a_0}\times\bar{\Omega}} \sum_{j=1}^{m} |A'_{ij}(x, y, w, v)| \Big],$$

$$(81) \qquad \sigma_2 = \max_i \Big[\max_{D_{a_0}\times\bar{\Omega}} \sum_{j=1}^{m} |\alpha'_{ij}(x, y, w, v)| \Big],$$

where $\bar{\Omega} = [-\Omega, \Omega]^{m+n}$.

Let us formulate some assumptions needed for Corollary 3 below. We retain Assumption H_6 (with σ_1, σ_2 understood according to (80), (81)).

Now we introduce a function class $\hat{K}$. As in Sec. 6, choosing any $k \in (\zeta, 1)$, $C, C', \omega_0 > 0$, $\omega_0 < \Omega(1-\zeta)(1+\sigma_2)^{-1}$ we assume that the products $C\omega_0$, $C'\omega_0$ are so small that (74) holds. Then we take any $\tilde{\varrho} \in (\varrho_0, \Omega]$ such that $\delta_0(\tilde{\varrho}) < k-\zeta$, and, as in Sec. 6, define Q by (76). Further, let $\hat{\chi}$ be given by (59) with any numbers $R_0, \ldots, R_3$ satisfying (78), (79), and let $\chi\colon I_{a_0} \to [0, \infty)$ be any summable function such that

$$(82) \qquad \chi(x) \geqslant b\hat{\chi}(x), \qquad b > 1.$$

Now $\hat{K}$ will stand for the class of all functions $z\colon D_a \to \mathbf{R}^m$ such that $|z| \leqslant \tilde{\varrho}$ and (10), (11) (with Q, χ defined by (76), (82)) are satisfied.

In the sequel, we retain Assumption H_4 provided that $\hat{K}$ is understood as in this section.

ASSUMPTION H_7. Suppose C, C', ω_0 are not greater than in the definition of $\hat{K}$ and satisfy (aside from (74)) the inequality

$$(83) \qquad \delta_0(\varrho_0) < (b-1)((1+q_1)b-1)^{-1}.$$

Assume $|\psi_i(y)| \leqslant \omega_0$.

COROLLARY 3. *Let Assumptions* H_6, H_4, H_7 *(with* $\sigma_1, \sigma_2, \hat{K}$ *redefined in this section) be satisfied. Moreover, let Assumption* H_3 *hold for* $\hat{C} = CC_1^{-1}$, $\hat{C}' = C'C_1^{-1}$, *where* $C_1 = \max(1+Q+Q_1, 1+r_1)$. *If* a *is sufficiently small,* $0 < a \leqslant a_0$, *then for any* $a_1, \ldots, a_N \in [0, a]$ *there exists a unique function* $z \in \hat{K}$ *satisfying* (51) *almost everywhere in* D_a *and* (3) *everywhere in* $\mathbf{R}^r$. *This function depends continuously on* ψ_i.

Proof. This corollary will be derived from Theorem 2. Obviously Assumptions H_1, H_6 are fulfilled. As in the proof of Corollary 1, applying Assumptions H_3 and H_4 one can show that (13)–(18) hold for $C = \hat{C}C_1$, $C' = \hat{C}'C_1$ and m_2, m_3 defined by (66).

As we know, in order to prove (19) it is sufficient to prove (67). Decreasing $\tilde{\varrho}$ (in $(\varrho_0, \Omega]$) if necessary (which does not extend the $\hat{K}$) we may assume that (83) holds with ϱ_0 replaced by $\tilde{\varrho}$. The latter inequality is equivalent to

$$q_1\delta_0(\tilde{\varrho})(1-\delta_0(\tilde{\varrho}))^{-1} < 1-b^{-1}.$$

By (79),

$$R_2 C + R_3 C' > \delta_0(\tilde{\varrho})(1-\delta_0(\tilde{\varrho}))^{-1}.$$

Hence, decreasing R_2, R_3 if necessary (in such a way that they still satisfy (79)) one can get

$$q_1 \delta_0(\tilde{\varrho})(1-\delta_0(\tilde{\varrho}))^{-1} < (R_2 C + R_3 C') q_1 \leqslant 1 - b^{-1}. \tag{84}$$

This procedure preserves inequality (82), which, in view of (84), implies (67). Therefore Assumption H_2 (with K_1 replaced by $\hat{K}$) is satisfied. This completes the proof.

The final corollary concerns a system with retarded argument, i.e. we treat the problem (51), (3) with the operators $V^{(k)}$ given by (69).

COROLLARY 4. *Let Assumption* H_6 (*with* σ_1, σ_2 *defined by* (80), (81)) *be satisfied. We retain the definitions of* Q, χ *and* $\hat{K}$ *given in this section. Further, let Assumptions* H_5 (*with* Q, χ *defined here*) *and* H_7, *with* $q_1 = \bar{N} + Q\bar{b}_1$, *hold. Finally, suppose Assumption* H_3 *holds for* $\hat{C} = CC_0^{-1}$, $\hat{C}' = C'C_0^{-1}$, $C_0 = \max(1 + Q + rN_1 + Qb_1, 2)$.

Under these assumptions, for sufficiently small $a \in (0, a_0]$ *and for any* $a_1, \ldots, a_N \in [0, a]$, *there exists a unique function* $z\colon D_a \to \mathbf{R}^m$ *satisfying* (51) (*with* $V^{(k)}$ *given by* (69)) *almost everywhere in* D_a *and* (3) *everywhere in* $\mathbf{R}^r$, *and belonging to* $\hat{K}$. *The function* z *depends continuously on* ψ_i.

The proof is an easy combination of the proofs of Corollaries 2 and 3; we omit the details.

8. Final remarks.

Remark 2. In [8], [10], [17], [19] the system (51) was treated in the particular case when the functions $\hat{A}_{ij}$ do not depend on the last variable, i.e. the $V_{ij}^{(1)}$ do not occur in any form. On the other hand, in [9], [11], [18], [20] the operators $V_{ij}^{(1)}$ are allowed to appear (in [9], [11], as in [8], [10], the operators $V^{(k)}$ ($k = 1, 2, 3$) are of the form (69)), but the assumptions on all functions $\hat{A}_{ij}$, $\hat{\varrho}_{ik}$, $\hat{f}_i$ are stronger than those in [8], [10], [17], [19]. Our corollaries join together these two kinds of results under considerably weaker assumptions.

Let us give the following example, which, in particular, reveals the generality of conditions (72), (73).

EXAMPLE. Suppose that

$$m_0(x),\ m_1(x),\ \hat{m}_2(x),\ \hat{m}_3(x) \leqslant \varkappa_0 \cdot x^{\varkappa}, \qquad 0 \leqslant x \leqslant 1,$$

with some constants $\varkappa_0 > 0$, $\varkappa > -1$. Then the function $\chi(x) = \varkappa_1 x^{\varkappa}$ with large $\varkappa_1 > 0$ satisfies (60) (and Corollaries 1 and 3 hold). Additionally we choose

$$\alpha_i(x, y) = x^{\lambda_i} |\sin y|^{\nu_i}, \qquad \lambda_i \geqslant 1,\ \nu_i \geqslant 1/(\varkappa + 1),$$

$i = 1, 2, 3$; $r = 1$. Then inequalities (72) (with $N_k = \varkappa_1 \nu_k$) and (73) (with $\bar{N} = \lambda_1$) are satisfied (and Corollaries 2 and 4 hold).

On the other hand, no case when $\hat{m}_2$ is not essentially bounded (as in this example) and, at the same time, a certain $V_{ij}^{(1)}$ is not the identity operator, is treated in [8]–[11], [17]–[20].

Remark 3. One of the purposes of this paper is to present a method allowing a generalization of some results known for quasi-linear equations to the corresponding equations whose coefficients (and/or free terms) are functionals. The method exemplified by the proof of Theorem 1 is based, roughly, on the following observation. The coefficients of a quasi-linear equation are composite functions of a given function and the unknown one. Each of the two functions is assumed to satisfy some conditions. In the proofs, however, it is often sufficient to use the conditions satisfied by the composite function only. In such cases the composite function can be replaced by another function having the required properties, e.g. by a certain functional. Let us mention several further examples.

Remark 4. In the way described above also some other results, e.g. Theorem II of [4] and the results of [15], can be generalized to systems with functional coefficients. Such a generalization is especially easy for the results of [3], [5], concerning system (20) with $\bar{A}_{ij} = \delta_{ij}$ (the first canonical form).

In the same simple manner, starting from the existence theorem of [13] related to the so-called mixed problem for system (20) with $\bar{A}_{ij} = \delta_{ij}$, $r = 1$, one can get a corresponding theorem for system (1) with $A_{ij} = \delta_{ij}$, $r = 1$, containing the result of paper [21] which is devoted to the mixed problem for system (51) with $\hat{A}_{ij} = \delta_{ij}$, $r = 1$.

Note finally that certain results for quasi-linear differential equations (or systems) of parabolic or elliptic type, found in e.g. [6], [12], can be immediately transferred, in the way outlined in Remark 3, to the corresponding equations with functional coefficients.

Remark 5. It is obvious that each system of the form (51) is a system of the form (1). Note that, conversely, each system of the form (1) can be represented (in many different ways) in the form (51). The simplest way is to choose, in (51), the particular functions

$$\hat{A}_{ij}(x, y, w, v) \equiv v, \qquad \hat{\varrho}_{ik}(x, y, w, v) \equiv v,$$
$$\hat{f}_i(x, y, w, v) \equiv v, \qquad i, j = 1, \ldots, m;\ k = 1, \ldots, r$$

and $n = 1$ (i.e., v is a scalar variable). Then

$$\hat{A}_{ij}\big(x, y, z(x, y), (V_{ij}^{(1)}z)(x, y)\big) \equiv (V_{ij}^{(1)}z)(x, y),$$

and similar relations hold for $\hat{\varrho}_{ik}$, $\hat{f}_i$. Clearly each function $A_{ij}\big(x, y, z(\cdot)\big)$ is an operator of type $(V_{ij}^{(1)}z)(x, y)$. Thus, we have shown that the class of systems of the form (1) is identical with the class of systems of the form (51).

Let us also notice the following fact. In, say, Corollary 1, one can slightly change some assumptions on the operators $V_{ij}^{(1)}$, $V_{ik}^{(2)}$, $V_i^{(3)}$ (with $n = 1$, see their definitions) and on the functions $\hat{A}_{ij}$, $\hat{\varrho}_{ik}$, $\hat{f}_i$ in such a way that the operators satisfy conditions of the same type as the conditions needed for the whole corresponding coefficients of system (51). Consequently, instead of system (51), one can consider the system whose coefficients are the corresponding operators:

$$(85)\qquad \sum_{j=1}^{m} (V_{ij}^{(1)}z)(x, y)\left[\frac{\partial z_j(x, y)}{\partial x} + \sum_{k=1}^{r} (V_{ik}^{(2)}z)(x, y)\frac{\partial z_j(x, y)}{\partial y_k}\right] = (V_i^{(3)}z)(x, y), \quad i = 1, \ldots, m,$$

that is, a suitable system (1).

Even a direct treatment of the obtained system (85) or (1) (the application of the method used in the proof of Theorem 1 is not always possible) is much simpler than for system (51) and, obviously, leads to the same results.

Similar observations can be made for Corollary 3 and the papers [17]–[22], directly treating system (51) or its particular cases.

References

[1] P. Bassanini, *Su una recente dimostrazione circa il problema di Cauchy per sistemi quasi lineari iperbolici*, Boll. Un. Mat. Ital. (5) 13-B (1976), 322–335.

[2] —, *Iterative methods for quasi-linear hyperbolic systems*, ibid. (6) 1-B (1982), 225–250.

[3] —, *Iterative methods for quasi-linear hyperbolic systems in the first canonic form*, Applicable Anal. 12 (1981), 105–112.

[4] L. Cesari, *A boundary value problem for quasi-linear hyperbolic systems in the Schauder canonic form*, Ann. Scuola Norm. Sup. Pisa (4) 1 (1974), 311–358.

[5] —, *A boundary value problem for quasi-linear hyperbolic systems*, Riv. Mat. Univ. Parma 3 (1974), 107–131.

[6] S. D. Eidelman, *Parabolic Systems*, Nauka, Moscow 1964 (in Russian).

[7] L. M. Graves, *The Theory of Functions of Real Variables*, Mc Graw-Hill, New York 1956.

[8] Z. Kamont and J. Turo, *On the Cauchy problem for quasi-linear hyperbolic system of partial differential equations with a retarded argument*, Boll. Un. Mat. Ital. (6) 4-B (1985), 901–916.

[9] —, —, *On the Cauchy problem for quasi-linear hyperbolic systems with a retarded argument*, Ann. Mat. Pura Appl. 143 (1986), 235–246.

[10] —, —, *A boundary value problem for quasi-linear hyperbolic systems with a retarded argument*, Ann. Polon. Math. 47 (1987), 347–360.

[11] —, —, *Generalized solutions of boundary value problems for quasi-linear hyperbolic systems with a retarded argument*, Rad. Mat. 4 (1988), 239–260.

[12] E. M. Landis, *Second order equations of elliptic and parabolic types*, Nauka, Moscow 1971 (in Russian).

[13] A. D. Myshkis and A. M. Filimonov, *Continuous solutions of quasi-linear hyperbolic systems in two independent variables*, Differentsial'nye Urawneniya 17 (1981), 488–500 (in Russian).

[14] A. Pliś, *Generalization of the Cauchy problem to a system of partial differential equations*, Bull. Acad. Polon. Sci. 4 (1956), 741–744.

[15] P. Pucci, *Problemi ai limiti per sistemi di equazioni iperboliche*, Boll. Un. Mat. Ital. (5) 16-B (1979), 87–99.
[16] J. Szarski, *Generalized Cauchy problem for differential-functional equations with first order partial derivatives*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), 575–580.
[17] J. Turo, *On some class of quasi-linear hyperbolic systems of partial differential-functional equations of the first order*, Czechoslovak Math. J. 36 (111) (1986), 185–197.
[18] —, *Existence and uniqueness of solutions of quasi-linear hyperbolic systems of partial differential-functional equations*, Math. Slovaca 37 (1987), 375–389.
[19] —, *A boundary value problem for hyperbolic systems of differential-functional equations*, Nonlinear Anal. Theory Methods Appl. 13 (1989), 7–18.
[20] —, *A boundary value problem for quasi-linear hyperbolic systems of hereditary partial differential equations*, Atti Sem. Mat. Fis. Univ. Modena 34 (1985–86), 1–19.
[21] —, *Local generalized solutions of mixed problems for quasi-linear hyperbolic systems of functional partial differential equations in two independent variables*, Ann. Polon. Math. 49 (1989), 259–278.
[22] —, *Generalized solutions of mixed problems for quasi-linear hyperbolic systems of functional partial differential equations in the Schauder canonic form*, ibid. 50 (1989), 157–183.

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