INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

ROZPRAWY MATEMATYCZNE

XLIII

M. KUCHARZEWSKI and M. KUCZMA

Basic concepts of the theory of geometric objects

WARSZAWA 1964
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Introduction

The theory of geometric objects arose not long ago. The first correct definition of a geometric object was given by A. Wundheiler [1] at the 1st International Conference on Tensor Differential Geometry in Moscow in the year 1934. In the year 1953, J. Haantjes and G. Laman [1], [2] gave another definition of a geometric object, based on the notion of a fibre space introduced by N. Steenrod [1] in the year 1951. This definition is rather difficult for readers who are not familiar with the notion of a fibre space and with its properties.

The purpose of this paper is to present J. Haantjes and G. Laman's definition of a geometric object in a way which does not require a use of the notion of a fibre space. Thus this definition will become easily intelligible to readers knowing only the elements of topology and group theory. On the other hand, the following considerations point to the connection existing between the definitions given by A. Wundheiler and by J. Haantjes and G. Laman, which is hardly noticeable to those not concerned with the theory of geometric objects, even after they have read the popular and clearly written article on fibre spaces and geometric objects by A. Goetz [1].

We shall proceed as follows. We shall start from Wundheiler's definition of a geometric object, which will be called a particular geometric object. Next we shall introduce the notion of an abstract geometric object, which is already closely connected with the geometric object as defined by J. Haantjes and G. Laman. Further we shall discuss the concept of the equivalence of geometric objects and the notion of a concomitant (and in particular, the notion of a relative concomitant, introduced by the first of the authors of the present paper, M. Kucharzewski [5]). We shall also touch the problem of an algebra of geometric objects and some questions connected with differential concomitants of geometric objects (the covariant derivative, the Lie derivative, etc.).

Thus the reader, even not familiar with the problems of differential geometry, will be able to learn from the present paper the basic concepts and problems of the theory of geometric objects. In comparison with article [21] of S. Gołąb, who set himself a similar task, the present paper touches a greater variety of problems, treats them in more detail and, above all, takes account of the latest results of this fast developing theory (1).

(1) S. Gołąb's article was written in 1955, and so it was not very up to date even when it appeared in 1959.

§ 1. Historical development of the concept of a geometric object

Before we pass to the proper subject of our considerations, we shall give here an outline of the historical development of the concept of a geometric object and of its theory. Such an outline not only gives a certain amount of knowledge of the theory discussed, but also helps one to understand its problems better. Therefore we regard this outline as a useful introduction to our further considerations.

The concept of a geometric object is already inherent in the definition of the geometry of a group of transformations given by F. Klein in his famous Erlangen program (Klein [1]). According to that definition the geometry of a group of transformations can be described as follows: Suppose we are given a manifold with a certain group of transformations. The set of all invariants of this group is called the *geometry of the group of transformations*. ("Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben, man entwickle die auf Gruppe bezügliche Invariantentheorie"). Now, every such invariant is a geometric object. But the question arises how to understand the word *invariant* and how to express it analytically.

In order to avoid ambiguities and to refer to the further considerations, we shall slightly modify the above definition of a geometry. Namely, instead of speaking about a group of transformations of a manifold, we shall consider a group of transformations of coordinate systems. Then the invariants, or in the modern terminology the geometric objects of a given geometry, are the expressions which do not change with a transformation of the coordinate system, or, in other words, do not depend on the choice of the coordinate system and thus represent a certain geometric content which is independent of the analytical representation introduced.

How should one understand an invariant of a given group of transformations of the coordinate system? The simplest definition one can think of is that an invariant is defined by a set of numbers which do not change with a transformation of the coordinate system. But then the notion of a geometric object would be restricted to that of a scalar. With such a definition a vector for instance would not be a geometric object of the affine geometry, because under the affine group of transformations of the coordinate system the components of a vector vary. But it is essential that they vary in a certain quite definite manner!

Many authors have made attempts to define a geometric object in a more or less exact manner. And so in the year 1926 O. Veblen and J. M. Thomas [1] wrote: "An invariant is an entity with definite determining components in any coordinate system such that the transforma-

tions of the components from one coordinate system to another form a group isomorphic with the group of analytic transformations of the coordinates". In the year 1927, O. Veblen writes: "An object of any sort which is not changed by transformations of coordinates is called an invariant".

In the year 1930 J. A. Schouten and E. R. van Kampen [1] used the term "geometric object" and gave the following definition: "Die Punkte einer Mannigfaltigkeit sind festgelegt durch Koordinaten, die 'Urvariablen', die den Transformationen einer gewissen Gruppe, der Basisgruppe, unterworfen sind. Ein geometrisches Objekt ist ein System von in irgendeinem Bereiche definierten Funktionen ('Bestimmungszahlen' genannt) der Urvariablen, das sich bei Transformationen der Basisgruppe 'in sich' mittransformiert, d.h. sich so transfomiert, daß die neuen Bestimmungszahlen lediglich von den alten Bestimmungszahlen und den Transformationsfunktionen abhängen". Lastly in the year 1932 O. Veblen and J. H. C. Whitehead [1] gave a curious definition of a geometric object, which evoked numerous protests: "Anything which is unaltered by transformations of coordinates is called an invariant. Thus a point is an invariant and so is a curve or a system of curves. Also, strictly speaking, anything, such as a plant or an animal, which is unrelated to the space which we are talking about, is an invariant. For an invariant which is related to the space, i. e. a property of the space ... we shall also use the term geometric object"(²).

The fact that in a relatively short period (1926-1932) so many authors endeavoured to define a geometric object shows on one hand that in that time there arose an urgent need of a definition of this notion, and on the other hand that none of the former attempts was regarded as satisfactory, even by the authors themselves. It was not until 1934 that A. Wundheiler gave a correct definition of a geometric object in a lecture held at the 1st International Conference on Tensor Differential Geometry in Moscow. This definition is contained in his paper [1], which appeared as late as 1937. The theory of geometric objects was further developed by J. A. Schouten and J. Haantjes [1], [2] and by S. Gołąb [1], [2], [3], [4]. It was then dealt with by more and more authors, among which one should mention first of all S. Gołąb [6], [7], [8], [9], [10], [11], [12], W. W. Wagner [1], [2], [3], [4], [5], [6], [8], [9], J. Pienzow [1], [2], [3], [4] and others (Bompiani [1], Kuiper–Yano [1], Rashevski [1], [2], Vranceanu [1]). In those papers the authors were mainly concerned with establishing the basic notions of the theory of geometric objects and with determining geometric objects of some types, i. e. with the classi-

(²) The above quotations have been taken from paper [1] of J. A. Schouten and D. van Dantzig.

fication of geometric objects. S. Gołąb obtained his results investigating directly the solution of some functional equations. On the other hand, W. W. Wagner connected the theory of geometric objects with the theory of Lie groups (Wagner [1]). He and J. Pienzow obtained their results by making use of the corresponding theorems of the latter theory. However, this method requires very strong assumptions.

In the year 1950 S. Gołąb introduced the notion of the equivalence of geometric objects (Gołąb [13], [14], [15]).

In the year 1952 a monograph by A. Nijenhuis [2] appeared, dealing with the theory of geometric objects. The author developed this theory basing himself on the notion of a groupoid, introduced by H. Brandt [1]. The notion of a groupoid replaced the previously used notion of a group or a pseudogroup of transformations of coordinates (Veblen–Whitehead [1], p. 38, Gołąb [5], Wagner [7]).

Lastly, in the year 1953, J. Haantjes and G. Laman [1], [2] gave a new definition of a geometric object and of the equivalence of geometric objects, based on the notion of a fibre space introduced by N. Steenrod [1], and they determined all one-dimensional objects of the first class in an n-dimensional space. With the use of this definition J. Pienzow has determined one-dimensional objects of higher classes (cf. Pienzow [5]) and two-dimensional objects (cf. Pienzow [6]) in a one-dimensional space.

Independently, the theory of geometric objects based on Wundheiler's definition continued to develop fast. Many results on these lines were obtained under very weak regularity suppositions (mainly by the use of functional equations) by S. Gołąb and his disciples: A. Jakubowicz, M. Kucharzewski, M. Kuczma, H. Pidek-Łopuszańska, E. Siwek and A. Zajtz, as well as by representatives of the Hungarian school: J. Aczél, A. Balogh, M. Hosszú, I. Makai, A. Moór, and of the Rumanian school: O. E. Gheorghiu and B. Crstici (cf. Aczél [1]-[10], Balogh [1], Gheorghiu [1]-[23], Gheorghiu–Crstici [1], Gołąb [22]-[25], Gołąb–Jakubowicz–Kucharzewski–Kuczma [1], Gołąb–Makai [1], Gołąb–Pidek [1], Gołąb–Siwek [1], Hosszú [1]-[4], Jakubowicz [1], Kucharzewski [1]-[5], Kucharzewski–Kuczma [1]-[5], [8], Kuczma [1], Makai [1]-[3], Moór [1]-[7], Pidek [1]-[3], Siwek [1], Zajtz [1]). The majority of these results have been presented in a monograph by J. Aczél and S. Gołąb [1], where one can also find a fairly complete bibliography concerning this subject up to the year 1960.

In the year 1962 the authors of the present work wrote two joint papers (Kucharzewski–Kuczma [6], [7]), in which they attempted to compare Wundheiler's definition with the Haantjes-Laman definition of a geometric object. Here we should like to present the main ideas of those papers in a more detailed manner.

§ 2. Manifold, coordinate system, transformations of the coordinate system

Before we pass to the definition of a geometric object, we shall introduce some auxiliary notions: the notion of a manifold, of a coordinate system and of transformations of coordinates.

DEFINITION 1. The set of all sequences of n real numbers $(\xi^1, \xi^2, \ldots, \xi^n)$ will be called the *n-dimensional arithmetical space.*

The n-dimensional arithmetical space will be denoted in the sequel by A_n and its points will be denoted by small letters of the Greek alphabet, e. g. ξ will denote the point $(\xi^1, \ldots, \xi^n)$. In the space A_n a topology can be introduced in the usual way, and so in the sequel we shall regard A_n as a topological space.

DEFINITION 2. An *n-dimensional manifold* X_n is a connected and separable topological space whose every point has a neighbourhood homeomorphic to an open domain of the n-dimensional arithmetical space A_n (cf. Haantjes-Laman [1], p. 210, Aczél-Gołąb [1], p. 7, and Lichnerowicz [1]).

Thus a circle is a one-dimensional manifold (although the whole circle is not homeomorphic to an open interval, every partial arc of a circle is). On the other hand, the figure drawn in fig. 1 (with the topology induced by that of the plane) is not a one-dimensional manifold, for no neighbourhood of the point x_0 is homeomorphic to an open interval.

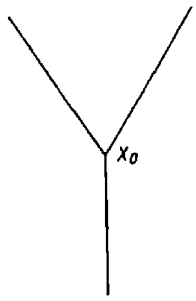

Fig. 1

The elements of the space X_n will be called *points* and will be denoted by small letters of the Latin alphabet: x, y, z, etc.

DEFINITION 3. We say that in a neighbourhood of a point $x_0 \in X_n$ a *local coordinate system* U has been defined provided we are given a homeomorphism of this neighbourhood onto a certain domain of the space A_n, i. e. if to each point x of this neighbourhood there corresponds exactly one point $\xi \in A_n$ and this correspondence is one-to-one and con-

tinuous and the inverse correspondence is also continuous (cf. fig. 2). The numbers $(\xi^1, \ldots, \xi^n)$ are called the *coordinates* of the point x in the coordinate system U.

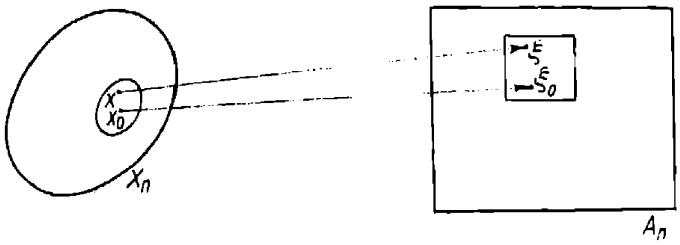

Fig. 2

Of course, in various neighbourhoods of the same point x_0 of a manifold X_n many coordinate systems can be defined. Let a coordinate system U_i be defined in a neighbourhood V_i and a coordinate system $U_{i'}$ be defined in a neighbourhood $V_{i'}$ of the same point $x_0 \in X_n$ (fig. 3). Since the

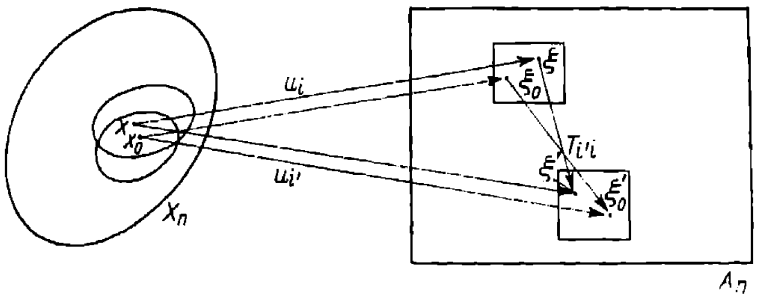

Fig. 3

common part of V_i and $V_{i'}$ is non-empty, it is also a certain neighbourhood of the point x_0. Consequently the systems U_i and $U_{i'}$ determine a one-to-one transformation of a certain domain of the space A_n into a certain domain of this space. This transformation will be denoted shortly by $T_{i'i}$:

$$T_{i'i} = U_{i'} U_i^{-1}.$$

If we denote by ξ^i and $\xi^{i'}$ the coordinates of the same point $x \in V_i \cap V_{i'}$ in the coordinate systems U_i and $U_{i'}$, respectively, then the transformation $T_{i'i}$ can be written in the form

$$T_{i'i}\colon\ \xi^{i'} = \varphi^{i'}(\xi^1, \ldots, \xi^n), \qquad i' = 1', 2', \ldots, n'. \tag{2.1}$$

A neighbourhood V_i together with a coordinate system U_i defined in it is called a *local map* (Lichnerowicz [1]), and a set of local maps whose neighbourhoods cover the whole manifold X_n is called an *atlas*

of this manifold. An atlas is said to be of class C^r, $r \geqslant 1$, if all the functions (2.1), corresponding to points belonging to two different neighbourhoods, are of class C^r and their Jacobian J is different from zero:

$$J = \det\left(\frac{\partial \xi^{i'}}{\partial \xi^i}\right) \neq 0. \tag{2.2}$$

The union of two atlases of a manifold X_n is again an atlas of this manifold. Two atlases of class C^r will be called *r-equivalent* if their union is again an atlas of class C^r. The r-equivalence of atlases is a logical equivalence (i. e. this relation is reflexive, symmetric and transitive), and thus we can divide the set of all the C^r atlases of a given manifold into classes of r-equivalent atlases.

DEFINITION 4. A manifold X_n with a chosen class of r-equivalent atlases is called a *differentiable manifold* of class C^r and will be denoted by X_n^r (Haantjes–Laman [1], Lichnerowicz [1]).

In practice, in order to determine a manifold X_n^r it is enough to indicate a manifold X_n and to define on it a single atlas of class C^r. This atlas will determine the class of r-equivalent atlases, and thus also the manifold X_n^r.

Since in the sequel we shall deal only with geometric objects fastened at a fixed point x_0 of a manifold X_n, we may confine ourselves only to coordinate systems defined in a certain neighbourhood of the point x_0 and determined by a class of r-equivalent atlases. Every two such atlases in a neighbourhood of the point x_0 determine a transformation (2.1) which is defined and one-to-one in a certain neighbourhood of the point ξ_0 and fulfils condition (2.2). Conversely, applying to a given coordinate system all possible transformations of class C^r with condition (2.2) which are defined and one-to-one in a certain domain we shall obtain all r-equivalent atlases restricted to a certain neighbourhood of the point x_0.

Consequently, if we deal only with objects fastened at a fixed point x_0, we may confine ourselves to a set $\boldsymbol{Z}^r$ of coordinate systems which are defined in an certain neighbourhood of the point x_0 and may be obtained from such a system by applying all the possible C^r transformations, fulfilling condition (2.2) and defined and one-to-one in a certain domain. The set of such transformations of coordinates forms a pseudogroup according to the definition given by S. Gołąb [5].

§ 3. Group, pseudogroup, groupoid

In the above considerations as the set $\boldsymbol{Z}$ of allowable coordinate systems we have taken the set $\boldsymbol{Z}^r$ obtained from a certain system by applying all the possible transformations (2.1) of class C^r, with

the Jacobian different from zero, defined and one-to-one in a certain domain.

One may consider, however, geometric objects also with respect to another set $\boldsymbol{Z}$ of the allowable coordinate systems. The set $\boldsymbol{Z}$ determines a set of the transformations of these coordinate systems, which will be denoted by $\boldsymbol{T}$. Conversely, if we know a set $\boldsymbol{T}$ of the transformations of coordinates and a coordinate system $U_0 \in \boldsymbol{Z}$, then the set $\boldsymbol{Z}$ of the allowable coordinate systems can be obtained as follows. Applying to the system U_0 all the possible transformations from the set $\boldsymbol{T}$ (those which can be applied) we obtain a set of coordinate systems. Applying again the transformations from the set $\boldsymbol{T}$ to the systems obtained, we get a new, wider set of coordinate systems, etc. In the set $\boldsymbol{Z}$ we include all the coordinate systems obtained by applying to the system U_0 an arbitrary but finite number of transformations from the set $\boldsymbol{T}$. This can be represented on a diagram, where the points denote coordinate systems from the set $\boldsymbol{Z}$ and the arrows denote transformations of coordinates belonging to the set $\boldsymbol{T}$ (fig. 4).

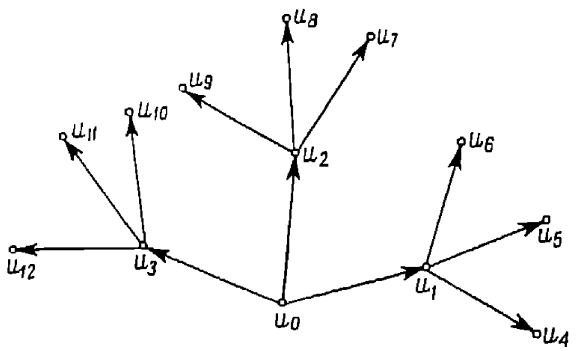

Fig. 4

Now let us try to decide what properties should characterize a set $\boldsymbol{T}$ of the transformations of an arbitrary set $\boldsymbol{Z}$ of allowable coordinate systems. For the sake of simplification let us assume at first that the manifold X_n and the set $\boldsymbol{Z}$ have the following properties:

(1) *Every coordinate system of the set $\boldsymbol{Z}$ is defined on the whole manifold.*

(2) *The ranges of all the coordinate systems are the same region D of the space A_n* (i. e., every coordinate system establishes a homeomorphism between the points of the manifold X_n and the same region D of the arithmetical space A_n).

If these conditions are fulfilled, we say that the manifold X_n is endowed with a *Klein coordinate structure* (cf. Veblen-Whitehead [1], Wagner [5], p. 137, Spivak [1]). The set $\boldsymbol{T}$ is then a set of one-to-one transformations of the region D onto itself. The set $\boldsymbol{T}$ has the following properties:

(α) *If* $T_1 \in \boldsymbol{T}$ *and* $T_2 \in \boldsymbol{T}$, *then* $T_2 T_1$ *exists and also belongs to* $\boldsymbol{T}$.

(β) *To every transformation* $T \in \boldsymbol{T}$ *there exists an inverse transformation* T^{-1} *belonging to* $\boldsymbol{T}$.

In fact, condition (α) can be obtained in the following manner. Let $T_1 \in \boldsymbol{T}$ and $T_2 \in \boldsymbol{T}$. This means that there exist coordinate systems U_1, U_2, U_3 belonging to $\boldsymbol{Z}$ such that U_2 is obtained by applying the transformation T_1 to the system U_1 and U_3 is obtained by applying the transformation T_2 to the system U_2. Let T_3 denote the transformation which leads from the system U_1 to U_3. By definition the transformation T_3 belongs to $\boldsymbol{T}$ and, on the other hand, evidently T_3 is the superposition of T_1 and T_2. Thus property (α) has been proved.

Similarly we shall prove property (β). Let $T \in \boldsymbol{T}$. By definition there exist coordinate systems U_1 and U_2 belonging to $\boldsymbol{Z}$ such that U_2 is obtained by applying the transformation T to U_1. But the set $\boldsymbol{T}$ must also include the transformation which leads from the system U_2 to the system U_1. This transformation is, as can easily be verified, inverse to the transformation T. Thus property (β) has also been proved.

Conditions (α) and (β) express the fact that the set of transformations $\boldsymbol{T}$ must form a group.

Now we may proceed conversely. If we are given an initial coordinate system U_0 and a group $\boldsymbol{T}$ of transformations of the range of the system U_0 onto itself, we can obtain the set $\boldsymbol{Z}$ of allowable coordinate systems by applying to U_0 all the transformations of the group $\boldsymbol{T}$. The set $\boldsymbol{Z}$ does not depend on the choice of the initial coordinate system, even if we continued arbitrarily long the process, described above, of forming allowable coordinate systems. This follows from the fact that every system from the set $\boldsymbol{Z}$ can be obtained by applying only one transformation from the group $\boldsymbol{T}$.

In this manner the choice of a transformation group $\boldsymbol{T}$ and of an initial coordinate system U_0 determines a set $\boldsymbol{Z}$ of allowable coordinate systems with regard to which one can determine geometric objects. Thus the transformation group $\boldsymbol{T}$ may be taken as the basis of a classification of geometries (Klein [1]).

The matter gets complicated when the ranges of the allowable coordinate systems do not coincide. In such a case we say that the manifold X_n is endowed with a *Veblen-Whitehead coordinate structure* (cf. Veblen-Whitehead [1], Wagner [5], Spivak [1]). Then, of course, the set $\boldsymbol{T}$ of the transformations of the allowable coordinate systems does not form a group. O. Veblen and J. W. C. Whitehead [1] were the first to call attention to this fact. They introduced the notion of a pseudogroup of transformations as a set $\boldsymbol{T}$ of transformations with the following properties:

(α_1) *If* $T_1 \in \boldsymbol{T}$ *and* $T_2 \in \boldsymbol{T}$ *and* $T_2 T_1$ *exists, then* $T_2 T_1 \in \boldsymbol{T}$.

(β_1) *To every transformation* $T \in \boldsymbol{T}$ *there exists an inverse transformation* T^{-1} *belonging to* $\boldsymbol{T}$.

Here the existence of the composition of transformations T_1 and T_2 is understood so that the range of the first transformation coincides with the domain of definition of the second transformation.

J. A. Schouten and J. Haantjes generalized a little the notion of a pseudogroup, assuming the existence of the composition of transformations T_1 and T_2 also in the case where the range of the first transformation and the domain of definition of the second transformation have a common region.

Finally S. Gołąb extended the notion of a pseudogroup of transformations, making it more precise. According to S. Gołąb a pseudogroup of transformations is a set $\boldsymbol{T}$ of transformations with the following properties:

(a) *Every transformation is defined in an open set.*

(b) *A restriction of any transformation belonging to* $\boldsymbol{T}$ *to an open set also belongs to* $\boldsymbol{T}$.

(c) *The composition of two transformations* T_1 *and* T_2 *belonging to* $\boldsymbol{T}$, *if it exists, also belongs to* $\boldsymbol{T}$. *The existence is here understood so that the range of the first transformation and the domain of definition of the second have a common point.*

(d) *If* $T_1 \in \boldsymbol{T}$ *and* ξ_0 *belongs to the domain of definition of* T_1, *then there exists a transformation* T_2, *locally inverse to* T_1 *at the point* ξ_0, *and also belongs to* $\boldsymbol{T}$. (T_2 is *locally inverse to* T_1 *at the point* ξ_0 if the transformation $T_2 T_1$ is the identity transformation in a certain neighbourhood of the point ξ_0).

An example of a pseudogroup in the above sense is provided for instance by the set of all the transformations of class C^r in a certain domain which have the Jacobian different from zero. This pseudogroup will be denoted, after S. Gołąb [18], by $\boldsymbol{G}_r$.

S. Gołąb's definition of a pseudogroup has a local character, which is expressed by the postulate of local invertibility, which has replaced the former postulate of global invertibility, and by a weaker condition of the existence of the composition of transformations.

If follows from the conditions defining a pseudogroup that if we are given a pseudogroup $\boldsymbol{T}$ of transformations and an initial coordinate system U_0, then the set $\boldsymbol{Z}$ of allowable coordinate systems formed in the manner described at the beginning of the present section has the following property: from an arbitrary system of $\boldsymbol{Z}$ one can pass to any other system of $\boldsymbol{Z}$ with the aid of a certain transformation from the pseudogroup $\boldsymbol{T}$.

(Hence it follows in particular that the set Z of allowable coordinate systems does not depend on the choice of the initial system.) In the case of a pseudogroup as defined by S. Gołąb or by J. A. Schouten and J. Haantjes, one must make an identification of some systems, regarding as identical every two systems which coincide in a certain common region.

If we are talking only about coordinate systems defined in a neighbourhood of a fixed point x_0 of the manifold X_n, then it is advisable to define the composition of transformations in a somewhat different manner. Namely, we shall say that the composition of two transformations T_1 and T_2 exists with respect to the point ξ_0 if the image of ξ_0 by the transformation T_1 belongs to the domain of definition of the transformation T_2 (or, in other words, the common part of the range of the transformation T_1 and of the domain of definition of the transformation T_2 contains $T_1(\xi_0)$). The idea is that the systems defined in a neighbourhood of the point x_0 should pass by transformations from the set T again into systems defined in a neighbourhood of x_0.

DEFINITION 5. A pseudogroup of transformations of a set of coordinate systems as defined by S. Gołąb, where the existence of the composition of transformations T_1 and T_2 is understood as the existence of the composition with respect to the coordinates of the point x_0, will be called a *pseudogroup of transformations with respect to the point* x_0.

A. Nijenhuis in his monograph (Nijenhuis [2]) introduced, instead of transformations, the notion of a *transformation element*, defined as a pair consisting of a point ξ_0 and of a transformation T: $\xi' = \varphi(\xi)$ defined in a neighbourhood of this point. Such a transformation element is denoted by

$$(\xi_0, T). \tag{3.1}$$

In the set of transformation elements (3.1) one can define a composition of elements as follows: the composition of two elements (ξ_0, T_1) and (η_0, T_2), where

$$T_1\colon\ \xi' = \varphi(\xi) \quad \text{and} \quad T_2\colon\ \xi' = \psi(\xi),$$

exists if $\varphi(\xi_0) = \eta_0$ and the resulting element equals (ξ_0, T_2T_1). The transformation element (ξ_0, ξ) plays the part of the identity of this operation.

The set of transformation elements (3.1), where T is an analytic transformation ([3]) in a neighbourhood of the point ξ_0 with a non-vanishing Jacobian, in which the operation of composing has been defined in the above manner, forms a groupoid in the sense of H. Brandt (Brandt [1]).

([3]) Hence it follows that two transformations equal in a neighbourhood of the point ξ_0 must be identical.

Namely, a groupoid is defined as a set $\mathfrak{G}$ of elements (with an operation defined for some couples of elements) with the following properties:

(1) *If for three elements, a, b, c of the set $\mathfrak{G}$ the relation*

$$ab = c$$

holds, then each of them is uniquely determined by the remaining two.

(2) *If ab and bc exist, then there exist also $(ab)c$ and $a(bc)$; if ab and $(ab)c$ exist, then there exist also bc and $a(bc)$; if ac and $a(bc)$ exist, then there exist also ab and $(ab)c$. In all the three cases the equality*

$$(ab)c = a(bc)$$

holds.

(3) *To every element $a \in \mathfrak{G}$ there exists a uniquely determined element $e_r(a)$, the right-sided unity, a uniquely determined element $e_l(a)$, the left-sided unity, and a uniquely determined inverse element a^{-1} such that*

$$ae_r(a) = a, \qquad e_l(a)a = a,$$
$$a^{-1}a = e_r(a), \qquad aa^{-1} = e_l(a).$$

(4) *For any two unities $e(a)$ and $e(b)$ there exists at least one element $c \in \mathfrak{G}$ such that $ce(a)$ and $ce(b)$ exist.*

An example of a groupoid is provided by the set $\mathfrak{G}$ of the pairs of numbers (a, b) with the composition law $(a, b)(b, c) = (a, c)$. Another example can be obtained as follows. Let $\mathscr{G}$ be an arbitrary group and $\mathfrak{G}$ the groupoid of the preceding example. The set of triplets $(a, b; g)$, where $(a, b) \in \mathfrak{G}$ and $g \in \mathscr{G}$ with the operation

$$(a, b; g_1)(b, c; g_2) = (a, c; g_1 g_2)$$

(where $g_1 g_2$ is the product of the elements g_1 and g_2 in the sense of the operation of the group $\mathscr{G}$) is again a Brandt groupoid.

Suppose that we are given an arbitrary groupoid of transformation elements and a local coordinate system U_0 defined in a neighbourhood of a point x_0 of a manifold X_n. Let ξ_0 be the coordinates of the point x_0 in the coordinate system U_0. With the aid of elements of the groupoid of the form (ξ_0, T) we can obtain other local coordinate systems defined in a neighbourhood of the point x_0. Applying to the coordinate systems thus obtained further transformation elements of the groupoid (those which can be applied), we shall obtain further systems. In this manner the groupoid of transformation elements and the initial coordinate system determine a set $\boldsymbol{Z}$ of allowable coordinate systems defined in a neighbourhood of the point x_0. According to the properties of the

groupoid one can pass from an arbitrary system of the set $\boldsymbol{Z}$ to any other system of this set with the aid of a certain transformation element of the groupoid. Hence it follows further that the set $\boldsymbol{Z}$ does not depend on the choice of the initial system.

In order to speak about geometric objects at a point x_0 of a manifold X_n (for the present we do not assume that X_n is a differentiable manifold) we must have a fixed set $\boldsymbol{Z}$ of coordinate systems defined in a neighbourhood of the point x_0. Two systems will be regarded as equal if they are identical in a neighbourhood of x_0. The set $\boldsymbol{Z}$ will be called a *set of allowable coordinate systems* at the point x_0. The set $\boldsymbol{T}$ of all the transformations leading from systems of the set $\boldsymbol{Z}$ to all the other systems of the set $\boldsymbol{Z}$ forms a pseudogroup of transformations with respect to the point x_0. And the corresponding set of the transformation elements forms a Brandt groupoid.

In practice, in order to define a set of allowable coordinate systems at a point x_0 one gives an initial system U_0 and a certain set of transformations $\boldsymbol{T}$ which must form a pseudogroup with respect to the point x_0. Instead of giving a pseudogroup of transformations with respect to the point x_0 one can also give a groupoid of transformation elements.

Apparently we have established a very general notion of a coordinate system. Nevertheless even this wide concept of a system of coordinates is not sufficient in all the problems of geometry. E. g. in projective geometry we deal with coordinate systems which consist in assigning to the elements of the manifold some subsets of the space A_n. More detailed investigations of such coordinate systems may be found in a paper by A. Pforr [1].

§ 4. Wundheiler's definition of a geometric object

Now we pass to defining a geometric object. Suppose that we are given a set $\boldsymbol{Z}$ of allowable coordinate systems defined in a neighbourhood of a point x_0 of a manifold X_n.

DEFINITION 6 (Wundheiler [1]). At the point x_0 there is defined an *object* ω *with* m *components*

$$\omega(\omega^1, \ldots, \omega^m), \tag{4.1}$$

if to every allowable coordinate system a sequence of m numbers $\omega^1, \ldots, \omega^m$ (called *components of the object*) has been assigned, i. e. if there is defined a function

$$\omega = \Phi(U), \qquad U \in \boldsymbol{Z}. \tag{4.2}$$

The object is called *geometric* if its components $\omega^1, \ldots, \omega^m$ in an arbitrary new coordinate system $U' \in \boldsymbol{Z}$ can be expressed by its components (4.1) in the old system U and by transformation (2.1). The dependence of $\omega^{i'}$ on ω^i and $T_{i'i}$ has thus the form

$$\omega' = F(\omega, T_{i'i}) \tag{4.3}$$

and is called the *transformation formula* (or *transformation law*) of the object ω.

A geometric object is determined if we know its components ω_0 in a certain coordinate system $U_0 \in \boldsymbol{Z}$ and its transformation formula (4.3). Function (4.2) may then be expressed as

$$\Phi(U) = F\big(\omega_0, T(U)\big), \tag{4.4}$$

where $T(U)$ is the transformation leading from the system U_0 to the system U.

In order that transformation formula (4.3) should define the components of an object uniquely in every coordinate system, the function F cannot be quite arbitrary, but must fulfil the conditions

$$F\big(F(\omega, T_{i'i}), T_{i''i'}\big) = F(\omega, T_{i''i}), \tag{4.5}$$

$$F(\omega, I) = \omega, \tag{4.6}$$

where $T_{i'i}$, $T_{i''i'}$ and $T_{i''i}$ denote the transformations which lead from the system U to U' from U' to U'' and from U to U'', respectively, and I denotes the identity transformation.

Whether a given object is geometric or not depends in an essential way on the set $\boldsymbol{Z}$ of allowable coordinate systems. We shall illustrate this fact by an example.

Let X_1 be a straight line on which a point x_0 has been fixed. Let U_0 be an arbitrary Cartesian coordinate system on the whole X_1 (i. e. an isometric homeomorphism of X_1 onto the whole space of real numbers A_1); as the set $\boldsymbol{Z}$ of the allowable coordinate systems we take the set of the systems obtained from U_0 by applying the group of the transformations

$$T_k\colon\ \xi' = \xi + k,$$

where k runs over the set of all the integers. If we denote by U_k the system obtained from U_0 by applying the transformation T_k, then from a system U_{k_1} to a system U_{k_2} leads the transformation $T_{k_2\ k_1}$. Now we define an object ω (with one component) by the formula

$$\omega = \Phi(U_k) = \begin{cases} 0 & \text{for} \quad k \neq 0, \\ 1 & \text{for} \quad k = 0. \end{cases}$$

This object is not geometric. For an indirect proof, let us suppose that ω is a geometric object. Then there exists a function $F(\omega, T)$ such that

$$\omega(U_{k_2}) = F\big(\omega(U_{k_1}), T_{k_2-k_1}\big), \tag{4.7}$$

where $\omega(U_k) = \Phi(U_k)$ denotes the value of the component of the object ω in the system U_k.

Now let us consider three coordinate systems: U_0, U_1 and U_2. The transformation T_{-1} leads from the system U_1 to U_0 and also from the system U_2 to U_1. In the coordinate systems U_1 and U_2 the component of the object ω equals zero, in the system U_0 it equals 1. Thus we have by (4.7), assuming first $k_1 = 2$, $k_2 = 1$, and then $k_1 = 1$, $k_2 = 0$,

$$0 = F(0, T_{-1}) \quad \text{and} \quad 1 = F(0, T_{-1}),$$

whence by comparing

$$0 = 1.$$

The contradiction obtained proves that ω is not a geometric object.

Now we shall introduce in X_1 another set $\boldsymbol{Z}^*$ of allowable coordinate systems. Let U_0^* be the system obtained from the system U_0 by applying the transformation

$$\xi^* = \frac{1}{\pi} \operatorname{arc\,tan} \xi.$$

Thus the range of the system U_0^* is the interval $(0, 1)$. As the set $\boldsymbol{Z}^*$ we take the set of the systems U_k^* obtained from U_0^* by applying the transformations

$$T_{k0}^*\colon\ \xi' = \xi + k, \qquad \xi \in (0,1), \tag{4.8}$$

where k runs over the set of all the integers. The pseudogroup $\boldsymbol{T}^*$ of the transformations of the coordinate systems of the set $\boldsymbol{Z}^*$ will contain, besides transformations (4.8), also all the transformations

$$T_{kl}^*\colon\ \xi' = \xi + k - l, \qquad \xi \in (l, l+1),\ k, l \text{ integers}.$$

The transformation T_{kl}^* leads from the system U_l^* to the system U_k^*.

This time the object

$$\omega^* \qquad \Phi(U_k^*) \quad \begin{cases} 0 & \text{for} \quad k \neq 0, \\ 1 & \text{for} \quad k = 0 \end{cases}$$

is geometric. The function $F(\omega^*, T^*)$ has the form

$$F(\omega^*, T_{kl}^*) = \begin{cases} 0 & \text{for} \quad k \neq 0, \\ 1 & \text{for} \quad k = 0. \end{cases}$$

We cannot repeat here the argument from the preceding case, because now a given transformation T^*_{kl} may be applied only to one coordinate system, namely to U^*_l. It is so, because the ranges of any two allowable coordinate systems are disjoint.

Further examples of objects, geometric as well as non-geometric, will be shown in the next section.

§ 5. Special geometric objects

Now we assume that $X_n = X^r_n$ is a differentiable manifold of class C^r and as the set Z of allowable coordinate systems at a point $x_0 \in X^r_n$ we take the set Z^r of the systems obtained from a certain system U_0 by applying the pseudogroup (with respect to the point x_0) G_r of transformations (2.1), defined and of class C^r in a neighbourhood of the point x_0 and with a Jacobian $J \neq 0$. To every such transformation (2.1) there corresponds in a one-to-one manner a system of numbers

$$(5.1) \qquad A^{i'}_i,\ A^{i'}_{i_1 i_2}, \ldots, A^{i'}_{i_1 \ldots i_s} \qquad \left(\begin{array}{c} i' = 1', \ldots, n', \\ i, i_1, \ldots, i_s = 1, \ldots, n, \\ s \leqslant r \end{array}\right),$$

where we have applied the shortened notation

$$A^{i'}_{i_1 \ldots i_k} = \frac{\partial^k \xi^{i'}}{\partial \xi^{i_k} \ldots \partial \xi^{i_1}}(\xi_0).$$

Moreover, according to (2.2),

$$(5.2) \qquad J = \det(A^{i'}_i) \neq 0$$

and $A^{i'}_{i_1 \ldots i_k}$ are symmetric with respect to the lower indices (independence of the order of taking derivatives).

To every system of parameters (5.1) we can assign in a one-to-one manner a sequence of n polynomials on n variables ([4])

$$(5.3) \quad w^{i'}(\xi^1, \ldots, \xi^n) = A^{i'}_i \xi^i + \frac{1}{2!} A^{i'}_{i_1 i_2} \xi^{i_1} \xi^{i_2} + \ldots + \frac{1}{s!} A^{i'}_{i_1 \ldots i_s} \xi^{i_1} \ldots \xi^{i_s},$$

$$i' = 1', \ldots, n'.$$

In the set W^s_n of all the sequences of n polynomials (5.3) we define an inner operation consisting in inserting the polynomials of the first se-

([4]) Here and in the sequel we use Einstein's summation convention: one should sum (from 1 to n) with respect to all the indices which appear on two different levels (i. e. as lower and upper indices). Thus e. g. $A^{i'}_i \xi^i$ denotes $\sum_{i=1}^{n} A^{i'}_i \xi^i$, etc.

quence in place of the variables of the polynomials of the second sequence and then deleting the terms of the orders higher than s. With respect to this operation, the set W_n^s forms a Lie group [5].

The operation in the group W_n^s induces an operation in the set of all the systems of numbers (5.1), which consequently also forms a group. This group will be denoted after Haantjes and Laman by $\mathscr{L}_n^s$ and its elements (systems of parameters (5.1)) will be denoted shortly by a single letter L. If L_1 is the system of parameters (5.1) and L_2 the system of the parameters

$$B_i^{i'},\ B_{i_1 i_2}^{i'},\ \ldots,\ B_{i_1,\ldots,i_s}^{i'},$$

then the parameters $C_i^{i'}, C_{i_1 i_2}^{i'}, \ldots, C_{i_1\ldots i_s}^{i'}$, representing the product (in the sense of the group operation) $L_2 L_1$ are given by the following formulae:

$$(5.4)\qquad \begin{cases} C_i^{i'} = B_k^{i'} A_i^k, \\ C_{i_1 i_2}^{i'} = B_{k_1 k_2}^{i'} A_{i_1}^{k_1} A_{i_2}^{k_2} + B_k^{i'} A_{i_1 i_2}^k, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Formulae (5.4) are identical with those expressing derivatives of a composite function.

On account of the symmetry of the parameters $A_{i_1\ldots i_k}^{i'}$ with respect to the lower indices, system (5.1) contains only

$$N = n\binom{n+s}{s} - n$$

independent parameters. Thus the set of systems (5.1) can be imbedded in the N-dimensional arithmetical space A_N. The topology of A_N induces a topology in $\mathscr{L}_n^s$. $\mathscr{L}_n^s$ endowed with this topology is a topological group.

The set of triplets $(\xi_0^i, \xi_0^{i'}, L)$, where ξ_0^i denotes the coordinates of the point x_0 in the coordinate system U, $\xi_0^{i'}$ denotes the coordinates of x_0 in the coordinate system U', and $L \in \mathscr{L}_n^s$, with the operation

$$(\xi_0^i, \xi_0^{i'}, L_1)(\xi_0^{i'}, \xi_0^{i''}, L_2) = (\xi_0^i, \xi_0^{i''}, L_2 L_1),$$

forms a Brandt groupoid (cf. the example from § 3, p. 14). This groupoid will be denoted by $\mathfrak{G}_n^s$.

Definition 7. Geometric object (4.1) is called a *special geometric object* if in its transformation formula (4.3) the dependence on the transformation $T_{i'i}$ is expressed only by the coordinates of the point x_0 in the old coordinate system and in the new one and by a finite number of parameters (5.1). The transformation formula of the object then has the form

$$(5.5)\qquad \omega' = F(\omega, \xi_0^i, \xi_0^{i'}, A_i^{i'}, A_{i_1 i_2}^{i'}, \ldots, A_{i_1\ldots i_s}^{i'}),$$

---

[5] Regarding the definition of a Lie group, cf. Pontriagin [1], p. 263.

Object ω is said to be of *class* s if the function F in (5.5) actually depends at least on one of the parameters $A^{i'}_{i_1\ldots i_s}$ (6).

A special geometric object is called a *purely differential geometric object* if its transformation formula does not contain ξ^i_0 and $\xi^{i'}_0$:

$$\omega' = F(\omega, A^{i'}_i, A^{i'}_{i_1 i_2}, \ldots, A^{i'}_{i_1\ldots i_s}). \tag{5.6}$$

Transformation formula (5.6) can be written shortly in the form

$$\omega' = F(\omega, L), \qquad L \in \mathscr{L}^s_n. \tag{5.7}$$

When one deals with not purely differential geometric objects it is more convenient to base the considerations on the groupoid $\mathfrak{G}^s_n$ instead of the group $\mathscr{L}^s_n$. But not purely differential geometric objects can be reduced to non-differential objects (i. e. objects whose transformation formula does not contain at all parameters (5.1)) and to purely differential objects (Wagner [2], Nijenhuis [2], Aczél [2]). In the sequel we shall be concerned only with the latter.

Now we shall give some examples of more important objects:

1. A geometric object with one component ω which does not change by the passage from one coordinate system to another,

$$\omega' = \omega, \tag{5.8}$$

is called a *scalar*. A scalar is a geometric object of class 0, at the same time purely differential and non-differential. (5.8) is its transformation formula.

2. A geometric object with one component ω with the transformation formula

$$\omega' = (\operatorname{sgn} J)\,\omega, \tag{5.9}$$

where $J = \det(A^{i'}_i)$, is called a *biscalar* (7). A biscalar is a purely differential geometric object of the first class.

3. The coordinates $\xi^1_0, \ldots, \xi^n_0$ of the point x_0 form an object with n components (so here $m = n$) with transformation formula (2.1). It is a non-differential geometric object of class zero.

4. An object with one component with the transformation formula

$$\omega' = J\omega \tag{5.10}$$

(6) The notion of the class of an object has been introduced by J. A. Schouten and J. Haantjes [2].

(7) More exactly, a biscalar is an arbitrary object with one component which can assume only two different values. The term is due to S. Gołąb.

is called an *ordinary density*. Generally, an object with the transformation formula

$$\omega' = |J|^a (\operatorname{sgn} J)\,\omega \tag{5.11}$$

is called an *ordinary density of weight* $-a$.

5. An object with one component with the transformation formula

$$\omega' = |J|\,\omega \tag{5.12}$$

is called a *Weyl density*. Generally, an object with the transformation formula

$$\omega' = |J|^a\,\omega \tag{5.13}$$

is called a *Weyl density of weight* $-a$. Densities (ordinary as well as Weyl densities) are purely differential geometric objects of the first class (with one component).

6. An object with n components $v^1, \ldots, v^n$ with the transformation formula

$$v^{i'} = A^{i'}_i v^i \tag{5.14}$$

is called a *contravariant vector*. A contravariant vector is a purely differential geometric object of the first class (with n components).

7. An object with n components $w_1, \ldots, w_n$ with the transformation formula

$$w_{i'} = A^i_{i'} w_i, \tag{5.15}$$

where the matrix $\|A^i_{i'}\|$ is inverse to the matrix $\|A^{i'}_i\|$, is called a *covariant vector*. A covariant vector is a purely differential geometric object of the first class (with n components). An example of a covariant vector is provided by the gradient of a scalar field; if $\sigma(\xi)$ is a field of scalars, defined and of class C^1 in a neighbourhood of a point ξ_0, then the numbers

$$w_i = \frac{\partial \sigma}{\partial \xi^i}(\xi_0) \tag{5.16}$$

change according to formula (5.15) when we pass to a new coordinate system U'.

8. An object with n^{p+q} components $a^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}$ $(i_1, \ldots, i_p, j_1, \ldots, j_q = 1, 2, \ldots, n)$ with the transformation formula

$$a^{i'_1 \ldots i'_p}{}_{j'_1 \ldots j'_q} = A^{i'_1}_{i_1} A^{i'_2}_{i_2} \ldots A^{i'_p}_{i_p} A^{j_1}_{j'_1} \ldots A^{j_q}_{j'_q} a^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \tag{5.17}$$

is called a *p-times contravariant* and *q-times covariant affinor*, or shortly an *affinor of valence* (p, q). Affinors are purely differential geometric objects of the first class. Affinors are also called *tensors* (8).

9. An object with n^{p+q} components $\mathfrak{g}^{i_1\ldots i_p}{}_{j_1\ldots j_q}$ with the transformation formula

$$\mathfrak{g}^{i'_1\ldots i'_p}{}_{j'_1\ldots j'_q} = |J|^a A^{i'_1}_{i_1}\ldots A^{i'_p}_{i_p} A^{j_1}_{j'_1}\ldots A^{j_q}_{j'_q}\mathfrak{g}^{i_1\ldots i_p}{}_{j_1\ldots j_q} \tag{5.18}$$

is called an *affinor* W-*density* (*Weyl density*) *of valence* (p, q) *and of weight* $-a$. Analogically, an object with the transformation formula

$$\mathfrak{g}^{i'_1\ldots i'_p}{}_{j'_1\ldots j'_q} = |J|^a(\operatorname{sgn} J) A^{i'_1}_{i_1}\ldots A^{i'_p}_{i_p} A^{j_1}_{j'_1}\ldots A^{j_q}_{j'_q}\mathfrak{g}^{i_1\ldots i_p}{}_{j_1\ldots j_q} \tag{5.19}$$

is called an *affinor* G-*density* (*ordinary density*) *of valence* (p, q) *and of weight* $-a$. Affinor densities are purely differential geometric objects of the first class. Affinor densities are also called *tensor densities*, and in the case where $p = 1$ and $q = 0$, or $p = 0$ and $q = 1$, they are called *vector densities* (contravariant or covariant, respectively).

10. An object with one component ω, defined at a point x_0 of a two-dimensional manifold X_2^r $(r \geqslant 1)$, with the transformation formula

$$\omega' = \frac{A_1^{2'} + A_2^{2'}\omega}{A_1^{1'} + A_2^{1'}\omega} \tag{5.20}$$

is called an *object of Pienzow*. It is a purely differential geometric object of the first class (with one component). An example of an object of Pienzow is provided by the ratio of the components v^2/v^1 of a two-dimensional contravariant vector. So an object of Pienzow can be regarded as representing the geometric concept of direction on the plane.

11. An object with n^3 components Γ^i_{jk} $(i, j, k = 1, \ldots, n)$ with the transformation formula

$$\Gamma^{i'}_{j'k'} = A^{i'}_i A^j_{j'} A^k_{k'}\Gamma^i_{jk} + A^{i'}_a A^a_{j'k'} \tag{5.21}$$

is called an *object of a linear connection*. It is a purely differential geometric object of the second class. An example of an object of a linear connection is provided by so-called Christoffel symbols of the second kind.

12. Given a field of scalars $\sigma(\xi)$ of class C^2 in a neighbourhood of a point ξ_0, we can form an object with n^2 components

$$u_{ij} = \frac{\partial^2\sigma}{\partial\xi^j\,\partial\xi^i}(\xi_0). \tag{5.22}$$

(8) Originally, in the terminology of J. A. Schouten, an affinor was called *tensor* whenever it was symmetric with respect to all the indices. But now most authors use the terms affinor and tensor interchangeably.

Object (5.22) is not a geometric object. When we pass to a new coordinate system U', the components u_{ij} of object (5.22) change according to the formula

$$u_{i'j'} = A^k_{i'j'} w_k + A^i_{i'} A^j_{j'} u_{ij}, \tag{5.23}$$

where w_k is the covariant vector (5.16) (the gradient of the field $\sigma(\xi)$). So we see that the components $u_{i'j'}$ of object (5.22) in the coordinate system U' depend not only on the components u_{ij} of object (5.22) in the coordinate system U and on the transformation $T_{i'i}$ (represented in (5.23) by the parameters $A^i_{i'}$ and $A^k_{i'j'}$), but also on a certain additional object (5.16), which means that object (5.22) is not geometric.

Let us note, however, that object (5.22) will become a geometric object if we restrict the set of allowable coordinate systems to the set $\boldsymbol{Z}_A$ of the systems obtained from a certain coordinate system with the aid of the group $\boldsymbol{A}$ of affine transformations

$$T_{i'i}\colon\ \xi^{i'} = A^{i'}_i \xi^i + B^{i'}, \qquad \det(A^{i'}_i) \neq 0.$$

In fact, then $\xi^i = A^i_{i'} \xi^{i'} - A^i_{i'} B^{i'}$ and $A^k_{i'j'} = \dfrac{\partial^2 \xi^k}{\partial \xi^{j'} \partial \xi^{i'}} (\xi_0) = 0$. Formula (5.23) gets simpler

$$u_{i'j'} = A^i_{i'} A^j_{j'} u_{ij},$$

i. e. object (5.22) becomes a twice covariant tensor. In other words, in the affine geometry (5.22) is a geometric object, although it is not in the geometry based on the general pseudogroup of transformations $\boldsymbol{G}_r$.

If we restrict the set of allowable coordinate systems to the set of the rectangular Cartesian systems, i. e. systems obtained with the aid of the group $\boldsymbol{O}$ of the orthogonal transformations

$$T_{i'i}\colon\ \xi^{i'} = A^{i'}_i \xi^i + B^{i'},$$

where $A = \|A^{i'}_i\|$ is an orthogonal matrix, then the difference between contravariant and covariant vectors disappears. In fact, using the matrix notation one can write transformation formula (5.14) of a contravariant vector shortly in the form

$$v' = Av, \tag{5.24}$$

and the transformation formula of a covariant vector can be written in the form

$$w' = (A^{-1})^T w \tag{5.25}$$

($\|A^i_{i'}\|$ is the inverse matrix to $\|A^{i'}_i\|$ and, moreover, in (5.15) the summation is with respect to the upper index in the matrix $\|A^i_{i'}\|$, which corresponds exactly to multiplying the vector w by the matrix $\|A^i_{i'}\|^T$). But since

for the orthogonal matrices we have $(A^{-1})^T = A$, formulae (5.24) and (5.25) are identical in this case.

Thus, in order to note a difference between contravariant and covariant vectors, one must pass to a more complicated geometry than the most familiar metric geometry, based on the group $\boldsymbol{O}$.

§ 6. Abstract geometric object. The Haantjes-Laman definition of a geometric object

The function $F(\omega, L)$, occurring in (5.7), can define a transformation formula for more than one geometric object. In order to obtain from (5.7) a geometric object we must know its components ω_0 in a certain coordinate system U_0 (cf. (4.4)).

Let $M \subset A_m$ be the set of the components in the coordinate system U_0 of all the geometric objects ω which have transformation law (5.7). Since we may apply to the system U_0 all the transformations belonging to the pseudogroup $\boldsymbol{G}_r$, the function $F(\omega, L)$ must be defined in $M \times \mathscr{L}_n^s$. On the other hand, if the section of the set $\{(\omega, L)\colon \omega = \omega_0, L \in \mathscr{L}_n^s\}$ with the domain of definition of the function F is not empty, i. e. if ω_0 is the set of components of an object with transformation formula (5.7) in an allowable coordinate system, then there exists an object whose components in the system U_0 are ω_0. Hence it follows that the domain of definition of the function $F(\omega, L)$ is the set $M \times \mathscr{L}_n^s$.

The function $F(\omega, L)$ must satisfy the equation (cf. (4.5))

$$F\big(F(\omega, L_1), L_2\big) = F(\omega, L_2 L_1) \quad \text{for} \quad \omega \in M,\ L_1, L_2 \in \mathscr{L}_n^s \tag{6.1}$$

and (cf. (4.6)) the condition

$$F(\omega, I) = \omega \quad \text{for} \quad \omega \in M, \tag{6.2}$$

where I is the unity of the group $\mathscr{L}_n^s$. Hence the following implication results:

$$\textit{if } \omega \in M, \textit{ then } F(\omega, L) \in M \textit{ for every } L \in \mathscr{L}_n^s. \tag{6.3}$$

Definition 8. The family of all the geometric objects ω with transformation formula (5.7) will be called an *abstract geometric object with transformation formula* (5.7).

Every geometric object (according to definition 6) will be called a *particular geometric object.*

The set M will be called (after Haantjes and Laman [1]) the *fibre* of the (abstract) geometric object ω.

An abstract geometric object is defined whenever its transformation law is given, i. e. whenever there is given a function $F(\omega, L)$ defined in

the set $M \times \mathscr{L}_n^s$ (where $M \subset A_m$) and fulfilling equation (6.1) and condition (6.2). In order to define a particular geometric object one must also specify its components in a certain fixed coordinate system U_0.

Let us note that a function is always understood as a domain of definition and a functional rule. A functional rule alone is not sufficient to determine a function and so it does not define an abstract geometric object. Thus, e. g., rule (5.12) may define as many as seven different abstract geometric objects (according to the form of the domain of definition of the function $F(\omega, L)$, in order to define this domain it is enough to give only the set M):

1) $\omega' = |J|\omega, \quad \omega \in (-\infty, +\infty)$,

2) $\omega' = |J|\omega, \quad \omega \in (0, +\infty)$,

3) $\omega' = |J|\omega, \quad \omega \in \langle 0, +\infty)$,

4) $\omega' = |J|\omega, \quad \omega \in (-\infty, 0)$,

5) $\omega' = |J|\omega, \quad \omega \in (-\infty, 0\rangle$,

6) $\omega' = |J|\omega, \quad \omega \in (-\infty, 0) \cup (0, +\infty)$,

7) $\omega' = |J|\omega, \quad \omega = 0$.

In view of condition (6.3) other sets M are impossible.

Each of the seven above-mentioned abstract geometric objects is called an *abstract Weyl density*. We shall obtain a particular Weyl density if we prescribe its value ω_0 in a certain coordinate system U_0, e. g. assuming $\omega_0 = 2$.

The same particular geometric object may be obtained from various abstract geometric objects. Assuming, for instance, in a coordinate system U_0 (9) the values of the components of a contravariant vector (5.14)

$$v^1 \overset{*}{=} v^2 \overset{*}{=} \ldots \overset{*}{=} v^n \overset{*}{=} 0,$$

we obtain an object whose components in all the coordinate systems are all zero. Similarly, assuming in U_0 the values of the components of a contravariant vector W-density $\mathfrak{v}^{i'} = |J| A_i^{i'} \mathfrak{v}^i$

$$\mathfrak{v}^1 \overset{*}{=} \mathfrak{v}^2 \overset{*}{=} \ldots \overset{*}{=} \mathfrak{v}^n \overset{*}{=} 0,$$

we obtain the same particular geometric object.

Usually authors make no distinction between an abstract geometric object and a particular geometric object. Speaking about a particular geometric object they give only its transformation formula, and thus,

(9) In order to express the fact that a certain equality holds in one coordinate system and need not have an invariant character under the transformations of the pseudogroup considered, we use (following Schouten) the symbol $\overset{*}{=}$.

in fact, they deal with an abstract geometric object. Sometimes it can be a source of misunderstanding and often causes ambiguities and difficulties for readers not quite familiar with the theory of geometric objects.

Let ω_0 denote the components of a particular geometric object with transformation formula (5.7) in a certain fixed coordinate system U_0.

DEFINITION 9. The set of all the elements of the fibre M of the form

$$\omega = F(\omega_0, L), \tag{6.4}$$

where L ranges over the whole group $\mathscr{L}_n^s$, will be called the *transitive fibre determined by* ω_0 and will be denoted by M_{ω_0}.

According to the above definition M_{ω_0} is the set of the components of the object ω_0 in all the allowable coordinate systems. But M_{ω_0} can also be interpreted in a different manner. Namely M_{ω_0} can be regarded as the set of those particular geometric objects which in the coordinate system U_0 have components ω defined by formula (6.4).

In order to investigate the structure of the fibre M of an abstract geometric object, we introduce in M the following relation of equivalence ([10]): two elements ω_1 and ω_2 of the fibre M are said to be *equivalent* if there exists an $L \in \mathscr{L}_n^s$ such that $\omega_2 = F(\omega_1, L)$. This relation is logical equivalence, i. e. it is reflexive, symmetric and transitive. Thus the fibre M can be divided into classes of equivalent elements. The set of all the elements equivalent to ω_0 is just the transitive fibre M_{ω_0}. Thus it is obvious that every fibre of an abstract geometric object can be divided into transitive fibres. Every two transitive fibres are either identical or disjoint, and the set M is the union of all the transitive fibres:

$$M = \bigcup_{\omega \in M} M_\omega.$$

DEFINITION 10. If the fibre M of an abstract geometric object ω consists of only one transitive fibre, then the object ω is called *transitive*.

Thus, for instance, among the seven abstract Weyl densities mentioned on p. 25, objects 2), 4) and 7) are transitive.

To every particular geometric object ω_0 there may be assigned a unique transitive abstract geometric object, namely the object (with the same transformation formula as ω_0) whose fibre is M_{ω_0}. It is the abstract geometric object containing the particular geometric object ω_0 and characterized by the smallest fibre.

Suppose that we are given an abstract geometric object ω with transformation formula (5.7) and a fibre M. Formula (5.7) defines a family of transformations of the set M onto itself which depend on the parame-

([10]) Not to be confused with the equivalence of geometric objects (cf. § 8)!

ters $L \in \mathscr{L}_n^s$. This family of transformations form a Lie group. In particular, according to relation (6.1), the composition of two transformations

$$\omega' = F(\omega, L_1) \quad \text{and} \quad \omega' = F(\omega, L_2), \quad \omega \in M,$$

which correspond to the parameters L_1 and L_2, respectively, is again a transformation belonging to this family,

$$\omega' = F(\omega, L_2 L_1),$$

and corresponding to the parameters $L_2 L_1$. Since the set $\mathscr{L}_n^s$ of the parameters L also forms a group, relation (5.7) may be regarded as a homomorphism h which assigns to the group $\mathscr{L}_n^s$ a group $\mathscr{G}$ of transformations of the set M onto itself. The determination of transformation formulae of geometric objects is, in fact, the determination of homomorphisms of the group $\mathscr{L}_n^s$ into a group of transformations of the set M onto itself.

Now suppose that we are given a Hausdorff space M and an effective ([11]) topological group $\mathscr{G}$ of transformations of the set M onto itself. For every subset $\mathscr{U}$ of the group $\mathscr{G}$ and every $\omega_0 \in M$ we denote by $\mathscr{U}(\omega_0)$ the set of the elements of M which are of the form $g\omega_0$, where $g \in \mathscr{U}$. Further we suppose that the set $\mathscr{U}(\omega_0)$ is open in the set $\mathscr{G}(\omega_0)$ for every subset $\mathscr{U}$ open in $\mathscr{G}$ and every $\omega_0 \in M$.

The Haantjes-Laman definition of a geometric object (cf. Haantjes-Laman [1]) at a fixed point x_0 of a manifold X_n^r can now be formulated as follows:

Definition 11. A geometric object is defined at a point x_0 of a manifold X_n^r if there is given a homomorphism h of the group $\mathscr{L}_n^s$ into a group $\mathscr{G}$ of transformations of the fibre M onto itself.

According to our preceding remarks, giving a homomorphism h corresponds to giving a transformation formula of the object ω. This transformation formula can be written in the form

$$\omega' = h(L)\omega, \quad \omega \in M, \; L \in \mathscr{L}_n^s,$$

where $h(L)$ is the element of the group $\mathscr{G}$ which corresponds by the homomorphism h to the element L of the group $\mathscr{L}_n^s$. Thus a geometric object, as defined by J. Haantjes and G. Laman, corresponds to an abstract geometric object, with the difference that in the Haantjes-Laman defi-

([11]) A group $\mathscr{G}$ of transformations of the set M is called *effective* if for $g \in \mathscr{G}$ from the condition

$$(*) \qquad g\omega = \omega \quad \text{for all} \quad \omega \in M$$

it follows that g is the unity of the group $\mathscr{G}$. An example of a non-effective group is provided by the group of rotations of a circle, where relation (*) holds for all the rotations by a multiple of 2π (cf. Pontriagin [1], p. 135, Haantjes-Laman [1], p. 209).

nition it was necessary to make some suppositions of a topological nature, which did not occur in definition 8. Thus J. Haantjes and G. Laman eliminate all the geometric objects with discontinuous transformation formulae.

In fact J. Haantjes and G. Laman do not define a geometric object at a single point but on the whole manifold X_n^r, and they call the set of geometric objects so defined a *bundle of geometric objects*. The homomorphism h depends then on a point x of the manifold X_n^r and it is assumed that this dependence should be continuous. In the terminology of J. Haantjes and G. Laman a bundle of geometric objects is a fibre space with the base X_n^r, group $\mathscr{G}$, fibre M and homomorphism $h(x)$. As we see, a bundle of geometric objects is in fact a set of abstract geometric objects defined at all the points of the manifold considered.

We shall illustrate this by an example. Let X be a surface which at every point has a tangent plane. For a given point $x_0 \in X$ the tangent plane at the point x_0 represents a fibre M of contravariant vectors (fibre over x_0, according to the terminology of J. Haantjes and G. Laman). The set of all the vectors of all the fibres, when x_0 ranges over the whole surface X, is a bundle of contravariant vectors. In the terminlogy of J. Haantjes and G. Laman this is a fibre space with the base X, fibre A_2 and group $\mathscr{G}$, where $\mathscr{G}$ is the group of the linear homogeneous non-singular transformations of the set A_2 onto itself. $\mathscr{L}_2^1$ is here the group of non-singular matrices $\|A_i^{i'}\|$ of the second order (with the multiplication of matrices as the group operation; cf. (5.4)), and the homomorphism h assigns to a matrix $\|A_i^{i'}\| \in \mathscr{L}_2^1$ the linear homogeneous transformation of the set A_2 onto itself

$$\omega^{i'} = A_i^{i'} \omega^i.$$

A bundle of geometric objects is not to be confused with a field of particular geometric objects. In the above example we shall obtain a field of contravariant vectors by choosing from each fibre a single contravariant vector in such a manner that it should depend continuously on the point x of the surface X.

§ 7. Classification of geometric objects

S. Gołąb has based the classification of special geometric objects on the following three characteristic parameters: the number m of components of the object, the dimension n of the manifold X_n^r, and the class s of the object. J. Aczél and S. Gołąb [1] unify these three parameters into one symbol $[m, n, s]$, which they call the *type* of the geometric object. Let us note that the notion of the type of a geometric object makes sense when applied to a particular as well as to an abstract geometric object.

The classification based on the above three parameters has been generally accepted in mathematical literature. J. Haantjes and G. Laman were the first to introduce the notion of the dimension of a geometric object, defined as the dimension of the fibre M, which replaces the number of components of the object. We shall slightly modify their original definition.

DEFINITION 12. The *dimension p of a particular geometric object ω* is the (topological) dimension of its transitive fibre M_ω. The *dimension p of an abstract geometric object* is the greatest of the dimensions of its transitive fibres.

The dimension of a geometric object seems to be a useful additional parameter, but it cannot replace any of the parameters forming the type of the object. One of its shortcomings is the fact that the dimension of an object is defined only if the transitive fibres have a sufficiently regular topological structure to enable us to speak about their dimension.

A biscalar (5.9) is a geometric object with one component in an n-dimensional space, of class one and dimension zero. For a pair of densities

$$\omega^{1'} = J\omega^1, \qquad \omega^{2'} = J\omega^2$$

in a two-dimensional space X_2^r $(r \geqslant 1)$ the corresponding parameters are $m = 2$, $n = 2$, $s = 1$, $p = 1$, whereas for a contravariant vector (5.14) in the space X_2^r the corresponding parameters are $m = 2$, $n = 2$, $s = 1$, $p = 2$. Let us note that the dimension p of an object is not determined by the remaining three parameters; in particular it need not coincide with the number of components of the object.

One of the main problems of the theory of geometric objects, the so-called *classification problem*, consists in determining all the geometric objects of particular types. This problem can be solved by solving functional equation (6.1) (or rather, in most cases, a system of functional equations if the number of components of the object is greater than one). But there are also other possible approaches to this problem, e. g. with a use of Lie groups (Wagner, Pienzow), or of topological groups (Haantjes and Laman, Pienzow). The former approach (used mainly by the Polish, Hungarian and Rumanian schools), however, has the advantage of allowing us to determine geometric objects under the weakest suppositions and so it leads to the strongest results.

The classification problem is very difficult and so far has been solved only for a few types of geometric objects, often under additional suppositions about the form of the function F (e. g. that F depends on ω linearly; we then speak about *linear geometric objects*). Geometric objects with one component were first determined by S. Gołąb [2] under the supposition that the function F is of class C^1. Then geometric objects with

one component were determined by J. S. Dubnow ([12]) (in a one-dimensional space) and J. Pienzow [1], under the supposition that the function F is analytical. Lastly, geometric objects with one component have been determined under very weak regularity suppositions by J. Aczél [1], [4]. Linear geometric objects with one component have been completely determined (without any regularity suppositions whatever) in the following papers: Gołąb-Kucharzewski [2], Kuczma [1], Kucharzewski-Kuczma [1], [2].

Objects with a number of components greater than one have been determined, under various suppositions regarding the number of components, dimension of the space, class of the object, as well as the regularity or the form of the functions occurring in the transformation formula, by J. Aczél, A. Balogh, J. S. Dubnow, O. E. Gheorghiu, B. Crstici, S. Gołąb, M. Hosszú, A. E. Liber, M. Kucharzewski, M. Kuczma, M. M. Mihaileanu, J. Pienzow, P. K. Rashevski and W. W. Wagner. The reader interested in details is referred to the papers quoted in the references at the end of the present paper as well as to the monograph of J. Aczél and S. Gołąb [1].

The classification problem has not been completely solved even for linear geometric objects. The transformation formula of a linear geometric object $\omega(\omega^1, \ldots, \omega^m)$ has the form

$$\omega' = f(L)\omega + g(L),$$

where $f = \|f_i^{i'}\|$ is a square matrix of order m and $g = \|g^{i'}\|$ is a matrix with m rows and one column. From (6.1) we obtain the system of functional equations

$$\begin{aligned} f(L_2L_1) &= f(L_2)f(L_1), \\ g(L_2L_1) &= f(L_2)g(L_1)+g(L_2). \end{aligned} \tag{7.1}$$

As we have said, system (7.1) has not been solved yet in the general case (for arbitrary m, n, s). But for linear geometric objects with one component the classification problem has been solved completely. The only linear geometric objects with one component, of class $s \neq 1$, are the objects with the transformation formula

$$\omega' = \varphi(J)\omega + c[\varphi(J) - 1] \tag{7.2}$$

or

$$\omega' = \omega + \log|\varphi(J)|, \tag{7.3}$$

where $\varphi(u)$ may be an arbitrary solution of the functional equation

$$\varphi(uv) = \varphi(u)\varphi(v),$$

([12]) The results of Dubnow have not been published; cf. Wagner [5].

and c is an arbitrary constant. Under the supposition of the measurability of the function φ objects (7.2) and (7.3) are equivalent (cf. § 8) to an ordinary density (5.10), or to a Weyl density (5.12), or to a biscalar (5.9), or to a scalar (5.8) (Kuczma [1]).

Linear geometric objects with one component, of class $s = 2$ and $s = 3$, exist only in a one-dimensional space. Their transformation formulae have the form (we introduce here the short notation: $a_1 = A^1_1$, $a_2 = A^1_{11}$, $a_3 = A^1_{111}$)

$$\omega' = \frac{1}{a_1}\omega + k\frac{a_2}{a_1^2} + c\left(\frac{1}{a_1} - 1\right), \tag{7.4}$$

$$\omega' = \frac{1}{a_1^2}\omega - \frac{3}{2}k\frac{a_2^2}{a_1^4} + k\frac{a_3}{a_1^3} + c\left(\frac{1}{a_1^2} - 1\right), \tag{7.5}$$

where k and c may be arbitrary constants (Kucharzewski–Kuczma [1], [2]). (7.4) and (7.5) are objects of a linear and a projective connection, respectively.

Objects with one component, of class $s \geqslant 4$, do not exist (Gołąb–Kucharzewski [2], Kucharzewski–Kuczma [1]). This theorem is true also for non-linear geometric objects (Dubnow [1], Pienzow [1], [3], Gołąb [8], [11], [26], Aczél [1], [6]).

Linear geometric objects with several components have been determined (for some values of the parameters m, n, s) by O. E. Gheorghiu [4], [8], [9], [18], [20], O. E. Gheorghiu and B. Crstici [1], A. Balogh [1], M. Kucharzewski and M. Kuczma [3], [4], [5], [8], and P. K. Rashevski [2].

J. Haantjes and G. Laman [2] have determined all the transitive geometric objects (according to their definition, this means, of course, abstract geometric objects ([13])) of the first class and dimension 1. Their results differ slightly from those of S. Gołąb [2], [12] and J. Pienzow [1] (concerning objects with one component), which is assigned by the authors to the difference in the definition of a geometric objects on which they have based their investigations. But the main reason lies here in the fact that S. Gołąb and J. Pienzow on one hand, and J. Haantjes and G. Laman on the other hand, have based their classification on different parameters:

([13]) J. Haantjes and G. Laman determine bundles of geometric objects, and thus "fields" of abstract geometric objects. But they assume that the homomorphism h does not depend on x. Thus we may confine ourselves to considering these objects at one fixed point x_0 of the manifold X^r_n.

J. Haantjes and G. Laman obtained their result regarding the determination of the geometric objects of the first class and dimension 1 from a certain theorem concerning closed subgroups of the multiplicative group of real numbers. They dit not give, however, a proof of that theorem, or a reference to a paper where this proof could be found. Consequently their results cannot be regarded as proved in a satisfactory manner.

the former on the number of components of the object, and the latter on the dimension of the object. Now we shall survey the objects found by J. Haantjes and G. Laman [2] and we shall compare them with some objects found by other authors on the ground of Wundheiler's definition of a geometric object. We shall see that in some cases these will be objects with one component, but in other cases–objects with two or even three components.

Object I.1 in paper [2] of J. Haantjes and G. Laman has as its fibre the set $M = (-\infty, 0) \cup (0, +\infty)$. The group $\mathscr{G}$ is the group of the transformations (14) $\omega' = g\omega$, where $g \neq 0$. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_n^1$ the transformation $\omega' = J\omega$, where $J = \det(A_i^{i'})$. Thus this object is the abstract ordinary density (cf. (5.10)) with the transitive fibre $M = (-\infty, 0) \cup (0, +\infty)$.

Object I.2 has the fibre $M = (0, +\infty)$. $\mathscr{G}$ is the group of the transformations $\omega' = g\omega$, where $g > 0$. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_n^1$ the transformation $\omega' = |J|\omega$. Thus it is the abstract Weyl density (cf. (5.12)) with the transitive fibre $M = (0, +\infty)$.

Object I.4 (15) has as its fibre M the unit circle

$$(\omega^1)^2 + (\omega^2)^2 = 1. \tag{7.6}$$

$\mathscr{G}$ is here the group of rotations (16) of circle (7.6). The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_n^1$ the rotation of the fibre M through the angle

$$\frac{2\pi \log|J|}{\log c} \pmod{2\pi}$$

where $c > 1$ is a constant. The homomorphism h can be written in the form (17)

$$\begin{cases} \omega^{1'} = \omega^1 \cos\dfrac{2\pi\log|J|}{\log c} - \omega^2 \sin\dfrac{2\pi\log|J|}{\log c}, \\[2ex] \omega^{2'} = \omega^1 \sin\dfrac{2\pi\log|J|}{\log c} + \omega^2 \cos\dfrac{2\pi\log|J|}{\log c}. \end{cases} \tag{7.7}$$

(14) In § 6 in the formula $\omega' = g\omega$, $g\omega$ denoted the result of the action of the element g of a group $\mathscr{G}$ on the point $\omega \in M$. Now $g\omega$ denotes the product of the numbers g and ω. It is not a contradiction, because in this case the number g determines uniquely a transformation from the group $\mathscr{G}$ consisting in multiplying all the $\omega \in M$ by g.

(15) Object I.3, as more complicated, will be discussed after objects I.4 and I.5.

(16) The group of rotations must be taken modulo 2π, for otherwise it would not be effective.

(17) On account of the periodicity of the functions cosine and sine we may omit $\bmod 2\pi$ in the arguments of these functions in (7.7), (7.8) and (7.10).

This object has been determined and classified in papers [3], [4] of Kucharzewski and Kuczma. (For the first time this object was found by A. Woźniacki in the year 1952; his result, however, has not been published.) Object I.4 is abstract object (7.7) with transitive fibre (7.6).

Object I.5 has, as its fibre M again circle (7.6). The group $\mathscr{G}$, however, is this time the group of the rotations of the circle M and symmetry of M with respect to its centre. The homomorphism H assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_n^1$ the rotation of M through the angle $(\pi \log J)/\log c \pmod{2\pi}$ if $J > 0$, and the rotation of M through the angle $(\pi \log(-J))/\log c \pmod{2\pi}$ and symmetry with respect to the centre if $J < 0$ (where $c > 1$ is a constant). The homomorphism h can be written in the form

$$(7.8)\qquad \begin{cases} \omega^{1'} = (\operatorname{sgn} J)\left(\omega^1 \cos\dfrac{\pi \log|J|}{\log c} - \omega^2 \sin\dfrac{\pi \log|J|}{\log c}\right), \\ \omega^{2'} = (\operatorname{sgn} J)\left(\omega^1 \sin\dfrac{\pi \log|J|}{\log c} + \omega^2 \cos\dfrac{\pi \log|J|}{\log c}\right). \end{cases}$$

This object has also been determined and classified in papers [3], [4] of Kucharzewski and Kuczma (and previously in an unpublished work of A. Woźniacki). Object I.5 is abstract object (7.8) with transitive fibre (7.6).

Object I.3 has as its fibre M two unit circles. We must put them in the three-dimensional space A_3. Their equations will have the form

$$(7.9)\qquad \begin{cases} (\omega^1)^2 + (\omega^2)^2 = 1, \quad \omega^3 = 1, \\ (\omega^1)^2 + (\omega^2)^2 = 1, \quad \omega^3 = -1. \end{cases}$$

The group $\mathscr{G}$ is the group of the rotations of circles (7.9) and symmetry with respect to the plane $\omega^3 = 0$. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_n^1$ the rotation of circles (7.9) through the angle $\dfrac{2\pi \log J}{\log c} \pmod{2\pi}$ if $J > 0$, and the rotation through the angle $\dfrac{2\pi \log(-J)}{\log c} \pmod{2\pi}$ and symmetry with respect to the plane $\omega^3 = 0$ if $J < 0$ (where $c > 1$ is a constant). The homomorphism h can be written in the form

$$(7.10)\qquad \begin{cases} \omega^{1'} = \omega^1 \cos\dfrac{2\pi \log|J|}{\log c} - \omega^2 \sin\dfrac{2\pi \log|J|}{\log c}, \\ \omega^{2'} = \omega^1 \sin\dfrac{2\pi \log|J|}{\log c} + \omega^2 \cos\dfrac{2\pi \log|J|}{\log c}, \\ \omega^{3'} = (\operatorname{sgn} J)\,\omega^3. \end{cases}$$

This object has been determined and classified in paper [8] of Kucharzewski and Kuczma. Object I.3 is abstract object (7.10) with transitive fibre (7.9).

Remark. Objects I.3, I.4 and I.5 are, in fact, families of (abstract) geometric objects. For different values of the parameter $c > 1$ we obtain different objects. Objects corresponding to different values of the parameter $c > 1$ are not equivalent! (cf. Kucharzewski–Kuczma [4], [8]; regarding the notion of the equivalence of geometric objects cf. § 8).

Objects II.1-II.5 exist only in a two-dimensional space. The group $\mathscr{L}_n^s = \mathscr{L}_2^1$ is here a group of non-singular matrices of the second order with the multiplication of matrices as the group operation.

Object II.2 ([18]) has as its fibre M unit circle (7.6). To every point of the circle M there corresponds exactly one semi-line drawn from the origin (on the plane of the variables ω^1, ω^2) and passing through this point. Every centro-affine transformation on the plane of the variables ω^1, ω^2 determines a transformation of the set of those semi-lines onto itself. This transformation, in turn induces a transformation of the circle M onto itself. The group $\mathscr{G}$ is in this case the group of the transformations of the circle M onto itself induced by the group of the centro-affine transformations of the plane of the variables ω^1, ω^2. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_2^1$ the transformation of the circle M induced by the centro-affine transformation of the plane of the variables ω^1, ω^2:

$$\begin{cases} \omega^{1'} = A_1^{1'}\omega^1 + A_2^{1'}\omega^2, \\ \omega^{2'} = A_1^{2'}\omega^1 + A_2^{2'}\omega^2. \end{cases} \tag{7.11}$$

This homomorphism can be written in the form

$$\begin{cases} \omega^{1'} = \varrho(\omega, L)[A_1^{1'}\omega^1 + A_2^{1'}\omega^2], \\ \omega^{2'} = \varrho(\omega, L)[A_1^{2'}\omega^1 + A_2^{2'}\omega^2], \end{cases} \tag{7.12}$$

where

$$\varrho(\omega, L) = \frac{[(\omega^1)^2 + (\omega^2)^2]^{1/2}}{[(A_1^{1'}\omega^1 + A_2^{1'}\omega^2)^2 + (A_1^{2'}\omega^1 + A_2^{2'}\omega^2)^2]^{1/2}}. \tag{7.13}$$

Object II.2 is the (non-linear) abstract object (7.12) with transitive fibre (7.6).

As is seen from the above description, object II.2 represents on the plane a direction with orientation. It can be put into yet another form ([19]).

([18]) Object II.1, as more complicated, will be discussed after objects II.2 and II.3.

([19]) The following considerations consist essentially in replacing object (7.12) by equivalent object (7.17).

We replace fibre (7.6) by two straight lines

$$\omega^1 = 1 \quad \text{and} \quad \omega^1 = -1. \tag{7.14}$$

Now we map the fibre M onto the new fibre (7.14), assigning to each point ω_0 of circle (7.6) that point of the set (7.14) at which the semi-line drawn from the origin and passing through the point ω_0 cuts the straight lines (7.14). In this manner the points $\omega^1 = 0$, $\omega^2 = 1$ and $\omega^1 = 0$, $\omega^2 = -1$ have no correspondents on the straight-lines (7.14). We shall include them into the new fibre as improper points (points at infinity). Thus the new fibre N will consist of two straight lines (the parameters on the fibre N will be denoted by σ^1, σ^2 to distinguish them from the points of the fibre M)

$$\begin{cases} \sigma^1 = 1, & \sigma^2 \in (-\infty, +\infty), \\ \sigma^1 = -1, & \sigma^2 \in (-\infty, +\infty), . \end{cases} \tag{7.15}$$

and of two improper points

$$\begin{cases} \sigma^1 = 0, & \sigma^2 = 1, \\ \sigma^1 = 0, & \sigma^2 = -1. \end{cases} \tag{7.16}$$

σ^1 and σ^2 are connected with ω^1 and ω^2 by the relations

$$\begin{aligned} \sigma^1 &= \operatorname{sgn}\omega^1, & \sigma^2 &= \omega^2/\omega^1 & &\text{if} & \omega^1 &\neq 0, \\ \sigma^1 &= \omega^1, & \sigma^2 &= \omega^2 & &\text{if} & \omega^1 &= 0. \end{aligned}$$

A centro-affine transformation of the plane of the variables σ^1, σ^2 induces a transformation of the fibre N onto itself, which on the lines (7.15) can be written as

$$\begin{cases} \sigma^{1'} = \sigma^1 \operatorname{sgn}(A_1^{1'} + A_2^{1'}\sigma^2), \\ \sigma^{2'} = \dfrac{A_1^2 + A_2^{2'}\sigma^2}{A_1^{1'} + A_2^{1'}\sigma^2}. \end{cases} \tag{7.17}$$

Object (7.17) has occurred in the paper of Gołąb–Jakubowicz–Kucharzewski–Kuczma [1] as an object determining (on the plane) a direction with orientation. Let us note, however, that formula (7.17) is not quite exact; it does not make allowance for the improper points (7.16) of the fibre N. This object may be included in the family of objects relative to the object of Pienzow (5.20). All the objects of this family are usually written in mathematical literature in this not quite exact manner.

Object II.3 has as its fibre the unit circle (7.6). The group $\mathscr{G}$ is the group of the transformations of the fibre M induced by the group of the centro-affine transformations of the plane of the variables ω^1, ω^2 and symmetry with respect to the centre of the circle. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_2^1$ the transformation of the fibre M

induced by the centro-affine transformation (7.11) of the plane of the variables ω^1, ω^2 if $J > 0$, and the transformation induced by (7.11) and symmetry with respect to the centre if $J < 0$. This homomorphism can be written in the form

$$(7.18)\qquad \begin{cases} \omega^{1'} = (\operatorname{sgn} J)\,\varrho(\omega, L)[A_1^{1'}\omega^1 + A_2^{1'}\omega^2], \\ \omega^{2'} = (\operatorname{sgn} J)\,\varrho(\omega, L)[A_1^{2'}\omega^1 + A_2^{2'}\omega^2], \end{cases}$$

where $\varrho(\omega, L)$ is given by formula (7.13). Object II.3 is the (non-linear) abstract object (7.18) with transitive fibre (7.6).

Introducing again the fibre N (given by formulae (7.15) and (7.16)) we can put object II.3 (apart from the improper points) into the form

$$\begin{cases} \sigma^{1'} = \sigma^1(\operatorname{sgn} J)[\operatorname{sgn}(A_1^{1'} + A_2^{1'}\sigma^2)], \\ \sigma^{2'} = \dfrac{A_1^{2'} + A_2^{2'}\sigma^2}{A_1^{1'} + A_2^{1'}\sigma^2}. \end{cases}$$

It is again an object related to objects of Pienzow. As far as we know, it has not occurred so far in other papers.

Object II.1 has as its fibre M two unit circles (7.9). The group $\mathscr{G}$ is the group of the transformations of circles (7.9) into themselves induced by the group of the centro-affine transformations of the planes $\omega^3 = 1$ and $\omega^3 = -1$ and symmetry with respect to the plane $\omega^3 = 0$. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_2^1$ the transformation of the fibre M induced by the centro-affine transformation of the planes $\omega^3 = 1$ and $\omega^3 = -1$:

$$(7.19)\qquad \begin{cases} \omega^{1'} = A_1^{1'}\omega^1 + A_2^{1'}\omega^2, \\ \omega^{2'} = A_1^{2'}\omega^1 + A_2^{2'}\omega^2, \\ \omega^{3'} = \omega^3 \end{cases}$$

if $J > 0$, and the transformation of the fibre M induced by transformation (7.19) and symmetry with respect to the plane $\omega^3 = 0$ if $J < 0$. This homomorphism can be written in the form

$$(7.20)\qquad \begin{cases} \omega^{1'} = \varrho(\omega, L)[A_1^{1'}\omega^1 + A_2^{1'}\omega^2], \\ \omega^{2'} = \varrho(\omega, L)[A_1^{2'}\omega^1 + A_2^{2'}\omega^2], \\ \omega^{3} = (\operatorname{sgn} J)\,\omega^3, \end{cases}$$

where $\varrho(\omega, L)$ is given by formula (7.13). Object II.1 is the (non-linear) abstract object (7.20) with transitive fibre (7.9).

Introducing again the fibres N_1 and N_2 on the planes $\sigma^3 = 1$ and $\sigma^3 = -1$, respectively, we can put object II.1 (apart from the improper

points) into the form

$$\begin{cases} \sigma^{1'} = \sigma^1 \operatorname{sgn}(A_1^{1'} + A_2^{1'}\sigma^2), \\ \sigma^{2'} = \dfrac{A_1^{2'} + A_2^{2'}\sigma^2}{A_1^{1'} + A_2^{1'}\sigma^2}, \\ \sigma^{3'} = (\operatorname{sgn} J)\sigma^3. \end{cases}$$

It is again an object related to objects of Pienzow. As far as we know' this object has not occurred so far in other papers.

Object II.5 has as its fibre M a unit circle on which points symmetric with respect to the centre have been identified:

$$(7.21) \qquad (\omega^1)^2 + (\omega^2)^2 = 1, \quad (\omega^1, \omega^2) \quad (-\omega^1, -\omega^2).$$

To every such point there corresponds in a one-to-one manner a whole straight line drawn through the centre of the circle. The group of the centro-affine transformations of the plane of the variables ω^1, ω^2 induces a group $\mathscr{G}$ of transformations of fibre (7.21) onto itself. The homomorphism h assigns to an element $L = \|A_i^{i'}\| \in \mathscr{L}_2^1$ the transformation of the fibre M induced by the centro-affine transformation (7.11) of the plane of the variables ω^1, ω^2. Because of the identification of the points on the circle, it would be difficult to write this homomorphism with the aid of effective formulae.

Fibre (7.21) can be replaced by a new fibre N, a projective line. Introducing on N homogeneous coordinates σ^1, σ^2, we can write the homomorphism h in the form

$$(7.22) \qquad \begin{cases} \sigma^{1'} = A_1^{1'}\sigma^1 + A_2^{1'}\sigma^2, \\ \sigma^{2'} = A_1^{2'}\sigma^1 + A_2^{2'}\sigma^2. \end{cases}$$

But formulae (7.22) do not define a geometric object in the sense of Wundheiler's definition, since to a given coordinate system there corresponds not a single sequence of numbers (components of the object) but a whole family of pairs (σ^1, σ^2) (pairs (σ^1, σ^2) and $(\bar{\sigma}^1, \bar{\sigma}^2)$ are components of the same object whenever $\begin{vmatrix} \bar{\sigma}^1 & \bar{\sigma}^2 \\ \sigma^1 & \sigma^2 \end{vmatrix} = 0$). Disregarding the improper point $(0, \sigma^2)$ of N, we can introduce on N a single parameter $\tau = \sigma^2/\sigma^1$. The homomorphism h can then be written in the form

$$(7.23) \qquad \tau' \quad \frac{A_1^{2'} + A_2^{2'}\tau}{A_1^{1'} + A_2^{1'}\tau}.$$

This is an object of Pienzow (5.20). But formula (7.23) is not quite exact (cf. the remark regarding object II.2 above).

Here we see an advantage of the approach of J. Haantjes and G.

Laman over the classical approach of A. Wundheiler. Making use of the latter it is difficult to write down an object of Pienzow in an exact and simple manner. A way out of this difficulty seems to be the following. One should generalize Wundheiler's definition of a geometric object as A. Pforr [1] generalized the notion of coordinates, i. e. assign to coordinate systems not uniquely determined sequences of numbers (components of the object) but uniquely determined subsets of an arithmetical space. An attempt at such an approach may be found in the paper of W. W. Wagner [5], p. 167 ([20]).

Object II.4 has as its fibre M two circles on which points symmetric with respect to the centre have been identified:

$$(7.24)\qquad \begin{cases}(\omega^1)^2+(\omega^2)^2=1, & \omega^3=1,\\ (\omega^1)^2+(\omega^2)^2=1, & \omega^3=-1,\end{cases}\qquad (\omega^1,\omega^2,\omega^3)\qquad (-\omega^1,-\omega^2,\omega^3).$$

The group $\mathscr{G}$ is the group of the transformations of the fibre M induced by the centro-affine group of transformations of the planes $\omega^3=1$ and $\omega^3=-1$ and the symmetry with respect to plane $\omega^3=0$. The homomorphism h assigns to an element $L=\|A_i^{i'}\|\in\mathscr{L}_2^1$ the transformation of the fibre M induced by the centro-affine transformation (7.19) if $J>0$, and the transformation induced by (7.19) and symmetry with respect to the plane $\omega^3=0$ if $J<0$.

Replacing the fibre M by two projective lines N_1 and N_2 corresponding to the parameters $\sigma^3=1$ and $\sigma^3=-1$, respectively, and introducing on N_1 and N_2 homogeneous coordinates (σ^1,σ^2), we can write the homomorphism h in the form

$$(7.25)\qquad \begin{cases}\sigma^{1'}=A_1^{1'}\sigma^1+A_2^{1'}\sigma^2,\\ \sigma^{2'}=A_1^{2'}\sigma^1+A_2^{2'}\sigma^2,\\ \sigma^{3'}=(\operatorname{sgn}J)\sigma^3.\end{cases}$$

As in the preceding case, formulae (7.25) do not define a geometric object in the sense of Wundheiler's definition. Disregarding the improper points on the lines N_1 and N_2 we can introduce on the fibre $N=N_1\cup N_2$ two parameters: $\tau^1=\sigma^2/\sigma^1$, $\tau^2=\sigma^3$. The homomorphism h can then be written in the form

$$(7.26)\qquad \tau^{1'}=(A_1^{2'}+A_2^{2'}\tau^1)/(A_1^{1'}+A_2^{1'}\tau^1),\qquad \tau^{2'}\qquad (\operatorname{sgn}J)\tau^2.$$

It is again an object related to objects of Pienzow. But also this time formula (7.26) is not quite exact.

([20]) Added in proof. This idea has recently been developed by E. Siwek [2], [3].

Let us note that among the above ten objects only I.1, I.2 and II.5 are objects with one component. They have been found by S. Gołąb [2], [12] and partially also by J. Pienzow [1] (owing to the analyticity supposition regarding the transformation formula of the required objects, Pienzow has not obtained Weyl density I.2). On the other hand, objects with one component (5.8) and (5.9) (i. e. a scalar and a biscalar) do not appear in the work of J. Haantjes and G. Laman. In fact, the dimension of these objects is not one but zero.

§ 8. Equivalence of geometric objects

As we have seen, two objects with quite different transformation formulae can represent the same geometrical meaning (e. g., objects (7.12) and (7.17), which both represent a direction with orientation in a two-dimensional space). This fact has induced S. Gołąb to introduce in the year 1950 the notion of equivalence (or similarity) of geometric objects (Gołąb [13], [14], [15]).

DEFINITION 13. Particular geometric objects ω and σ are called *equivalent* if there exists an invertible function H such that the relation

$$\sigma = H(\omega) \tag{8.1}$$

holds in every allowable coordinate system (i. e. it is invariant under transformations of the coordinates).

Since the above definition concerns particular geometric objects, the function H occurring in it need be defined and invertible only on the transitive fibre M_ω of the object ω. Hence it follows that, say, the particular pair of Weyl densities

$$\begin{gathered} \omega^{1'} = |J|\,\omega^1, \quad \omega^{2'} = |J|\,\omega^2, \\ \omega^1 \overset{*}{=} 1, \quad \omega^2 \overset{*}{=} 1 \quad \text{in a coordinate system } U_0, \end{gathered} \tag{8.2}$$

(regarded as one object with two components) is equivalent to the single particular Weyl density

$$\sigma' = |J|\,\sigma, \quad \sigma \overset{*}{=} 1 \quad \text{in the system } U_0. \tag{8.3}$$

In fact, the function

$$H(\omega^1, \omega^2) = \omega^1$$

is defined and uniquely invertible on the transitive fibre M_ω of object (8.2) (on M_ω we have $\omega^1 = \omega^2$) and the relation

$$\sigma = H(\omega^1, \omega^2)$$

holds in every allowable coordinate system.

The notion of the equivalence can be extended in a natural way onto the abstract geometric objects. Let ω and σ be abstract geometric objects with the transformation formulae

$$\omega' = F(\omega, L), \qquad \omega \in M, \ L \in \mathscr{L}_n^s, \tag{8.4}$$

and

$$\sigma' = G(\sigma, L), \qquad \sigma \in N, \ L \in \mathscr{L}_n^s, \tag{8.5}$$

and with the fibres M and N respectively.

DEFINITION 14. Abstract geometric objects ω and σ with transformation formulae (8.4) and (8.5) respectively are called *equivalent* if there exists a one-to-one mapping H of the fibre M onto the fibre N such that for $\omega \in M$ and $\sigma \in N$ relation (8.1) implies the relation

$$\sigma' = H(\omega'). \tag{8.6}$$

Although the particular pair of Weyl densities (8.2) is equivalent to the particular Weyl density (8.3), the abstract pair of Weyl densities

$$\omega^{1'} = |J|\,\omega^1, \qquad \omega^{2'} = |J|\,\omega^2, \qquad \begin{array}{l} \omega^1 \in (-\infty, 0) \cup (0, +\infty), \\ \omega^2 \in (-\infty, 0) \cup (0, +\infty) \end{array} \tag{8.7}$$

is not equivalent to the single abstract Weyl density

$$\sigma' = |J|\,\sigma, \qquad \sigma \in (-\infty, 0) \cup (0, +\infty) \tag{8.8}$$

(cf. § 9).

If we replace ω' and σ' in (8.6) by the corresponding expressions (8.4) and (8.5), then we obtain

$$G(\sigma, L) = H\big(F(\omega, L)\big),$$

whence by (8.1)

$$F(\omega, L) = H^{-1}\Big(G\big(H(\omega), L\big)\Big), \qquad \omega \in M, \ L \in \mathscr{L}_n^s. \tag{8.9}$$

Relation (8.9) says that the groups of transformations of the fibres M and N given by transformation formulae (8.4) and (8.5) are isomorphic and the isomorphism is determined by the function $H(\omega)$.

According to the definition given by J. Haantjes and G. Laman [1], two geometric objects (in the sense of the Haantjes–Laman definition) ω and σ at the point x_0 are equivalent if there exists a homeomorphism $H(\omega)$ of the fibre M onto the fibre N fulfilling relation (8.9). In the case of bundles of objects the homeomorphism H is assumed to exist for every point x of the manifold X_n^r and to depend on x in a continuous manner.

So the difference between the equivalence of abstract geometric objects (definition 14) and the equivalence of geometric objects defined

by J. Haantjes and G. Laman is that in definition 14 the function H is not required to be a homeomorphism. Now we shall introduce a notion which will correspond to the Haantjes–Laman concept of the equivalence of geometric objects.

DEFINITION 15. If the fibres M and N of abstract geometric objects (8.4) and (8.5) are topological spaces and if there exists a homeomorphism H of the fibre M onto the fibre N such that for $\omega \in M$ and $\sigma \in N$ relation (8.1) implies relation (8.6), then objects (8.4) and (8.5) are called *strictly equivalent.*

Of course, if two abstract geometric objects are strictly equivalent, they are also equivalent. But two equivalent abstract geometric objects need not be strictly equivalent. We shall show this by an example.

Let $\sigma = K(\omega)$ be a one-to-one map of the unit circle (7.6) onto the surface of the unit sphere

$$(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 = 1. \tag{8.10}$$

(Such a map exists since sets (7.6) and (8.10) have the same cardinality.) The function

$$\sigma = |\omega| K\left(\frac{\omega}{|\omega|}\right), \qquad |\omega| = [(\omega^1)^2 + (\omega^2)^2]^{1/2},$$

establishes equivalence between the abstract pair of Weyl densities (8.7) and the abstract triple of Weyl densities

$$\sigma^{1'} = |J|\sigma^1, \qquad \sigma^{2'} = |J|\sigma^2, \qquad \sigma^{3'} = |J|\sigma^3, \tag{8.11}$$
$$\sigma^i \in (-\infty, 0) \cup (0, \infty), \qquad i = 1, 2, 3.$$

But the fibre of object (8.7) (the two-dimensional space A_2 with the point $(0, 0)$ removed) and the fibre of object (8.11) (the three-dimensional space A_3 with the point $(0, 0, 0)$ removed) are not homeomorphic. Consequently objects (8.7) and (8.11) are not strictly equivalent.

If abstract geometric objects (8.4) and (8.5) are equivalent, then every particular geometric object with transformation formula (8.4) is equivalent to a particular geometric object with transformation formla (8.5). The converse theorem is not true, as it is obvious from the example of objects (8.7) and (8.8). Although every particular pair of Weyl densities is equivalent to a single particular Weyl density, nevertheless the abstract Weyl density (8.8) and the abstract pair of Weyl densities (8.7) are not equivalent.

But if the abstract geometric objects (8.4) and (8.5) are transitive, then from the equivalence of particular geometric objects with transformation formulae (8.4) and (8.5) follows the equivalence of the abstract geometric objects (8.4) and (8.5).

Finally let us note that two particular geometric objects with the same transformation formula by no means need to be equivalent. E. g., the particular Weyl density (8.3) and the particular Weyl density

$$\omega' = |J|\omega, \quad \omega \overset{*}{=} 0 \text{ in the system } U_0,$$

are not equivalent.

§ 9. Fibres of equivalent geometric objects

The study of the equivalence of geometric objects is an important problem of the theory of geometric objects (it forms a part of the general classification problem). Establishing the equivalence or non-equivalence of geometric objects ω and σ with transformation formulae (8.4) and (8.5) respectively consists usually in solving functional equation (8.9) or in proving that this equation has no invertible solution. Often it is a rather difficult task. Some help can be obtained from theorems concerning relations between the transitive fibres of equivalent objects([21]).

THEOREM 1 (Kucharzewski–Kuczma [7]). *If the abstract geometric objects* (8.4) *and* (8.5) *are equivalent, then the function H which establishes their equivalence maps the transitive fibres of object* (8.4) *onto the transitive fibres of object* (8.5). *Thus the function H establishes a one-to-one correspondence between the transitive fibres of object* (8.4) *and the transitive fibres of object* (8.5).

It follows from the above theorem that the sets of the transitive fibres of equivalent geometric objects must be of the same cardinality. Thus, for instance, the single abstract Weyl density which has three transitive fibres: the interval $(-\infty, 0)$, the point 0, and the interval $(0, +\infty)$, cannot be equivalent to the abstract pair of Weyl densities whose set of the transitive fibres is infinite (consists of the set of the semilines beginning at the origin and of one single point: the origin itself).

Since the function H is invertible, the corresponding transitive fibres of equivalent objects must also be of the same cardinality. Hence it follows for instance that a scalar (whose transitive fibre consists of a single point) and a biscalar (whose transitive fibre consists of two points) cannot be equivalent. But the above properties do not allow us to decide about the equivalence of the transitive ordinary density (with the fibre $(-\infty, 0) \cup (0, +\infty)$) and the transitive Weyl density (with the fibre $(0, +\infty)$).

One can prove, however, a stronger theorem concerning the relation between the fibres of geometric objects whose transformation formulae are sufficiently regular.

([21]) Thus the determination of the transitive fibres of geometric objects (cf. Gołąb-Siwek [1], Siwek [1]) is of great importance.

DEFINITION 16. We shall say that transformation formula (8.4) of an object ω is *regular* if for every sequence $L_\nu \in \mathscr{L}_n^s$, $L_\nu \to I$ (I is the unity of the group $\mathscr{L}_n^s$) and for every $\omega_0 \in M$ we have

$$\lim_{\nu\to\infty} F(\omega_0, L_\nu) = \omega_0$$

and conversely, for every sequence $\omega_\nu \in M_{\omega_0}$, $\omega_\nu \to \omega_0$, there exists a sequence $L_\nu \in \mathscr{L}_n^s$, $L_\nu \to I$ such that $\omega_\nu = F(\omega_0, L_\nu)$.

The condition of the regularity (in the above sense) of the transformation formula of an object is somewhat stronger than the condition of the continuity of the function F. It certainly will not be fulfilled for objects with a discontinuous transformation formula. But we shall show here an example of an object with a continuous transformation formula which is not regular in the sense of the above definition.

For this purpose we assume $s = 1$ and we restrict the set Z of the allowable coordinate systems to the set of the coordinate systems obtained from a certain system U_0 by applying the transformations of class C^1 in a neighbourhood of the point x_0 and with the Jacobian

$$J = e^k, \quad k \text{ an integer.} \tag{9.1}$$

Thus the group $\mathscr{L}_n^1$ will be restricted to the subgroup $\mathscr{K}$ consisting of the matrices $\|A_i^{i'}\|$ with the determinant $J = \det\|A_i^{i'}\|$ fulfilling (9.1).

Then the transformation formula of object (7.7), though continuous, is not regular. In fact, let us choose arbitrarily an $\omega_0 = (\omega_0^1, \omega_0^2)$, $(\omega_0^1)^2 + (\omega_0^2)^2 = 1$. The transitive fibre M_{ω_0} of the object ω_0 with transformation formula (7.7) forms a dense set on circle (7.6). Thus we can choose a sequence $\omega_\nu \in M_{\omega_0}$ such that $\omega_\nu \to \omega_0$, $\omega_\nu \neq \omega_0$. Suppose that there exists a sequence $L_\nu \in \mathscr{K}$ such that $L_\nu \to I$, $\omega_\nu = F(\omega_0, L_\nu)$. To each L_ν there corresponds a matrix $\|_\nu A_i^{i'}\|$, to I there corresponds the unit matrix $E = \|\delta_i^{i'}\|$ (where $\delta_i^{i'}$ is the Kronecker symbol). Let $J_\nu = \det\|_\nu A_i^{i'}\|$, $J_0 = \det E$. From the convergence $L_\nu \to I$ it follows that $J_\nu \to J_0$. But, according to (9.1), $J_\nu = e^{k_\nu}$, $J_0 = 1$, and so we would have $k_\nu \to 0$, which is impossible, since k_ν are integers different from zero (otherwise we would have $\omega_\nu = \omega_0$).

For objects with regular transformation formulae we have the following

THEOREM 2 (Kucharzewski–Kuczma [7]). *If geometric objects ω and σ with regular transformation formulae are equivalent, then their transitive fibres are homeomorphic.*

The above theorem is valid for particular as well as abstract geometric objects. From theorem 2 it follows in particular that the transitive ordinary density and the transitive Weyl density are not equivalent, since the sets $(-\infty, 0) \cup (0, +\infty)$ and $(0, +\infty)$ are not homeomorphic.

Theorem 2 is valid only for transitive fibres. Intransitive fibres of equivalent geometric objects with a regular transformation formulae need not be homeomorphic (i. e., equivalent abstract geometric objects with a regular transformation formulae need not be strictly equivalent), as is seen from the example of objects (8.7) and (8.11).

Theorems 1 and 2 often allow us to decide about the non-equivalence of geometric objects without solving functional equation (8.9). They can also help to establish the equivalence of equivalent geometric objects. Namely they can sometimes suggest a construction of a function H which establishes the equivalence of the objects considered. (Let us note that when we investigate the equivalence of geometric objects it is enough to find one invertible function satisfying (8.9); it is not necessary to solve the functional equation (8.9) completely).

We shall illustrate this by an example (cf. Kucharzewski–Kuczma [7]). Namely we shall prove that the objects (cf. Kucharzewski–Kuczma [4])

$$\left\{\begin{aligned} \omega^{1'} &= |J|^{\alpha}\omega^1 + |J|^{\alpha}\omega^2 \log|J|^{\beta} \\ \omega^{2'} &= |J|^{\alpha}\omega^2, \end{aligned}\right. \qquad \alpha \neq 0,\ \beta \neq 0, \tag{9.2}$$

and

$$\left\{\begin{aligned} \sigma^{1'} &= |J|^{\alpha}\sigma^1, \\ \sigma^{2'} &= |J|^{\alpha}\sigma^2 \end{aligned}\right. \tag{9.3}$$

are equivalent.

The transitive fibres of object (9.3) are the open semilines beginning at the origin and the origin itself. We must determine the transitive fibres of object (9.2). We obtain their parametric equations writing $|J| = t$ in formulae (9.2):

$$\left\{\begin{aligned} \omega^{1'} &= t^{\alpha}(\omega^1 + \omega^2\beta \log t), \\ \omega^{2'} &= t^{\alpha}\omega^2, \end{aligned}\right. \qquad 0 < t < \infty. \tag{9.4}$$

ω^1 and ω^2 play here the part of parameters determining single curves of the family. We distinguish three cases:

I. $\omega^1 = 0$, $\omega^2 = 0$. Then (9.4) represents one point: the origin (on the plane of the variables $\omega^{1'}$, $\omega^{2'}$).

II. $\omega^1 \neq 0$, $\omega^2 = 0$. Then (9.4) represents two semi-lines: the positive and the negative part of the $\omega^{1'}$-axis, according as ω^1 is positive or negative.

III. $\omega^1 \neq 0$, $\omega^2 \neq 0$. Then each of the curves (9.4) meets $\omega^{2'}$-axis exactly at one point corresponding to the value of the parameter t:

$$t = \exp\left(-\frac{\omega^1}{\omega^2\beta}\right).$$

Accordingly we may take as ω^1, ω^2 in (9.4) the coordinates of the point of the intersection of the corresponding curve with the $\omega^{2'}$-axis, i. e. we may put $\omega^1 = 0$. Then equations (9.4) obtain a simpler form

$$(9.5) \qquad \begin{cases} \omega^{1'} = t^\alpha \omega^2 \beta \log t, \\ \omega^{2'} = t^\alpha \omega^2, \end{cases} \qquad 0 < t < \infty.$$

The required function H must map curves (9.5) onto semi-lines beginning at the origin. In particular, one of the curves (9.5) must be mapped onto the positive half of the σ^2-axis. This may be, for example, the curve passing through the point (0,1). It has the equations

$$(9.6) \qquad \begin{cases} \omega^{1'} = t^\alpha \beta \log t, \\ \omega^{2'} = t^\alpha, \end{cases} \qquad 0 < t < \infty.$$

After eliminating t, the equation of curve (9.6) can be written in the form

$$(9.7) \qquad \omega^1 = \omega^2 \frac{\beta}{\alpha} \log \omega^2$$

(we write ω^1, ω^2 instead of $\omega^{1'}$, $\omega^{2'}$). In order to map curve (9.7) onto the positive half-axis σ^2 we put

$$(9.8) \qquad \begin{cases} \sigma^1 = \omega^1 - \omega^2 \frac{\beta}{\alpha} \log \omega^2, \\ \sigma^2 = \omega^2, \end{cases} \qquad \omega^2 > 0.$$

Analogically, the curve of family (9.5), passing through the point $(0, -1)$ has the equation

$$(9.9) \qquad \omega^1 = \omega^2 \frac{\beta}{\alpha} \log(-\omega^2).$$

The transformation

$$(9.10) \qquad \begin{cases} \sigma^1 = \omega^1 - \omega^2 \frac{\beta}{\alpha} \log(-\omega^2), \\ \sigma^2 = \omega^2, \end{cases} \qquad \omega^2 < 0,$$

maps curve (9.9) onto the negative half-axis σ^2. (9.8) and (9.10) can be unified in the form

$$(9.11) \qquad \begin{cases} \sigma^1 = \omega^1 - \omega^2 \frac{\beta}{\alpha} \log|\omega^2|, \\ \sigma^2 = \omega^2, \end{cases} \qquad \omega^2 \neq 0.$$

Moreover, we put

$$(9.12) \qquad \sigma^1 = \omega^1, \quad \sigma^2 = 0 \quad \text{for} \quad \omega^2 = 0.$$

Formulae (9.11) and (9.12) define the desired function $\sigma = H(\omega)$, which can easily be verified. The function H found above can also serve to show the equivalence of the object with the transformation formula (cf. Kucharzewski–Kuczma [4])

$$\begin{cases} \omega^{1'} = (\operatorname{sgn} J)\,|J|^{\alpha}(\omega^1 + \omega^2 \beta \log|J|), \\ \omega^{2'} = (\operatorname{sgn} J)\,|J|^{\alpha}\omega^2, \end{cases} \qquad \alpha \neq 0,\ \beta \neq 0,$$

and the object consisting of a pair of ordinary densities of weight $-\alpha$

$$\begin{cases} \sigma^{1'} = (\operatorname{sgn} J)\,|J|^{\alpha}\sigma^1, \\ \sigma^{2'} = (\operatorname{sgn} J)\,|J|^{\alpha}\sigma^2. \end{cases}$$

In papers [3], [4], [5] of Kucharzewski and Kuczma transitive linear geometric objects of type $[2, 2, 1]$ have been determined and classified. Every such object is equivalent to one of the following objects restricted to a single transitive fibre:

1. A contravariant vector: $\omega^{i'} = A^{i'}_{i}\omega^i$ $(i' = 1', 2')$.
2. A contravariant G-vector: $\omega^{i'} = (\operatorname{sgn} J) A^{i'}_{i}\omega^i$.
3. A contravariant vector W-density of a weight $-\alpha$: $\omega^{i'} = |J|^{\alpha} A^{i'}_{i}\omega^i$.
4. A contravariant vector G-density of a weight $-\alpha$: $\omega^{i'} = (\operatorname{sgn} J)|J|^{\alpha} A^{i'}_{i}\omega^i$.
5. A pair of Weyl densities with one component: $\omega^{i'} = |J|\,\omega^i$.
6. A pair of ordinary densities with one component: $\omega^{i'} = J\omega^i$.
7. A pair of biscalars: $\omega^{i'} = (\operatorname{sgn} J)\,\omega^i$.
8. An object with the transformation formula (cf. (7.7)):

$$\begin{cases} \omega^{1'} = \omega^1 \cos(\alpha \log|J|) - \omega^2 \sin(\alpha \log|J|), \\ \omega^{2'} = \omega^1 \sin(\alpha \log|J|) + \omega^2 \cos(\alpha \log|J|), \end{cases} \qquad \alpha > 0.$$

9. An object with the transformation formula (cf. (7.8)):

$$\begin{cases} \omega^{1'} = (\operatorname{sgn} J)[\omega^1 \cos(\alpha \log|J|) - \omega^2 \sin(\alpha \log|J|)], \\ \omega^{2'} = (\operatorname{sgn} J)[\omega^1 \sin(\alpha \log|J|) + \omega^2 \cos(\alpha \log|J|)], \end{cases} \qquad \alpha > 0.$$

Objects 3, 4, 8, 9 corresponding to different values of the parameter α are not equivalent. Thus they are rather whole families of non-equivalent geometric objects. The dimension of objects 1, 2, 3, 4 is 2, the dimension of objects 5, 6, 8, 9 is 1, the dimension of object 7 is zero.

Let us also note that in a two-dimensional space covariant vectors are equivalent to contravariant vector G-densities of weight 1. In a space of a greater dimension covariant vectors cannot be reduced to contravariant vector densities.

Under the assumption of class C^1 of the transformation formula objects with one component have been determined and classified (Gołąb [7], [12], [8], [6], [11], Dubnow [1], Pienzow [1], [3], Aczél [1], [4], [6]). Every transitive object with one component, of class $s \leqslant 1$, is equivalent to a scalar, or to a biscalar, or to an ordinary density, or to a Weyl density, or to an object of Pienzow. Objects of higher classes exist only in a one-dimensional space. They are equivalent to an object of a linear connection (7.4), or to an object of a projective connection (7.5), or to an object with the transformation formula

$$\omega' = |\omega|^{1/a_1}\left(\exp\frac{a_2}{a_1}\right)[\operatorname{sgn}(\omega a_1)],$$

or to an object with the transformation formula

$$\omega' = |\omega|^{1/a_1^2}\left[\exp\left(\frac{a_3}{a_1^3} - \frac{3a_2^2}{2a_1^4}\right)\right][\operatorname{sgn}(\omega a_1)]$$

(cf. also Aczél–Gołąb [1], pp. 39, 43, 56).

§ 10. Concomitants

Suppose that we are given an abstract geometric object

$$\omega(\omega^1, \ldots, \omega^m), \tag{10.1}$$

with a fibre M and with the transformation formula

$$\omega' = f(\omega, T_{i'i}). \tag{10.2}$$

Let

$$\Omega = \Psi(\omega) = \{\psi^1(\omega^1, \ldots, \omega^m), \ldots, \psi^q(\omega^1, \ldots, \omega^m)\} \tag{10.3}$$

be a function of object (10.1), defined in a set $N \subset M$, which does not depend on the choice of the coordinate system. Then the function Ψ defines an object Ω with q components

$$\Omega(\Omega^1, \ldots, \Omega^q). \tag{10.4}$$

Definition 17. Object Ω is called a *relative concomitant of the object ω in the set N*. If Ω is a geometric object, then Ω is called a *geometric concomitant*.

Since in the sequel we shall be concerned only with geometric concomitants, we shall usually omit the adjective "geometric".

The notion of a concomitant was introduced by J. A. Schouten [1], [2]. It was then developed by S. Gołąb [4] (cf. also Aczél-Gołąb [1],

p. 16). A relative concomitant has been defined in the above manner in paper [5] of Kucharzewski.

Let Ω be a geometric concomitant of an object ω and let

$$\Omega' = F(\Omega, T_{i'i}) \tag{10.5}$$

be its transformation formula. From the fact that the form of the function Ψ in (10.3) cannot depend on the choice of the coordinate system we obtain the relation

$$\Omega' = \Psi(\omega'). \tag{10.6}$$

Inserting (10.6) and (10.3) to (10.5) we obtain

$$\Psi(\omega') = F\big(\Psi(\omega), T_{i'i}\big),$$

whence we have, after taking into account (10.2),

$$\Psi\big(f(\omega, T_{i'i})\big) = F\big(\Psi(\omega), T_{i'i}\big). \tag{10.7}$$

Every concomitant Ω of the object ω having transformation formula (10.5) must satisfy equation (10.7) for every $\omega \in N$ and every $T_{i'i}$ belonging to the pseudogroup of transformations of the allowable coordinate systems.

If we are given transformation formula (10.2) of the object ω and transformation formula (10.5) of the concomitant Ω, then (10.7) is a functional equation with the unknown function Ψ. Solving this equation we can determine all the concomitants of a given type of the object ω.

Let us also note that the set N cannot be quite arbitrary; it must fulfil the condition

$$\textit{if } \omega \in N, \textit{ then also } \omega' = f(\omega, T_{i'i}) \in N \tag{10.8}$$

for every $T_{i'i}$ belonging to the above mentioned pseudogroup of transformations of the allowable coordinate systems.

Definition 18. If the set N coincides with the whole fibre M, then the concomitant is called *absolute*.

Concomitants defined in definitions 17 and 18 are also called *algebraic concomitants* to distinguish them from so-called differential concomitants (cf. § 12).

One of the main problems of the theory of geometric objects is the determination of all the concomitants of objects of a given type. For the present this problem cannot be solved completely, since geometric objects of particular types have not yet all been determined. So far only concomitants of some geometric objects have been found. S. Gołąb [4], [22], J. Aczél [8], M. Ikeda [1], M. Ikeda-S. Abe [1], M. Kucharzewski [1]-[5], I. Leroy [1], A. E. Liber [2], I. Makai [1], A. Moór [2] and A. Zajtz [1] have dealt with these problems.

As follows from a principle given by J. Aczél [8] (cf. also Aczél–Gołąb [1], p. 19), the knowledge of all the concomitants of a geometric object ω allows us to determine all the concomitants of objects equivalent to ω. Therefore, of the whole family of equivalent objects, it is enough to determine the concomitants for a single object. Usually we choose an object with a possibly simple transformation formula, e.g. a linear one.

Now we shall present some results from the theory of algebraic concomitants of purely differential special geometric objects. Equation (10.7) then becomes

$$\Psi[f(\omega, L)] = F[\Psi(\omega), L] \quad \text{for} \quad \omega \in N \subset M \text{ and } L \in \mathscr{L}_n^s. \tag{10.9}$$

Every function of a scalar (5.8)

$$\sigma = \psi(\omega)$$

is again a scalar and this is the most general algebraic concomitant of that object.

S. Gołąb [4] (cf. also Aczél–Gołąb [1], p. 132) has determined all the absolute concomitants of an ordinary density (5.10)

$$\Omega = \Psi(\omega),$$

where the function Ψ is continuous except for zero. The only continuous concomitants of an ordinary density are scalars, biscalars, objects equivalent to an ordinary density and objects equivalent to a Weyl density. In view of the above-mentioned principle, this allows one to determine also all the continuous concomitants of an ordinary density of an arbitrary weight $-\alpha$ (5.11).

For vectors the situation is more complicated. There may exist scalar, vector as well as tensor concomitants of vectors and even objects of Pienzow (5.20) may be concomitants of vectors. We shall not give here the detailed results, since they may be found in the literature quoted above. We shall only illustrate by one simple example the method of determining relative concomitants of vectors. Namely we shall determine all the relative scalar concomitants of a pair of contravariant vectors. For simplification we shall confine ourselves to the case of a two-dimensional space. The group $\mathscr{L}_n^s = \mathscr{L}_2^1$ will then be a group of non-singular square matrices $\|A_i^{i'}\|$ of the second order.

Suppose that at a point $x_0 \in X_2$ there are given two contravariant vectors, which may be unified into one object ω with four components

$$\omega[\underset{1}{v}(\underset{1}{v}^1, \underset{1}{v}^2), \underset{2}{v}(\underset{2}{v}^1, \underset{2}{v}^2)]. \tag{10.10}$$

We want to determine all the scalar concomitants of object (10.10), i. e.

to find all the functions

$$\Psi(\omega) = \Psi(\underset{1}{v}, \underset{2}{v}) = \Psi\left(\left\|\begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix}\right\|\right)$$

which satisfy equation (10.9). In our case this equation has the form

$$\Psi(\underset{1}{Av}, \underset{2}{Av}) = \Psi(\underset{1}{v}, \underset{2}{v}), \tag{10.11}$$

where $A = \|A_i^{i'}\|$ is an arbitrary non-singular 2×2 matrix.

The fibre of the object ω is the whole four-dimensional space A_4. A function Ψ which satisfies equation (10.11) for all the non-singular 2×2 matrices A and for all $(\underset{1}{v}, \underset{2}{v}) \in A_4$ is an absolute concomitant of the pair of contravariant vectors. But we shall try to determine first the relative concomitants of the object ω. Accordingly we shall determine the sets N which fulfil condition (10.8). These sets must be a union of transitive fibres of the object ω. Thus we see that we are again led to the problem of determining the transitive fibres of an object, this time for object (10.10).

The origin, i. e. the point $(0, 0, 0, 0)$, is a transitive fibre of ω and we shall denote it by N_0. The next transitive fibre is the set of those

$$\left\|\begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix}\right\| \in A_4$$

which fulfil for a certain λ the condition $\underset{2}{v} = \lambda \underset{1}{v}$, $\underset{1}{v} \neq (0, 0)$. We shall denote this set by $N_{1\lambda}$. The vectors in this set are linearly dependent with the same coefficient of proportionality. The next transitive fibre is the set N_{01} of all the pairs $(\underset{1}{v}, \underset{2}{v})$ in which $\underset{1}{v}$ is the null vector. Lastly, we denote by N_2 the set of those quadruples of real numbers

$$\left\|\begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix}\right\|$$

in which the vectors $\underset{1}{v}$ and $\underset{2}{v}$ are linearly independent.

The sets $N_{1\lambda}$ are planes in the space A_4, given as the intersection of two three-dimensional hyperplanes with the equations

$$\begin{aligned} \underset{2}{v}^1 - \lambda \underset{1}{v}^1 &= 0, \\ \underset{2}{v}^2 - \lambda \underset{1}{v}^2 &= 0, \end{aligned} \tag{10.12}$$

with the origin removed. Similarly the set N_{01} is such a plane.

It can easily be verified that two planes defined by system (10.12) for different values of the parameter λ have no points in common besides the origin, and so we have (in accordance with the general theory)

$$N_{1\lambda} \cap N_{1\mu} = \emptyset \quad \text{for} \quad \lambda \neq \mu.$$

Similarly we have

$$N_{1\lambda} \cap N_{01} = \emptyset \quad \text{for} \quad -\infty < \lambda < +\infty.$$

The sets N_0, $N_{1\lambda}$ for $-\infty < \lambda < +\infty$ and N_{01} do not exhaust the whole space A_4. The rest of A_4 forms the set N_2 consisting of the points to which there correspond linearly independent vectors.

In order to obtain the relative concomitants of object (10.10) we solve functional equation (10.11) in particular sets $N_0, N_{1\lambda}, N_{01}, N_2$.

In the set N_0 equation (10.11) becomes

$$\Psi\left(\left\|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\|\right) = \Psi\left(\left\|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\|\right),$$

which means that at this point Ψ may be an arbitrary constant:

$$\Psi\left(\left\|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\|\right) = c_0.$$

In the set $N_{1\lambda}$ we choose a matrix A as follows. Since the vector $\underset{1}{v}(\underset{1}{v}^1, \underset{1}{v}^2)$ is not the null vector, one of its components must be different from zero. Without loss of generality we may assume that it is $\underset{1}{v}^1$:

$$\underset{1}{v}^1 \neq 0.$$

Then we take as the matrix A

$$A = \left\|\begin{matrix} 1/\underset{1}{v}^1 & 0 \\ -\underset{1}{v}^2 & \underset{1}{v}^1 \end{matrix}\right\|. \tag{10.13}$$

After inserting (10.13) into equation (10.11) we obtain

$$\Psi\left(\left\|\begin{matrix} 1 & \lambda \\ 0 & 0 \end{matrix}\right\|\right) = \Psi\left(\left\|\begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix}\right\|\right).$$

Writing

$$\varphi(\lambda) = \Psi\left(\left\|\begin{matrix} 1 & \lambda \\ 0 & 0 \end{matrix}\right\|\right)$$

we get hence

$$\Psi(\omega) = \varphi(\lambda) \quad \text{for} \quad \omega \in N_{1\lambda}.$$

Similarly it can be shown that in the set N_{01} the function Ψ must be constant:

$$\Psi(\omega) = c_1 \quad \text{for} \quad \omega \in N_{01}.$$

In the set N_2 the vectors $\underset{1}{v}$ and $\underset{2}{v}$ are linearly independent and thus the matrix

$$\left\| \begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix} \right\| \tag{10.14}$$

is non-singular. Assuming as the matrix A in equation (10.11) the inverse of matrix (10.14) we obtain

$$\Psi\left(\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|\right) = \Psi\left(\left\| \begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix} \right\|\right),$$

which means that the value of the function Ψ in the set N_2 must be constant:

$$\Psi(\omega) = c_2 \quad \text{for} \quad \omega \in N_2.$$

Thus the function

$$\Psi(\omega) = \begin{cases} c_0 & \text{for} \quad \omega \in N_0, \\ \varphi(\lambda) & \text{for} \quad \omega \in N_{1\lambda}, \\ c_1 & \text{for} \quad \omega \in N_{01}, \\ c_2 & \text{for} \quad \omega \in N_2 \end{cases} \tag{10.15}$$

is the most general absolute concomitant of a pair of contravariant vectors. This function restricted to an arbitrary set N, which is a union of a number of transitive fibres of object (10.10), is the most general relative concomitant of that object in the set N.

Function (10.15) is in general discontinuous. If we require only that the function Ψ be continuous at the point $\left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\|$, then it follows that it must be constant in the whole space A_4. This results from the fact that in each of the sets $N_{1\lambda}$, N_{01} and N_2 there exist pairs of vectors $\underset{1}{v}, \underset{2}{v}$ such that the point $\left\| \begin{matrix} \underset{1}{v}^1 & \underset{2}{v}^1 \\ \underset{1}{v}^2 & \underset{2}{v}^2 \end{matrix} \right\|$ is situated arbitrarily near the point $\left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\|$.

It is enough to assume

$$\underset{1}{v} = \left(\frac{1}{n}, 0\right), \quad \underset{2}{v} \quad \left(\lambda \frac{1}{n}, 0\right) \quad \text{in } N_{1\lambda},$$

$$\underset{1}{v} = (0, 0), \quad \underset{2}{v} = \left(\frac{1}{n}, 0\right) \quad \text{in } N_{01},$$

$$\underset{1}{v} = \left(\frac{1}{n}, 0\right), \quad \underset{2}{v} \quad \left(0, \frac{1}{n}\right) \quad \text{in } N_2.$$

Then

$$\left\| \begin{matrix} v^1_1 & v^1_2 \\ v^2_1 & v^2_2 \end{matrix} \right\| \to \left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\| \quad \text{as} \quad n \to \infty.$$

In geometry an important part is played by the relative concomitant of a pair of vectors (v_1, v_2) in the set

$$N = \bigcup_{-\infty<\lambda<\infty} N_{1\lambda}.$$

In the affine space E_2 it is closely connected with the ratio of segment section (AB, C). As we know, the ratio of a section of segment AB by a point C is the number λ such that

$$AC = \lambda BC.$$

It can be shown that in the affine space E_2 the ratio of a segment section for three collinear points is a relative concomitant of a triple of points in the set N of collinear triples and there are no other essentially different concomitants of this kind. The problem of finding concomitants of a triple of points can be reduced to that of finding concomitants of a pair of contravariant vectors (Kucharzewski [1]).

Similarly one can consider concomitants of tensors of higher valences. The most fully developed is the theory of concomitants of affinors of valence two, i. e. mixed affinors a^i_j, covariant tensors a_{ij} and contravariant tensors a^{ij}. S. Gołąb [22], S. Gołąb–E. Siwek [1], E. Siwek [1], I. Leroy [1], M. Ikeda [1], M. Ikeda–S. Abe [1] and A. Zajtz [1] have dealt with these problems. A recent paper of Zajtz [1] has virtually closed the question of determining concomitants of affinors of valence two.

We shall give here some results concerning scalar concomitants of affinors of valence two. For the mixed affinor

$$a^{i'}_{j'} = A^{i'}_i A^j_{j'} a^i_j \tag{10.16}$$

the fibre M is the whole n^2-dimensional arithmetical space. The transitive fibres are the sets of similar matrices

$$\mathfrak{A} = \|a^i_j\|. \tag{10.17}$$

Since similar matrices have the same Jordan form and a scalar concomitant must be constant on the transitive fibres, the most general scalar concomitant of affinor (10.16) is an arbitrary function of the Jordan form of matrix (10.17). If, moreover, we require that the concomitant be a continuous function, then it must be a function only of the characteristic roots of the matrix $\mathfrak{A}$:

$$\Psi(\mathfrak{A}) = \varphi(\lambda_1, \ldots, \lambda_n).$$

Here φ must be a symmetric function of its arguments. The function φ can also be expressed in terms of the coefficients of the characteristic equation of the matrix $\mathfrak{A}$ (cf. Aczél–Gołąb [1], p. 140). Thus in particular, in the case of a two-dimensional space, the general continuous scalar concomitant of affinor (10.16) is an arbitrary function of the determinant and the trace of the matrix $\mathfrak{A}$ (Gołąb [22]).

For a covariant ([22]) and symmetric tensor

$$a_{i'j'} = A^i_{i'} A^j_{j'} a_{ij}, \tag{10.18}$$

$$a_{ij} = a_{ji}, \tag{10.19}$$

the scalar concomitant is an arbitrary function of the signs of the characteristic roots of the matrix

$$\mathfrak{A} = \|a_{ij}\|,$$

or, in other words, a function of the signature of the quadratic form with the matrix $\mathfrak{A}$ (cf. Schouten–Struik [1]):

$$\Psi(\mathfrak{A}) = \varphi(\operatorname{sgn}\lambda_1, \ldots, \operatorname{sgn}\lambda_n)$$

($\operatorname{sgn}\lambda = 0$ for $\lambda = 0$). Here φ must again be a symmetric function of its arguments. If we assume that Ψ is continuous for the null affinor, then it must be constant.

For a skew symmetric covariant tensor

$$a_{ij} = -a_{ji}$$

the scalar concomitant is a function of the rank of the matrix $\mathfrak{A}$. Also in this case, assuming the continuity of Ψ for the null affinor, we obtain only constant functions.

In the general case, where tensor (10.18) is neither symmetric nor skew symmetric, the problem is more involved and even the results are so complicated that it would be difficult to present them here in a concise manner. The interested reader is referred to the paper of A. Zajtz [1].

§ 11. Algebra of geometric objects

Another important problem in the theory of geometric objects is the so called problem of an algebra of geometric objects. This problem may be formulated as follows.

We are given two geometric objects: an object ω with the transfor-

([22]) For a contravariant tensor the results are quite analogous.

mation formula

$$\underset{1}{\omega}' = f_1(\underset{1}{\omega}, L), \qquad L \in \mathscr{L}_n^s, \tag{11.1}$$

and an object $\underset{2}{\omega}$ with the transformation formula

$$\underset{2}{\omega}' = f_2(\underset{2}{\omega}, L), \qquad L \in \mathscr{L}_n^s. \tag{11.2}$$

We seek a function of these objects

$$\Omega = \Phi(\underset{1}{\omega}, \underset{2}{\omega}) \tag{11.3}$$

which is independent of the choice of the coordinate system and is again a geometric object (cf. Aczél–Gołąb [1], p. 92).

DEFINITION 19. If there exists a function $\Phi(\underset{1}{\omega}, \underset{2}{\omega})$ of geometric objects (11.1) and (11.2) such that (11.3) is again a geometric object, then we say that *the objects $\underset{1}{\omega}$ and $\underset{2}{\omega}$ admit an algebra and the operation Φ can be effected on them.*

From the fact that the function Φ should be independent of the choice of the coordinate system follows the relation

$$\Omega' = \Phi(\underset{1}{\omega}', \underset{2}{\omega}'). \tag{11.4}$$

Let

$$\Omega' = F(\Omega, L), \qquad L \in \mathscr{L}_n^s, \tag{11.5}$$

be the transformation formula of the object Ω. Inserting (11.1), (11.2) and (11.5) into (11.4) and taking into account (11.3), we obtain the functional equation

$$F\big(\Phi(\underset{1}{\omega}, \underset{2}{\omega}), L\big) = \Phi[f_1(\underset{1}{\omega}, L), f_2(\underset{2}{\omega}, L)], \tag{11.6}$$

which must be satisfied by a function Φ defining an algebra of the objects $\underset{1}{\omega}$ and $\underset{2}{\omega}$. Solving this equation allows us to decide whether a given pair of objects $\underset{1}{\omega}$, $\underset{2}{\omega}$ admits an algebra or not, and what can be the form of the operations in this algebra.

Since a pair of geometric objects $\underset{1}{\omega}$, $\underset{2}{\omega}$ can always be regarded as one object $\omega(\underset{1}{\omega}, \underset{2}{\omega})$, function (11.3) can be viewed as an algebraic concomitant of the object ω (cf. Aczél–Gołąb [1], p. 92). The problem of finding all the algebras for a given pair of geometric objects $\underset{1}{\omega}$ and $\underset{2}{\omega}$ is thus reduces to the problem of determining all the algebraic concomitants of

the object ω. Hence it follows in particular that for the algebras of geometric objects analogous notions as for concomitants can be introduced. Thus, e. g., we can consider relative algebras in a set N, where N is the domain of definition of function (11.3). As follows from (11.6), the set N cannot be quite arbitrary but must fulfil the condition

$$\textit{if } (\underset{1}{\omega}, \underset{2}{\omega}) \in N, \textit{ then also } \big(f_1(\underset{1}{\omega}, L), f_2(\underset{2}{\omega}, L)\big) \in N \textit{ for every } L \in \mathscr{L}_n^s.$$

Functional equation (11.6) must then be satisfied for all $(\underset{1}{\omega}, \underset{2}{\omega}) \in N$ and all $L \in \mathscr{L}_n^s$.

The problem of algebras of geometric objects has been dealt with by J. Aczél [8], [9], M. Hosszú [1], [2], [3], [4], S. Gołąb–H. Pidek [1], H. Pidek [1], [2], [3].

According to the general principle concerning concomitants of equivalent objects (Aczél [8], Aczél–Gołąb [1], p. 19), if we know an algebra for a certain pair of objects $\underset{1}{\omega}$ and $\underset{2}{\omega}$, then we can determine an algebra for all the pairs of objects equivalent to the objects $\underset{1}{\omega}$ and $\underset{2}{\omega}$, respectively.

So far the problem of algebras of geometric objects has been solved for objects of class zero, for scalars, biscalars, ordinary densities and Weyl densities (Aczél–Gołąb [1], p. 97), and for objects with one component (Aczél–Gołąb [1], p. 112), under some additional hypotheses.

In paper [2] of Kucharzewski all the concomitants have been determined of a pair of contravariant vectors which are themselves contravariant vector. Denoting, as previously, by N_0, $N_{1\lambda}$, N_{01} and N_2 the transitive fibres of object (10.10), we may formulate the result as follows:

The most general vector concomitant of vectors $\underset{1}{v}$ and $\underset{2}{v}$ has the form

$$\Phi(\underset{1}{v}, \underset{2}{v}) = \begin{cases} 0 & \textit{for} \quad (\underset{1}{v}, \underset{2}{v}) \in N_0, \\ \varphi(\lambda)\underset{1}{v} & \textit{for} \quad (\underset{1}{v}, \underset{2}{v}) \in N_{1\lambda}, \\ c_0\underset{2}{v} & \textit{for} \quad (\underset{1}{v}, \underset{2}{v}) \in N_{01}, \\ c_1\underset{1}{v} + c_2\underset{2}{v} & \textit{for} \quad (\underset{1}{v}, \underset{2}{v}) \in N_2, \end{cases} \tag{11.7}$$

where 0 is the null vector. If, however, we assume the continuity of the function Φ, then $\Phi(\underset{1}{v}, \underset{2}{v})$ must have the form

$$\Phi(\underset{1}{v}, \underset{2}{v}) = c_1\underset{1}{v} + c_2\underset{2}{v} \quad \textit{for every} \quad (\underset{1}{v}, \underset{2}{v}) \in A_{2n}. \tag{11.8}$$

In fact, let us take an arbitrary pair of vectors $(\underset{1}{v}, \lambda\underset{1}{v})$ belonging to $N_{1\lambda}$

and an arbitrary vector $\underset{2}{v}$ linearly independent of $\underset{1}{v}$. The pairs of vectors $\left(\underset{1}{v}, \lambda \underset{1}{v} + \frac{1}{k} \underset{2}{v}\right)$ belong to N_2 and tend as $n \to \infty$ to the pair $(\underset{1}{v}, \lambda \underset{1}{v})$. It follows by (11.7) from the continuity of the function Φ that

$$\varphi(\lambda) \underset{1}{v} = \Phi(\underset{1}{v}, \lambda \underset{1}{v}) = \lim_{k \to \infty} \Phi\left(\underset{1}{v}, \lambda \underset{1}{v} + \frac{1}{k} \underset{2}{v}\right)$$

$$= \lim_{k \to \infty} \left[c_1 \underset{1}{v} + c_2 \left(\lambda \underset{1}{v} + \frac{1}{k} \underset{2}{v}\right)\right] = c_1 \underset{1}{v} + c_2 \lambda \underset{1}{v},$$

whence we obtain

$$\varphi(\lambda) = c_1 + c_2 \lambda, \tag{11.9}$$

since $\underset{1}{v}$ is not the null vector. Taking into account (11.9) in (11.7) we see that

$$\Phi(\underset{1}{v}, \underset{2}{v}) = c_1 \underset{1}{v} + c_2 \underset{2}{v} \quad \text{for} \quad (\underset{1}{v}, \underset{2}{v}) \in N_{1\lambda}.$$

Similarly it can be shown that formula (11.8) is valid also in the sets N_0 and N_{01}.

Hence it follows for the algebra of contravariant vectors that a pair of contravariant vectors admits an algebra and that the most general continuous operation whose result is again a contravariant vector is function (11.8) (i. e. a linear combination of vectors). Furthermore, one may define operation on vectors only in the set of pairs of linearly independent vectors, or only in the set $\bigcup_{-\infty < \lambda < \infty} N_{1\lambda} \cup N_{01}$, etc. Then we obtain relative operations. They can all be obtained from formula (11.7).

§ 12. Differential concomitants

Suppose that we are given a field of geometric objects $\omega(\xi)$ of class C^q. This means that to each point ξ of a neighbourhood of a point ξ_0 there corresponds an object $\omega(\xi)$ whose transformation formula does not depend on ξ and that the components $\omega^1(\xi), \ldots, \omega^m(\xi)$ of the object $\omega(\xi)$ are functions of class C^q in this neighbourhood.

Definition 20. If the relation

$$\Omega = \Psi(\omega, \partial_i \omega, \partial_{i_1 i_2} \omega, \ldots, \partial_{i_1 \ldots i_q} \omega),$$

where $\partial_{i_1 \ldots i_k} = \partial^k / \partial \xi^{i_1} \ldots \partial \xi^{i_k}$, holds independently of the choice of the coordinate system and Ω is a geometric object, then the object Ω is called a *geometric differential concomitant of order q of the object ω.*

Differential concomitants play an important part in geometry. They are closely connected with algebraic concomitants, for there is much in favour of the conjecture that every differential concomitant can be reduced in a rather simple manner to an algebraic concomitant. S. Gołąb [9], [16], [17], [20], S. Gołąb–I. Makai [1], O. E. Gheorghiu [13], C. Jankiewicz [1], [2], I. Makai [2], [3], A. Moór [1], [4], [5], [6], [7], A. Nijenhuis [1], [3], [4], J. A. Schouten [2], [3], [4], [5], W. Ślebodziński [1], Y. Tashiro [1], [2], W. W. Wagner [1] and K. Yano [1], [2] have dealt with differential concomitants.

Some special differential concomitants, namely a Lie derivative and a covariant derivative, have been dealt with particularly often. The notion of a Lie derivative of an affinor field with respect to a given field of contravariant vectors has been introduced by W. Ślebodziński [1]. Recently it has become a subject of an extensive monograph by Yano [1].

A Lie derivative is connected with the study of motions in Riemannian spaces(23) and in some even more general spaces. Motions are transformations of the space into itself which do not alter the distance. A necessary and sufficient condition for an infinitesimal transformation(24)

$$'\xi^i = \xi^i + v^i(\xi)dt$$

to be a motion is that the Lie derivative of the field of the metric tensor $g_{ij}(\xi)$ with respect to the field of vectors $v^i(\xi)$ vanish (cf. Yano [1]).

The notion of a Lie derivative has been approached by S. Gołąb [20] in a different manner, strictly connected with the notion of a differential concomitant. According to S. Gołąb (cf. Aczél–Gołąb [1], p. 145), the Lie derivative of a field of contravariant vectors with respect to another field of contravariant vectors may be defined as follows.

Suppose we are given two fields of contravariant vectors:

$$u^i(\xi) \quad \text{and} \quad v^i(\xi).$$

DEFINITION 21. The *Lie derivative* $\underset{v}{L}u$ of the vector field $u^i(\xi)$ with respect to the vector field $v^i(\xi)$ is a differential concomitant of the first order of these fields, which is again a contravariant vector.

Thus

$$\underset{v}{L}u = \Psi(u, \partial u, v, \partial v), \tag{12.1}$$

(23) These are spaces in which at every point a metric tensor g_{ij} is defined, which allows one to define in a certain manner a metric in these spaces (cf. Gołąb [18], p. 113).

(24) We write here $'\xi^i$ instead of $\xi^{i\prime}$ in order to express the fact that it is not a change of coordinates but a transformation of the space into itself.

where the function Ψ should be independent of the choice of the coordinate system and (12.1) should be a contravariant vector. Relation (12.1) written in components has the form

$$(\underset{v}{\pounds} u)^k = \psi^k(u^i, \partial_j u^i, v^i, \partial_j v^i).$$

From the fact that the form of function (12.1) should be independent of the choice of the coordinate system follows the functional equation for the functions ψ^k:

$$A^{k'}_k \psi^k(u^i, \partial_j u^i, v^i, \partial_j v^i) = \psi^{k'}(u^{i'}, \partial_{j'} u^{i'}, v^{i'}, \partial_{j'} v^{i'}). \tag{12.2}$$

The arguments occurring on the right-hand side of relation (12.2) can be expressed as follows:

$$\begin{aligned} u^{i'} &= A^{i'}_i u^i, \qquad \partial_{j'} u^{i'} = A^{i'}_i A^j_{j'} \partial_j u^i + A^{i'}_{ij} A^j_{j'} u^i, \\ v^{i'} &= A^{i'}_i v^i, \qquad \partial_{j'} v^{i'} = A^{i'}_i A^j_{j'} \partial_j v^i + A^{i'}_{ij} A^j_{j'} v^i. \end{aligned} \tag{12.3}$$

Inserting relations (12.3) into the right-hand side of equation (12.2) we obtain a functional equation which must be satisfied for all u^i, v^i, $\partial_j u^i$, $\partial_j v^i$, for arbitrary non-singular matrices $\|A^{i'}_i\|$ and for arbitrary systems of parameters A^i_{ij} which are symmetric with respect to the lower indices.

Equation (12.2) in a two-dimensional space has been solved under the assumption of class C^1 regarding the functions ψ^k by S. Gołąb [26]. The general solution has the form

$$\psi^k = C_1 v^k + C_2 u^k + C_3(v^i \partial_i u^k - u^i \partial_i v^k), \qquad k = 1, 2, \tag{12.4}$$

where C_1, C_2 and C_3 are arbitrary functions of the scalars

$$\varrho_1 = \frac{\begin{vmatrix} v^1 & w^1 \\ v^2 & w^2 \end{vmatrix}}{\begin{vmatrix} v^1 & u^1 \\ v^2 & u^2 \end{vmatrix}}, \qquad \varrho_2 = \frac{\begin{vmatrix} u^1 & w^1 \\ u^2 & w^2 \end{vmatrix}}{\begin{vmatrix} v^1 & u^1 \\ v^2 & u^2 \end{vmatrix}},$$

and w^k is the contravariant vector

$$w^k = v^i \partial_i u^k - u^i \partial_i v^k. \tag{12.5}$$

Thus formula (12.4) represents the most general Lie derivative of a field of contravariant vectors $u^i(\xi)$ with respect to a field of contravariant vectors $v^i(\xi)$, of class C^1, in a two-dimensional space. I. Makai [3] has extended this theorem to the case of an n-dimensional space.

Usually in mathematical literature (Ślebodziński [1], Yano [1]) the Lie derivative of a field of contravariant vectors $u^i(\xi)$ with res-

pect to a field of contravariant vectors $v^i(\xi)$ is assumed to be vector (12.5).

Analogically one may define Lie derivatives for fields of covariant vectors and fields of densities (Aczél-Gołąb [1], p. 150), and even for arbitrary fields of geometric objects (Makai [3]).

Now we pass to the introduction of a covariant derivative as a differential concomitant of certain fields of geometric objects.

The notion of a covariant derivative is closely connected with that of a parallel displacement, introduced in the year 1917 by T. Levi-Civita [1] and independently, in the year 1918, by J. A. Schouten [1]. A definition of a covariant derivative was given for the first time by J. A. Schouten and D. J. Struik [1].

A covariant derivative is usually introduced in connection with the following fact. If we are given, for instance, a field of contravariant vectors

$$v^i(\xi) \tag{12.6}$$

of class C^1, we may build a new field of objects $v^i_j(\xi) = \partial_j v^i(\xi)$. Now, $v^i_j(\xi)$ is not a geometric object; its components $v^{i'}_{j'}$ in a new coordinate system are given by

$$v^{i'}_{j'} = A^{i'}_i A^j_{j'} v^i_j + A^{i'}_{ij} A^j_{j'} v^i,$$

and thus they depend not only on the components v^i_j in the old coordinate system and on the parameters L, but also on object (12.6) ([25]). The covariant derivative $\nabla_j v^i$ is thus introduced as a geometric object characterizing the rate of change of vector field (12.6) and it replaces in geometrical investigations the usual derivative $\partial_j v^i$ of the field v^i.

A covariant derivative may be defined in many ways. In the year 1948 S. Gołąb [9] (cf. also Gołąb [16], [18], p. 182) defined the covariant derivative as a differential concomitant in the following manner.

Suppose that we are given a field of contravariant vectors (12.6) and a field $\Gamma^i_{jk}(\xi)$ of objects of a linear connection (5.21).

DEFINITON 22. The *covariant derivative* $\nabla_j v^i$ of vector field (12.6) is a mixed affinor T^i_j which is a function of the field of the vectors v^i, of their derivatives $\partial_j v^i$ and of the object Γ^i_{jk}:

$$\nabla_j v^i = T^i_j(v^l, \partial_p v^l, \Gamma^l_{pr}). \tag{12.7}$$

(Thus T^i_j is a differential concomitant of the field of the vectors $v^i(\xi)$ and of the object $\Gamma^i_{jk}(\xi)$.)

([25]) But $v^i_j(\xi) = \partial_j v^i(\xi)$ represents a geometric object in the affine geometry; cf. the closing remarks in § 5.

Analogically one can define the covariant derivative $\nabla_j w_i$ of a field of covariant vectors $w_i(\xi)$.

A. Moór [1] has proved that covariant derivative (12.7) must be a function of the vectors v^i, of the affinors

$$V^i_j = \partial_j v^i + \Gamma^i_{jk} v^k, \tag{12.8}$$

and of the affinors

$$S^i_{jk} = \Gamma^i_{[jk]} = \tfrac{1}{2}[\Gamma^i_{jk} - \Gamma^i_{kj}]. \tag{12.9}$$

If we assume that the functions T^i_j are continuous and that the parameters of a linear connection are symmetric (i.e. $S^i_{jk} = 0$), then covariant derivative (12.7) may be expressed only by affinor (12.8), i.e.

$$\nabla_j v^i = T^i_j(V^k_l).$$

If, on the other hand, we assume that the functions T^i_j are linear and homogeneous with respect to v^i and $\partial_j v^i$, then covariant derivative (12.7) must have the form

$$\nabla_j v^i = C_1 V^i_j + C_2 \delta^i_j V^k_k + b^i_{kj}(S^l_{pr}) v^k,$$

where C_1 and C_2 are scalars, b^i_{kj} is an arbitrary affinor dependent on affinor (12.9) and δ^i_j is the Kronecker symbol (Moór [1]).

A. Moór has also obtained results concerning the covariant derivative of fields of covariant vectors.

Usually in mathematical literature the covariant derivative of a field of contravariant vectors (12.6) is assumed to be affinor (12.8). From the theorems of A. Moór quoted above it follows that if we assume that functions (12.7) are continuous and linear (with respect to v^i and $\partial_j v^i$) and that the parameters of a linear connection are symmetric, then the covariant derivative of a field of contravariant vectors (12.6) has the form

$$\nabla_j v^i = C_1 V^i_j + C_2 \delta^i_j V^k_k.$$

However, the question is still open what conditions should be imposed on functions (12.7) in order to obtain affinor (12.8) as the covariant derivative $\nabla_j v^i$ of field (12.6).

In a space in which there is given a metric tensor g_{ij} we can define in a natural way an object of a parallel displacement with the aid of the tensor g_{ij}:

$$\Gamma^i_{jk} \quad \begin{Bmatrix} i \\ jk \end{Bmatrix} \quad \tfrac{1}{2} g^{il}(\partial_k g_{lj} + \partial_j g_{kl} - \partial_l g_{jk}), \tag{12.10}$$

where g^{ij} are the elements of the matrix inverse to the matrix $\|g_{ij}\|$. ((12.10) is the so called *Christoffel symbol of the second kind*; cf. Gołąb [18], p. 219). In such a space the covariant derivative of a field of contra-

variant vectors $v^i(\xi)$ may be defined as a differential concomitant of the first order of field (12.6) and of the field of the metric tensors $g_{ij}(\xi)$ (Moór [4]). In other words, the covariant derivative $\nabla_j v^i$ is to be a mixed affinor depending on v^i, $\partial_j v^i$, g_{ik}, $\partial_j g_{ik}$:

$$\nabla_j v^i = T^i_j(v^l, \partial_p v^l, g_{ik}, \partial_p g_{ik}). \tag{12.11}$$

A. Moór [4] has proved under the assumption of the continuity of function (12.11) that the covariant derivative $\nabla_j v^i$ must then be a function only of the vector v^i, tensor g_{ij} and affinor (12.8), where the object Γ^i_{jk} is given by formula (12.10). Let us assume furthermore that T^i_j are linear homogeneous functions of v^i and $\partial_j v^i$

$$T^i_j = \Lambda^i_{jk}(g_{lr}, \partial_p g_{lr}) v^k + \Xi^{ih}_{jk}(g_{lr}, \partial_p g_{lr}) \partial_h v^k$$

and that the coefficients Λ^i_{jk} are symmetric:

$$\Lambda^i_{jk} = \Lambda^i_{kj}.$$

Then, under a certain initial condition, the covariant derivative $\nabla_j v^i$ of field (12.6) must be equal to affinor (12.8) (Moór [2]).

Further generalizations of the covariant derivative defined in this way to the case of more general spaces have also been worked out by A. Moór [5], [6].

References

J. Aczél

[1] *Beiträge zur Theorie der geometrischen Objekte. I-II*, Acta Math. Acad. Sci. Hung. 7 (1956), pp. 333-354.

[2] *Beiträge zur Theorie der geometrischen Objekte. III-IV*, Acta Math. Acad. Sci. Hung. 8 (1957), pp. 19-52.

[3] *Beiträge zur Theorie der geometrischen Objekte. V*, Acta Math. Acad. Sci. Hung. 8 (1957), pp. 53-64.

[4] *Beiträge zur Theorie der geometrischen Objekte. VI*, Acta Math. Acad. Sci. Hung. 10 (1959), pp. 1-12.

[5] *Beiträge zur Theorie der geometrischen Objekte. VII*, Acta Math. Acad. Sci. Hung. 10 (1959), pp. 251-268.

[6] *A geometriai objektumok elméletéhez. I*, M. Tud. Akad. III Osztály közleményei 8 (1958), pp. 41-65.

[7] *A geometriai objektumok elméletéhez. II*, M. Tud. Akad. III Osztály közleményei 8 (1958), pp. 211-243.

[8] *Ein allgemeines Prinzip bezüglich Komitanten, Differentialkomitanten, kovarianten Ableitungen und Algebren von äquivalenten geometrischen Objekten*, Acta Univ. Debrecen 6 (1960), pp. 5-13.

[9] *Verallgemeinerte Addition von Dichten*, Publ. Math. Debrecen 7 (1960), pp. 10-15.

[10] *Sur des comitants différentiels des champs de scalaires*, Mathematica, Cluj, 4 (27) (1962), pp. 5-8.

[11] *Remarques sur les relations d'équivalence*, Fund. Math. 5 (1962), pp. 267-269.

J. Aczél, S. Gołąb

[1] *Funktionalgleichungen der Theorie der geometrischen Objekte*, Monografie Matematyczne 39, Warszawa 1960.

A. Balogh

[1] *On the determination of geometric objects with special transformation formulae*, Mathematica, Cluj, 1 (24) (1959), pp. 199-219.

E. Bompiani

[1] *Enti geometrici definiti da sistemi differenziali*, Atti dell'Acad. Naz. dei Lincei Rendiconti (8) 1 (1946), pp. 887-894.

H. Brandt

[1] *Über eine Verallgemeinerung des Gruppenbegriffs*, Math. Ann. 96 (1926), pp. 360-366.

J. S. Dubnow

[1] *Tensoren und geometrische Objekte im eindimensionalen Raum* (Russ.), Abhandlungen Sem. Vekt. Tens. An. Mosk. 7 (1949), p. 3.

[2] *Differentialgeometrische Objekte und Imprimitivitätssysteme* (Russ.), Abhandlungen Sem. Vekt. Tens. An. Mosk. 8 (1950), p. 9.

O. E. Gheorghiu

[1] *Equations aux dérivées partielles et objets géométriques*, C. R. Acad. Sci. Paris 227 (1948), pp. 613-615.

[2] *Equations aux dérivées partielles et objets géométriques*, Bull. Sci. Polyt. Timişoara 13 (1948), pp. 223-233.

[3] *Les lois de transformation des objets géométriques spéciaux linéaires de classe v avec une composante en X_1*, C. R. Acad. Sci. Paris 229 (1949), pp. 611-613.

[4] *Determinarea legii de transformare a obiectelor diferenţial geometrice de clasa a II-a cu două componente în X_1*, Comunicările Acad. R. P. Rom. 1 (1951), pp. 1017-1020.

[5] *Un obiect geometric pseudolinear de clasa 1 cu două componente*, Comunicările Acad. R. P. Rom. 2 (1952), pp. 1-4.

[6] *Asupra teoriei obiectelor geometrice*, Acad. R. P. Rom. Buletin şti. mat.-fiz. 4 (1952), pp. 273-284.

[7] *Determinarea obiectelor geometrice speziale de clasa II*, Bul. şti. Timişoara 2 (1954), pp. 37-40.

[8] *Obiecte geometrice diferenţiale de clasa I. cu două componente în X_1*, Studii şi cerc. ştii. Timişoara 2 (1955), pp. 21-25.

[9] *Determinarea obiectelor geometrice liniara de clasa II în X_n*, Studii şi cerc. ştii. Timişoara 2 (1955), pp. 37-40.

[10] *Obiecte geometrice de lege fracţionară*, Bul. şti. Timişoara 6 (1955), pp. 965-988.

[11] *Obiecte geometrice speciale avind legea de transformare proiective*, Consf. Geom. Dif. 1955, Timişoara 1956, pp. 201-208.

[12] *Sisteme de ecuaţii cu derivate parţiale liniare şi obiectele geometrice asociate*, Consf. Geom. Dif. 1955, Timişoara 1956, pp. 231-237.

[13] *Determinarea derivatei covariante a pseudotensorilor*, Buletinul şti. tehn. Inst. Polit. Timişoara 1 (15) (1956), pp. 27-30.

[14] *Obiecte geometrice de lege fracţionară*, Studii şi cerc. ştii. Timişoara 3 (1956), no. 3-4, pp. 9-13.

[15] *Contributions à la théorie des objets géométriques spéciaux non-différentiels avec plusieurs composantes dans l'espace X_m. I*, C. R. Acad. Sci. Paris 245 (1957), pp. 822-824.

[16] *Contributions à la théorie des objets géométriques spéciaux non-différentiels avec plusieurs composantes dans l'espace X_m. II*, C. R. Acad. Sci. Paris 245 (1957), pp. 887-889.

[17] *Sur les objets géométriques associés à un système linéaire d'équations aux dérivées partielles du premier ordre*, C. R. Acad. Sci. Paris 247 (1958), 26-28.

[18] *Despre un sistem de ecuaţii funcţionale ce se întîlneşte în teoria obiectelor geometrice speciale*, Comunicările Acad. R. P. Rom. 8 (1958), pp. 133-139.

[19] *Obiecte geometrice în raport cu grupul proiectiv neolonom*, Bull. şti. tehn. Inst. Polit. Timişoara 3 (17) (1958), pp. 9-12.

[20] *Über die Theorie der speziellen geometrischen Objekte mit zwei Komponenten* (Russ.), Rev. math. pures appl. 4 (1959), pp. 77-93.

[21] *Asupra obiectelor geometrice asociate unui sistem liniar de ecuaţii cu derivate parţiali de ordinul I*, Studii cerc. mat. Acad. R. P. Rom. In. Mat. 10 (1959), pp. 145-158.

[22] *Geometrische Objekte II. Klasse bezüglich der nichtholonomen projektiven Gruppe* (Russ.), Rev. math. pures appl. 6 (1961), pp. 65-68.

[23] *Geometrische Objekte bezüglich der nichtholonomen projektiven Gruppe des Raumes X_n* (Russ.), Rev. math. pures appl. 7 (1962), pp. 417-424.

O. E. Gheorghiu, B. Crstici

[1] *Asupra unor obiecte geometrice ou două componente*, Studii cerc. mat. Acad. R. P. Rom. In. Mat. 9 (1958), pp. 311-331.

A. Goetz

[1] *O iloczynach skośnych*, Roczniki P. T. M., Prace Mat. 3 (1959), pp. 83-91.

S. Gołąb

[1] *Zur Theorie des affinen Zusammenhanges im eindimensionalen Raume. I*, Opuscula math. Kraków 2 (1938), pp. 7-9.

[2] *Über die Klassifikation der geometrischen Objekte*, Math. Zeitschrift 44 (1938), pp. 104-114.

[3] *Über eine Funktionalgleichung der Theorie der geometrischen Objekte*, Wiadomości Mat. 45 (1938), pp. 97-137.

[4] *Sur quelques points concernant la notion du comitant*, Ann. Soc. Polon. Math. 17 (1938), pp. 177-192.

[5] *Über den Begriff der Pseudogruppe von Transformationen*, Math. Ann. 116 (1939), pp. 768-780.

[6] *Sur la théorie des objets géométriques*, Ann. Soc. Polon. Math. 19 (1946), pp. 7-35.

[7] *Sur la théorie des objets géométriques. (Réduction des objets géométriques spéciaux de première classe aux objets du type Δ)*, Ann. Soc. Polon. Math. 20 (1947), pp. 10-27.

[8] *Alcuni teoremi della teoria degli oggetti geometrici*, Atti dell' Acad. Naz. dei Lincei Rendiconti (8) 5 (1948), pp. 120-122.

[9] *Sur la notion de dérivée covariante* (Congrès Pol. Math. Wrocław 1946), Colloquium Math. 1 (1948), p. 160.

[10] *Sur les objets géométriques non-différentiels*, Bull. Int. Ac. Pol. Sci., Cl. Sci. Math. Nat. ser. A. sci. math. (1949), pp. 67-72.

[11] *Contribution à la théorie des objets géométriques*, Prace mat-fiz. 47 (1949), pp. 1-15.

[12] *Sur les objets géométriques à une composante*, Ann. Soc. Polon. Math. 23 (1950), pp. 79-89.

[13] *La notion de similitude parmi les objets géométriques*, Bull. Int. Ac. Pol. Sci., Cl. Sci. Math. Nat. ser. A sci. math. (1950), pp. 1-7.

[14] *La notion de similitude parmi les objets géométriques*, Acad. Pol. Sci. Lettres C. R. M. Cl. Math. Nat. (1950), p. 10.

[15] *O pojęciu podobieństwa obiektów geometrycznych*, Sprawozdania Pol. Akad. Umiejętności 51 (1950), p. 115.

[16] *Über den Begriff der kovarianten Ableitung*, Nieuw. Archiew voor Wisk. (3) 2 (1954), pp. 90-96.

[17] *Sur la dérivée covariante des objets géométriques de deuxième classe*, Ann. Polon. Math. 1 (1954), pp. 107-113.

[18] *Rachunek tensorowy*, Warszawa 1956.

[19] *Zur Theorie der Übertragungen*, Schriftenreihe des Inst. für Math. D. Akad. Wiss. Berlin 1 (1957), pp. 162-177.

[20] *Differentialkomitanten und Liesche Ableitungen*, Mat. Lapok 10 (1959), p. 174.

[21] *Zagadnienia teorii obiektów geometrycznych*, Roczniki P. T. M., Prace Mat. 3 (1959), pp. 55-67.

[22] *Sur les comitants algébriques des tenseurs*, Ann. Polon. Math. 9 (1960), pp. 113-118.

[23] *Sur les comitants scalaires du premier ordre des champs de scalaires*, Mathematica, Cluj, 2 (25) (1960), pp. 253-255.

[24] *Sur l'équivalence des objets géométriques de deuxième classe dont le nombre de composantes est égal à la dimension de l'espace*, Czechoslovak Math. J. 11 (86) (1961), pp. 475-479.
[25] *La relation d'équivalence et les objets géométriques*, Fund. Math. 50 (1962), pp. 381-386.
[26] *Contribution à la théorie des objets géométriques (Non existence des objets de deuxième classe à une composante dans X_n pour $n \geqslant 2$)*, Tensor N. S. 14 (1963), pp. 122-131.

S. Gołąb, A. Jakubowicz, M. Kucharzewski, M. Kuczma

[1] *Sur l'objet géométrique représentant une direction munie d'un sens*, Ann. Polon. Math. 15 (1964), pp. 233-236.

S. Gołąb, M. Kucharzewski

[1] *Zur Theorie der geometrischen Objekte*, Ann. Polon. Math. 2 (1955), pp. 250-253.
[2] *Über den Begriff der Pseudogrößen*, Tensor N. S. 8 (1958), pp. 78-89.
[3] *Ein Beitrag zur Komitantentheorie*, Acta Math. Acad. Sci. Hung. 11 (1960), pp. 173-174.

S. Gołąb, I. Makai

[1] *Sur les comitants différentiels des champs vectoriels* (in preparation).

S. Gołąb, H. Pidek

[1] *Sur l'algèbre des objets géométriques de première classe à une composante*, Ann. Polon. Math. 4 (1957-1958), pp. 226-241.

S. Gołąb, E. Siwek

[1] *Sur les domaines de transitivité d'un groupe de transformations*, Ann. Polon. Math. 10 (1961), pp. 209-216.

J. Haantjes, G. Laman

[1] *On the definition of geometric objects. I*, Nederl. Akad. Wetensch. Proc. ser. A 56 = Indagationes Math. 15 (1953), pp. 208-215.
[2] *On the definition of geometric objects. II*, Nederl. Akad. Wetensch. Proc. ser. A 56 = Indagationes Math. 15 (1953), pp. 216-222.

M. Hosszú

[1] *Functional equations and algebraic methods in the theory of geometric objects*, Publ. Math. Debrecen 5 (1957-1958), pp. 294-329.
[2] *Függvényegyenletek és algebrai módszerek a geometriai objektumok elméletében. I*, M. Tud. Akad. III. Osztály közleményei 9 (1959), pp. 149-162
[3] *Függvényegyenletek és algebrai módszerek a geometriai objektumok elméletében. II*, M. Tud. Akad. III. Osztály közleményei 9 (1959), pp. 237-254.
[4] *Függvényegyenletek és algebrai módszerek a geometriai objektumok elméletében. III*, M. Tud. Akad. III. Osztály közleményei 9 (1959), pp. 333-346.

M. Ikeda

[1] *On tensorial concomitants of a nonsymmetric tensor $g_{\mu\nu}$. II*, Tensor N. S. 7 (1957), pp. 117-127.

M. Ikeda, S. Abe

[1] *On tensorial concomitants of a nonsymmetric tensor $g_{\mu\nu}$. I*, Tensor N. S. 7 (1957), pp. 59-69.

A. Jakubowicz

[1] *O pewnych obiektach geometrycznych równoważnych afinorom*, Zeszyty Naukowe Politechniki Szczecińskiej 17, 2 (1960), pp. 103-116.

C. Jankiewicz

[1] *La dérivée de Lie du comitant géométrique*, Ann. Polon. Math. 7 (1960), pp. 193-199.
[2] *Über Erhaltungssätze die aus Euler-Lagrangeschen Gleichungen folgen*, Ann. Polon. Math. 11 (1961), pp. 19-26.

F. Klein

[1] *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen 1872, Math. Ann. 43 (1893).

M. Kucharzewski

[1] *Über die skalaren Komitanten der Vektorfelder*, Ann. Polon. Math. 9 (1961), pp. 311-323.
[2] *Über die Vektorkomitanten der Vektorfelder*, Ann. Polon. Math. 9 (1961), pp. 299-309.
[3] *Die skalaren Komitanten, welche aus kovarianten und kontravarianten Vektoren gebildet sind*, Tensor N. S. 12 (1962), pp. 158-166.
[4] *Die kovarianten Vektorkomitanten, die aus kontravarianten Vektoren gebildet sind*, Tensor N. S. 12 (1962), pp. 140-150.
[5] *Zum Begriff der Komitante*, Ann. Polon. Math. 13 (1963), pp. 115-120.

M. Kucharzewski, M. Kuczma

[1] *On linear differential geometric objects with one component. I*, Tensor N. S. 10 (1960), pp. 245-254.
[2] *On linear differential geometric objects with one component. II*, Tensor N. S. 11 (1961), pp. 35-42.
[3] *Sur la classification des objets géométriques linéaires homogènes de la première classe à deux composantes dans l'espace à deux dimensions*, C. R. Acad. Sci. Paris 254 (1962), pp. 1562-1563.
[4] *Determination of geometric objects of the type* [2, 2, 1] *with a linear homogeneous transformation formula*, Ann. Polon. Math. 14 (1963), pp. 29-48.
[5] *Determination of linear differential geometric objects of the first class, with two components, in a two-dimensional space*, ibidem 15 (1964), pp. 77-84.
[6] *Some remarks on geometric objects and their equivalence. I*, Tensor N. S. 13 (1963), pp. 251-260.
[7] *Some remarks on geometric objects and their equivalence. II*, Tensor N. S. 13 (1963), pp. 261-268.
[8] *Klassifikation der linearen homogenen geometrischen Objekte des Typus J mit drei Komponenten*, Ann. Pol. Math. (to appear).

M. Kuczma

[1] *On linear differential geometric objects of the first class with one component*, Publ. Math. Debrecen 6 (1959), pp. 72-78.

M. N. Kuiper, K. Yano

[1] *On geometric objects and Lie groups of transformations*, Nederl. Akad. Wetensch. Proc. ser. A 58 = Indagationes Math. 17 (1955), pp. 411-420.

B. L. Laptew

[1] *Une forme invariante de la variation et la dérivée de S. Lie* (Russ.), Bull. Soc. Phys.-Math. Kazan (3) 12 (1940), pp. 3-8.

G. F. Laptew

[1] *La théorie de S. Lie des objets géométriques, qui dépendent de point et de direction* (Russ.), Bull. Soc. Phys.-Math. Kazan (3) 10 (1938), pp. 4-38.

[2] *On manifolds of geometric elements with a differential connection* (Russ.), Doklady Akad. Nauk SSSR 73 (1950), pp. 17-20.

[3] *On fields of geometric objects on imbedded manifolds* (Russ.), Doklady Akad. Nauk SSSR 78 (1951), pp. 197-200.

[4] *On a new invariant analytic method of differential geometric investigations* (Russ.), 125 years of the non-Euclidean geometry of Lobatschewsky, Moscow-Leningrad 1952, pp. 175-178.

I. Leroy

[1] *Sur les tenseurs symétriques et antisymétriques d'ordre deux, fonctions d'un tenseur asymétrique d'ordre deux*, Acad. Roy. Belg. Bull. Cl. Sci. (5) 47 (1961), pp. 25-41.

T. Levi-Civita

[1] *Nozione di parallelizmo in una varieta qualunque e consequente specificazione geometrica della curvatura Riemanniana*, Rend. Palermo 42 (1917), pp. 173-205.

A. E. Liber

[1] *On the classification of the affine connection in the two-dimensional space* (Russ.), Mat. Sbornik 27 (69) (1950), pp. 249-266.

[2] *On concomitants of geometric differential objects* (Russ.), Doklady Akad. Nauk SSSR 80 (1951), pp. 529-532.

[3] *Differential concomitants of various linear objects* (Russ.), Izv. Vyss. Ucebn. Zaved. Matematika 1960, no. 6 (19), pp. 158-162.

A. Lichnerowicz

[1] *Théorie globale des connexions et des groupes d'holonomie*, Roma 1955.

I. Makai

[1] *Über Invarianten, die aus gewissen Tensoren gebildet sind*, Publ. Math. Debrecen 7 (1960), pp. 359-363.

[2] *Über geometrische Objekte, die aus adjungierten n-Beinen gebildet sind*, Acta Math. Acad. Sci. Hung. 13 (1962), pp. 219-222.

[3] *Über die Lieschen Ableitungen der homogenen linearen geometrischen Objekte* (to appear).

N. N. Mihaileanu

[1] *Obiecte geometrice în geometria diferențială*, Buletinul ședințelor București 3 (1949), pp. 29-30.

[2] *Obiecte geometrice în geometria diferențială*, Studii cerc. mat. Acad. R. P. Rom. In. Mat. 1 (1950), pp. 318-373.

[3] *Obiecte geometrice asociate spațiilor cu conexiune proiectivă P_2*, Com. Acad. R. P. Rom 1 (1951), pp. 165-170.

A. Moór

[1] *Über die kovariante Ableitung der Vektoren*, Acta Sci. Math. Szeged 19 (1958), pp.237-246.

[2] *Über Tensoren, die aus angegebenen geometrischen Objekten gebildet sind*, Publ. Math. Debrecen 6 (1959), pp. 15-25.

[3] *Erweiterung des Begriffs der Räume skalarer und konstanter Krümmung*, Acta Sci. Math. Szeged 21 (1960), pp. 53-77.

[4] *Über die aus g_{ik} bestimmte kovariante Ableitung*, Acta Math. Acad. Sci. Hung. 11 (1960), pp. 175-186.

[5] *Untersuchungen über die kovariante Ableitung in Linienelementenräumen*, Publ. Math. Debrecen 7 (1960), pp. 41-53.

[6] *Nichtlineares invariantes Differential*, Ann. Mat. Pura Appl. 54 (IV) (1961), pp. 255-274.

[7] *Über projektive geometrische Invarianten*, Publ. Math. Debrecen 8 (1961), pp. 350-359.

M. Neumann

[1] *Obiecte geometrice asociate suprafețelor riglate*, Studii cerc. mat. Acad. R. P. Rom. In. Mat. 2 (1951), pp. 445-462.

A. Nijenhuis

[1] *X_{n-1}-forming sets of eigenvectors*, Nederl. Akad. Wetensch. Proc. ser. A 54 = Indagationes Math. 13 (1951), pp. 200-212.

[2] *Theory of the geometric object*, Amsterdam 1952.

[3] *Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I*, Nederl. Akad. Wetensch. Proc. ser. A 58 = Indagationes Math. 17 (1955), pp. 390-397.

[4] *Jacobi-type identities for bilinear differential concomitants of certain tensor fields. II*, Nederl. Akad. Wetensch. Proc. ser. A 58 = Indagationes Math. 17 (1955), pp. 398-403.

A. Pforr

[1] *Einfache geometrische Räume*, Math. Nachr. 20 (1959), pp. 265-301.

H. Pidek

[1] *Sur les objets géométriques de la classe zéro qui admettent une algèbre*, Ann. Soc. Polon. Math. 24 (1951), pp. 111-128.

[2] *Sur un problème de l'algèbre des objets géométriques de classe zéro dans l'espace X_1*, Ann. Polon. Math. 1 (1954), pp. 114-126.

[3] *Sur un problème de l'algèbre des objets géométriques de classe zéro dans l'espace X_m*, Ann. Polon. Math. 1 (1954), pp. 127-134.

J. Pienzow

[1] *Classification of differential geometric objects of the class v with one component* (Russ.), Doklady Akad. Nauk SSSR 54 (1946), pp. 563-566.

[2] *The classification of continuous pseudo-groups of Lie transformations in X_2 according to their characteristic objects* (Russ.), Abhandlungen Sem. Vekt. Tens. An. Mosk. 8 (1950), pp. 382-413.
[3] *On differential geometric objects of class v in X_1* (Russ.), Mat. Sbornik 26 (68) (1950), pp. 161-182.
[4] *The classification of geometric differential objects with two components* (Russ.), Doklady Akad. Nauk SSSR 80 (1951), pp. 537-540.
[5] *On bundles of one-dimensional geometric objects of class $n \geqslant 2$ in X_1^r* (Russ.), Doklady Akad. Nauk SSSR 104 (1955), pp. 356-359.
[6] *Two-dimensional bundles of geometric objects in X_1* (Russ.), Mat. Sbornik 46 (88) (1958), pp. 291-314.

L. S. Pontriagin

[1] *Grupy topologiczne*, Warszawa 1961.

P. K. Rashevski

[1] *Galois theory in fields of geometric objects* (Russ.), Abhandlungen Sem. Vekt. Tens. An. Mosk. 7 (1949), pp. 167-186.
[2] *Linear differential geometric objects* (Russ.), Doklady Akad. Nauk SSSR 97 (1954), pp. 609-611.
[3] *Mehrdimensionale δ-Funktionen und differentialgeometrische Objekte* (Russ.), Uspechi Mat. Nauk 10 (1955) no. 4 (66), pp. 145-152.

J. A. Schouten

[1] *Die direkte Analysis zur neueren Relativitätstheorie*, Verh. Kon. Akad. Amsterdam 12 Nr 6 95 (1918).
[2] *Über Differentialkomitanten zweier kontravarianter Größen*, Nederl. Akad. Wetensch. Proc. ser. A 43 (1940), pp. 3-6.
[3] *De differentiaaloperator van Lie*, Math. Centrum Amsterdam Report Z. W. 1949-10, pp. 1-7.
[4] *On the differential operators of first order in tensor calculus*, Math. Centrum Amsterdam Report Z. W. 1953-012, pp. 1-6.
[5] *On the differential operators of first order in tensor calculus*, Convegno Int. di Geom. Diff. Ital. 1953, Roma 1954, pp. 1-7.
[6] *Ricci-calculus. An introduction to tensor analysis and its geometrical applications*, Berlin-Göttingen-Heidelberg 1954.

J. A. Schouten, D. van Dantzig

[1] *Was ist Geometrie?* Abhandlungen Sem. Vekt. Tens. An. Mosk. 2-3 (1935), pp. 15-50.

J. A. Schouten, J. Haantjes

[1] *Zur Theorie des geometrischen Objektes*, Comptes rendus du Congr. Int. Math. Oslo 1936. II. Oslo 1937, pp. 155-159.
[2] *On the theory of the geometric object*, Proc. London Math. Soc. (2) 42 (1937), pp. 356-376.

J. A. Schouten, E. R. van Kampen

[1] *Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde*, Math. Ann. 103 (1930), pp. 752-783.

J. A. Schouten, D. J. Struik

[1] *Einführung in die neueren Methoden der Differentialgeometrie*, Groningen-Batavia 1934.

E. Siwek

[1] *Sur les domaines de transitivité du groupe de transformations des composantes d'un tenseur covariant du second ordre*, Ann. Polon. Math. 10 (1961), pp. 217-224.

[2] *Pseudoobjets géométriques*, Ann. Polon. Math. (to appear).

[3] *Sur la dérivée covariante des pseudoobjets géométriques*, ibidem (to appear).

M. A. Spivak

[1] *Algebraic theory of coordinate structures and geometric objects* (Russ.), Izv. Vyss. Ucebn. Zaved. Matematika 1958, no. 4 (5), pp. 236-247.

N. Steenrod

[1] *The topology of fibre bundles*, Princeton Math. Series 14, 6 (1951).

W. Ślebodziński

[1] *Sur les équations de Hamilton*, Bull. Acad. Roy. Belg. (5) 17 (1931), pp. 864-870.

Y. Tashiro

[1] *Sur la dérivée de Lie de l'être géométrique et son groupe d'invariance*, Tôhoku Math. J. (2) 2 (1950), pp. 166-181.

[2] *Note sur la dérivée de Lie d'un être géométrique*, Math. J. Okoyama Univ. 1 (1952), pp. 125-128.

N. Teodorescu

[1] *Equations aux dérivées partielles et objets géométriques*, Disquisitiones math. phys. 4 (1945), pp. 105-118.

O. Veblen

[1] *Invariants of quadratic differential forms*, Cambridge Tracts 27 (1927).

O. Veblen, J. M. Thomas

[1] *Projective invariants of affine geometry of paths*, Ann. of Math. 27 (1926), pp. 279-296.

O. Veblen, J. H. C. Whitehead

[1] *The foundations of differential geometry*, Cambridge Tracts 29 (1932).

G. Vranceanu

[1] *Obiecte geometrice de speţa a treia*, Buletinul şedinţelor Bucureşti 3 (1949), pp. 30-34.

W. W. Wagner

[1] *The absolute derivative of fields of local geometric objects in a compound manifold*, Doklady Akad. Nauk SSSR 40 (1943), pp. 94-97.

[2] *The theory of geometric objects and the theory of finite and infinite continuous groups of transformations*, Doklady Akad. Nauk SSSR 46 (1945), pp. 347-349.

[3] *Constant fields of local geometric objects in compound manifolds with a linear connection*, Doklady Akad. Nauk SSSR 53 (1946), pp. 183-186.

[4] *Classification of linear connections in a composite manifold $X_{n+(1)}$ according to their holonomy group* (Russ.), Abhandlungen Sem. Vekt. Tens. An. Mosk. 7 (1949), pp. 205-226.

[5] *Theorie der differentiellen Objekten und Grundlagen der Differentialgeometrie* (Russ.), Nachtrag zur russischen Übersetzung der Arbeit *The foundations of differential geometry*, Cambridge 1932, von O. Veblen — J. H. C. Whitehead, Moskau 1949.

[6] *Classification of simple differential geometric objects* (Russ.), Doklady Akad. Nauk SSSR 69 (1949), pp. 293-296.

[7] *On the theory of pseudogroups of transformations* (Russ.), Doklady Akad. Nauk SSSR 72 (1950), pp. 453-456.

[8] *Classification of simple differential geometric objects* (Russ.) Uspechi Mat. Nauk 5 (1950), no. 1 (35), pp. 213-214.

[9] *The geometry of the generalized Cartan spaces and the theory of geometric differential objects* (Russ.), Doklady Akad. Nauk SSSR 77 (1951), pp. 777-780.

A. G. Walker

[1] *Dérivation torsionnelle et seconde torsion pour une structure presque complexe*, C. R. Acad. Sci. Paris 245 (1957), pp. 1213-1215.

A. Wundheiler

[1] *Objekte, Invarianten und Klassifikation der Geometrien* (I. Intern. Konf. f. tens. Diff. Geom. u. i. Anw. Moskau 17-23. V. 1934), Abhandlungen Sem. Vekt. Tens. An. Mosk. 4 (1937), pp. 366-375.

K. Yano

[1] *The theory of Lie derivatives and its applications*, Amsterdam-Groningen 1957.

[2] *On Walker-differentiation in almost product or almost complex spaces*, Nederl. Akad. Wetensch. Proc. ser A 61 = Indagationes Math. 20 (1958), pp. 573-580.

K. Yano, Y. Tashiro

[1] *Some theorems on geometric objects and their applications*, Nieuw Archief voor Wisk. (3) 2 (1954), pp. 134-142.

A. Zajtz

[1] *Komitanten der Tensoren zweiter Ordnung*, Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace Mat. 8 (1964), pp. 1-52.

Received on 30 V 1963

Contents