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Small sets in C^n

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Abstract. The purpose of this note is to give a short survey of what is known about small sets in C^n.

In order to see an analogy with classical potential theory, let first Ω be an open subset of R^n ($n \geqslant 2$), K a closed subset of Ω. Consider the following properties that K may or may not have.

(i) For every $z_0 \in K$ there is a subharmonic function φ on Ω such that

$$\overline{\lim_{z \to z_0}} \varphi(z) < \varphi(z_0), \quad \text{where } z \in K,\ z \neq z_0.$$

(ii) There is a subharmonic function φ on Ω, $\varphi \not\equiv -\infty$, such that

$$K \subset \{z \in \Omega;\ \varphi(z) = -\infty\}.$$

(ii) There is a locally upper bounded family $(\varphi_i)_{i \in I}$ of subharmonic functions on Ω such that $K \subset \{z \in \Omega;\ \varphi(z) < \varphi^*(z)\}$, where $\varphi(z) = \sup_{i \in I} \varphi_i(z)$ and $\varphi^*(z) = \overline{\lim_{z' \to z}} \varphi(z')$.

(iv) If φ is subharmonic outside K and if

$$\overline{\lim_{z' \notin z}} \varphi(z') < +\infty, \quad \text{where } z' \notin K, \quad \forall z \in \Omega,$$

then φ extends to a uniquely determined subharmonic function on Ω.

In classical potential theory, it is a theorem that these properties are equivalent and the compact sets that have the properties are exactly those with vanishing Newton capacity.

To study the corresponding properties in C^n, R^n has to be replaced by C^n and the subharmonic functions by the plurisubharmonic functions. Conditions (i)–(iv) are then transformed into conditions (i′)–(iv′) and they are no longer equivalent but:

$$(\mathrm{i}') \substack{\Rightarrow \\ \not\Leftarrow} (\mathrm{ii}') \Leftrightarrow (\mathrm{iii}') \substack{\Rightarrow \\ \not\Leftarrow} (\mathrm{iii}) \substack{\Rightarrow \\ \not\Leftarrow} (\mathrm{iv}').$$

Removable singularity sets

Closed sets K having property (iv′) are called *removable singularity sets for bounded plurisubharmonic functions* and it was shown by Lelong [14] that every set with the equivalent properties (i)–(iv) (considered as a subset of $\boldsymbol{R}^{2n}$) also has property (iv′). A phenomenon that occurs in $\boldsymbol{C}^n$ ($n \geqslant 2$) is that a compact set can be removable in the sense that every plurisubharmonic function extends across it. These sets are called *removable singularity sets for plurisubharmonic functions* and it was shown in Cegrell [3] that there exist such sets with positive Newton capacity.

In particular, there is no converse of Lelong's result.

Here are some examples of conditions that guarantee K to be a removable singularity set for plurisubharmonic functions (note that only the first one is a local condition):

The $2n-2$ dimensional Hausdorff measure of K vanishes (Shiffman [21]).

The set K is a proper subset of an irreducible analytic subset (Siu [22], see also [4], [9] and [15]);

The set K is compact and has property (ii′) (Cegrell [4]).

On the other hand, it was shown in Cegrell [5] that if $K^0 \neq \emptyset$, where K is compact in Ω then the restriction map

$$\mathrm{PSH}(\Omega) \to \mathrm{PSH}(\Omega \setminus K)$$

cannot be surjective.

Negligible and pluripolar sets

Sets (not necessarily closed) having property (iii′) and (ii′) are called *negligible* and *pluripolar* respectively.

That condition (iii′) is a local condition and that (ii′)$\Rightarrow$(iii′) was proved by Lelong [16]. That (ii′) is a local condition was proved by Josefson [11] and the picture was completed by Bedford and Taylor [1] when they proved that (iii′)$\Rightarrow$(ii′). Thus, negligible and pluripolar sets are the same.

Thin sets

The (not necessarily closed) sets with property (i′) are called thin. Thinness is the most sensitive notion, every thin set is pluripolar according to Bedford and Taylor [1]. The converse is not true since the intersection with every complex line has to be thin considered as a subset of $\boldsymbol{C}$.

It follows from Leja's polynomial lemma [13] that if $[0, 1] \ni t \mapsto z(t) \in \boldsymbol{C}$ is a continuous curve, then it is not thin at any of its points. However, this is not true in higher dimension as shown by Sadullaev [19], [20] and Cegrell [6].

On the other hand, it was proved in Cegrell [6] that every polynomially convex and L-regular compact set is not thin at any of its points.

If there is a positive measure with support in K such that

$$K \subset \{\log * \mu = -\infty\},$$

then it follows as in the classical case that K is thin.

Small sets in the boundary

DEFINITION. Let Ω be a bounded, open and connected subset of C^n and let $z_0 \in \Omega$. We then define K to be

$$K = \{\mu \geqslant 0;\ \operatorname{supp}\mu \subset \partial\Omega,\ f(z_0) = \int f(z)\,d\mu(z),\ \forall f \in A(\Omega)\},$$

where $A(\Omega)$ is the set of analytic functions that extend continuously to $\partial\Omega$. We also define

$$Q(E) = \sup_{\mu \in K} \mu(E), \qquad E \subset \partial\Omega.$$

Note that, since K is weak*-compact, Q is a capacity.

DEFINITION. A complex measure μ on $\partial\Omega$ is called an *A-measure* (cf. Henkin [10]) if $\lim_{s\to+\infty} \int f_s\,d\mu = 0$ for every sequence $f_s \in A(\Omega)$, $s \in N$ with $|f_s| \leqslant 1$ and $\lim_{s\to+\infty} f_s(z) = 0$, $\forall z \in \Omega$.

The following lemma is a variant of Forelli [8]. See also König [12].

LEMMA 1. *Assume that E is an F_σ-set of vanishing Q-capacity. Then there is a sequence $f_s \in A(\Omega)$, $|f_s| \leqslant 2$, $s \in N$ such that*

1° $\lim_{s\to+\infty} f_s(z) = 0$, $\forall z \in \Omega$;

2° $\lim_{s\to+\infty} f_s(z) = 1$, $\forall z \in E$;

3° $\lim_{s\to+\infty} f_s(z) = 0$ *outside a set of vanishing Q-capacity.*

THEOREM 1. *If μ is an A-measure, then $\mu = f dv + \eta$, where $v \in K$, $f \in L^1(v)$ and η is orthogonal to $A(\Omega)$.*

Proof. Since K is convex and weak*-compact we can, by a theorem of Glicksberg, König and Seever (cf. Rudin [18]) write

$$\mu = f dv + \eta, \qquad \text{where } v \in K,\ f \in L^1(v)$$

and η is carried by an F_σ-set of vanishing Q-capacity. Choose a sequence $(f_s)_{s=1}^{\infty}$ as in Lemma 1. If f is any function in $A(\Omega)$ we get $0 = \lim_{s\to+\infty} \int f f_s\,d\mu$ by assumption. Moreover, 2° and 3° show that

$$\lim_{s\to+\infty} \int f f_s\,d\mu = \int f d\eta$$

and the theorem is proved.

Remark. It was shown by Henkin (see Henkin [10], Valskii [23] and Cole and Range [7]) that if Ω is strictly pseudoconvex, then the second term in the decomposition in Theorem 1 always vanishes for A-measures. In particular, every measure orthogonal to $A(\Omega)$ is absolutely continuous with respect to a measure in K. This means that Theorem 1 can be considered as a generalization of a theorem of F. and M. Riesz [17].

But in contradistinction to the case $n = 1$, measures orthogonal to $A(\Omega)$ may be singular with respect to the Lebesgue measure on the boundary.

The measure μ defined on the boundary of the unit ball in C^2 by

$$\int \varphi d\mu = \frac{1}{2\pi}\int_0^{2\pi} \varphi(0, e^{i\theta})\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}, 0)\,d\theta$$

shows this.

Consider the following properties of a compact subset L in the boundary of Ω.

(I) (interpolation). Every continuous function on L is the restriction of a function in $A(\Omega)$.

(N) If a measure μ is orthogonal to $A(\Omega)$, then $|\mu|(L) = 0$.

(PI) (peak-interpolation). To every continuous function g on L there is an f in $A(\Omega)$ which extends g and such that

$$|f(z)| < \sup_L |g|, \qquad z \in \bar{\Omega} \setminus L.$$

(P) (peak). There is an $f \in A(\Omega)$ such that $f = 1$ on L and $|f(z)| < 1$ on $\bar{\Omega} \setminus L$.

(Q) $Q(L) = 0$.

For the rest of this section, we assume that every point on $\partial\Omega$ has property (P). We then have:

THEOREM 2. (I) $\Leftrightarrow$ (N) $\Leftrightarrow$ (PI) $\Rightarrow$ (P) $\Rightarrow$ (Q).

Proof. That (I) $\Rightarrow$ (N) follows from Varopoulos [24]. The implication (N) $\Rightarrow$ (PI) is a theorem of Bishop [2]. That (PI) $\Rightarrow$ (P), (PI) $\Rightarrow$ (I) and that (P) $\Rightarrow$ (Q) is trivial.

It is a well-known fact that if Ω is strictly pseudoconvex, then all the concepts are equivalent (cf. Rudin [18], p. 204). This is also a consequence of the above mentioned result by Henkin and the following proposition.

PROPOSITION. *A necessary and sufficient condition that* (Q) $\Rightarrow$ (I) *is that every A-measure is absolutely continuous with respect to a measure in K.*

Proof. Assume that (Q) $\Rightarrow$ (I) and let μ be a given A-measure. By Theorem 1,

$$\mu = f d\nu + \eta, \qquad \nu \in K, \qquad \eta \perp A(\Omega),$$

where η is carried by an F_σ-set E of vanishing Q-capacity. If $\eta \not\equiv 0$ there is a compact subset E_1 of E with $\eta(E_1) \neq 0$. By assumption and Theorem 2, there is an $f \in A(\Omega)$; $f|_{E_1} = 1$, $|f| < 1$ on $\bar{\Omega} \setminus E_1$. Hence $0 = \lim_{m \to +\infty} \int f^m d\eta = \eta(E_1) \neq 0$ which is a contradiction and it follows that $\eta \equiv 0$.

Now, assume that every A-measure is absolutely continuous with respect to a measure in K. Let L be a given compact subset of $\partial\Omega$ with $Q(L) = 0$. To complete the proof, it is by Theorem 2 enough to prove that L has property (N). If μ is orthogonal to $A(\Omega)$, then

$$\mu = f dv, \qquad v \in K$$

by assumption and since $Q(L) = 0$ we have

$$|\mu|(L) = \int_L |f| \, dv = 0$$

and the proof is complete.

Remark. If the assumption that every point in $\partial\Omega$ has property (P) is removed, then (I) need not imply any of the other properties.

References

[1] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[2] E. Bishop, *A general Carleson–Rudin theorem*, Proc. Amer. Math. Soc. 13 (1962), 140–143.

[3] U. Cegrell, *Sur les ensembles singuliers impropres des fonctions plurisousharmoniques*, C. R. Acad. Sci. Paris Serie A 281 (1975).

[4] –, *Removable singularities for plurisubharmonic functions and related problems*, Proc. London Math. Soc. 36 (1978).

[5] –, *On the domains of existence for plurisubharmonic functions*, Banach Center Publications, Vol. 11, 1983; PWN–Polish Scientific Publishers.

[6] –, *Thin sets in C^n*, Zesz. Nauk Uniw. Jagellon. (to appear).

[7] B. Cole, M. Range, *A-measures on complex manifolds and some applications*, J. Func. An. 11 (1972), 393–400.

[8] F. Forelli, *Analytic measures*, Pacific J. Math. 13 (1963), 571–578.

[9] R. Harvey, J. Polking, *Extending analytic objects*, Comm. Pure Appl. Math. 28 (1975), 701–727.

[10] G. M. Henkin, *The Banach space of analytic functions on the ball and the bicylinder are not isomorphic*, Func. Ann. Appl. 2 (1968), 334–341.

[11] B. Josefson, *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions in C^n*, Ark. Mat. 16 (1978), 109–115.

[12] H. König, *On certain applications of the Hahn–Banach and Minimax theorems*, Arch. Math. 21 (1970), 583–591.

[13] F. Leja, *Sur les suites de polynômes bornés presque partout sur la frontière d'un domaine*, Math. Ann. 108 (1933), 517–524.

[14] P. Lelong, *Ensembles singuliers impropres des fonctions plurisousharmoniques*, J. Math. Pures Appl. 36 (1957), 263–303.

[15] –, *Lecture at the conference on analytic functions* (Kraków 1974), Ann. Polon. Math. 33 (1976–1977), 174.

[16] P. Lelong, *Fonctions entiers de type exponentiels dans C^n*, Ann. Inst. Fourier 16 (1966), 269–318.
[17] F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, C. R. 4ème Congr. Math. Scand. Stockholm 1916, 27–44.
[18] W. Rudin, *Function theory in the unit ball of C^n*, Springer-Verlag, New York, Heidelberg, Berlin 1980.
[19] A. Sadullaev, *P-regularity of sets in C^n*, Springer LNM 798 (1979), 402–408.
[20] —, *Plurisubharmonic measures and capacities on complex manifolds*, Uspekhi Math. Nauk. 36:4 (1981), 53–105.
[21] B. Shiffman, *Extension of positive line bundles and meromorphic maps*, Invent. Math. 15 (1972), 332–347.
[22] Y.-T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. 27 (1974), 53–156.
[23] R. E. Valskii, *On measures orthogonal to analytic functions in C^n*, Sov. Math. Dokl. 12 (1971), 808–812.
[24] N. Varopoulos, *Ensembles pics et ensembles d'interpolation pour les algèbres uniforms*, C. R. Acad. Sci. Paris 272 (1971), 866–867.

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