Indeterminate forms for multi-place functions

by A. I. Fine (Urbana) and S. Kass (Chicago)

§ 1. Introduction. A well-known theorem of Bernoulli, commonly called "l'Hospital's Rule" (cf. [1]), states that if a pair of differentiable 1-place functions \( f, g \) have a common zero or a common infinity at a point \( A \), which is not a limit point of zeros of \( g' \), then

\[
\lim_{A} \frac{f}{g} = L \quad \text{whenever} \quad \lim_{A} \frac{f'}{g'} = L.
\]

In this article we extend this theorem to higher place functions by proving that if a pair of differentiable \( n \)-place functions \( f, g \) have a common zero or a common infinity at a point \( A \), which is not a limit point of zeros of \( g_{a} \), then

\[
\lim_{A} \frac{f}{g} = L \quad \text{whenever} \quad \lim_{A} \frac{f_{a}}{g_{a}} = L,
\]

provided that, in the case where \( |g| \rightarrow \infty \), \( f \) and \( g \) are externally bounded. (Terminology and notation will be explained in § 2.) Just as in the classical case the theorem extends to the various "infinite cases" of \( L \) and \( A \).

For simplicity results are stated for 2-place functions; however each result holds for \( n \)-place functions: simply replace "2" by "\( n \)" in each proof and read the summation signs accordingly.

§ 2. Terminology. The symbol \( f(P) \) denotes the value of a function \( f \) at a point \( P : (p_{1}, p_{2}) \). Subscript notation will be used for partial derivatives. In particular,

\[
f_{a}(P) = (\cos a)f_{1}(P) + (\sin a)f_{2}(P) = \frac{\sum (p_{1}-a_{1})f(P)}{\sum (p_{1}-a_{1})^{2}}
\]

will be called the directional derivative of \( f \) with respect to \( A \). Here \( A : (a_{1}, a_{2}) \) is a fixed point, \( P : (p_{1}, p_{2}) \) a point variable, and \( a \) the angle between the positively directed \( x \)-axis and the directed line determined by segment \( (AP) \). The symbol \( (AP) \) \([AP]\) will be used to denote the open (closed) directed segment from \( A \) to \( P \).
It is important for what follows to observe that if \( X: (x_1, x_2) \) and \( Y: (y_1, y_2) \) are any pair of distinct points on \( \Delta P \), then \( \cos a = (x_1 - y_1)/\Delta \), and \( \sin a = (x_2 - y_2)/\Delta \), where \( \Delta = [\sum (x_i - y_i)^2]^{1/2} \). Thus for functions \( f \) and \( g \)

\[
(*) \quad \frac{f_i(P)}{g_i(P)} = \frac{\sum (p_i - a_i) f_i(P)}{\sum (p_i - a_i) g_i(P)} = \frac{\sum (x_i - y_i) f_i(P)}{\sum (x_i - y_i) g_i(P)}
\]

provided that these quotients are defined.

A point set \( S \subseteq E^2 \) is starlike with respect to point \( A \) if for each \( P \in S, (\Delta P) \subseteq S \).

A neighborhood \( N(A; \delta) \) in \( S \) is the intersection in \( E^2 \) of \( S \) with the open sphere of center \( A \) and radius \( \delta \).

Relative to a fixed point \( A \), we shall call a sequence of points \( \{Q_i\} \) external to a sequence of points \( \{Q_i'\} \) if for all but finitely many \( i \), \( Q_i \in (Q_i' A) \). We shall say that a function \( f \) is externally bounded with respect to \( A \) provided that the following condition holds for each neighborhood \( N(A; \delta) \) of \( A \): corresponding to each sequence \( \{Q_i'\} \subseteq N(A; \delta) \) which converges to \( A \), there exists a sequence \( \{Q_i\} \subseteq N(A; \delta) \), external to \( \{Q_i'\} \), on which \( f \) is bounded.

§ 3. Main results. We require the following extension of the Cauchy law of the mean to 2-place functions.

**Lemma 1.** Let \( f \) and \( g \) be 2-place functions defined on \( S \subseteq E^2 \). Suppose that \( S \) contains a line segment \( L \) directed from \( X: (x_1, x_2) \) to \( Y: (y_1, y_2) \) with \( a \) the angle between \( L \) and the positive \( x \)-axis.

If both \( f \) and \( g \) are continuous on the open segment and differentiable on the open segment, then there is some point \( P \in (XY) \) such that

\[
[f(X) - f(Y)] g_i(P) = [g(X) - g(Y)] f_i(P).
\]

**Proof.** Form the function

\[
h(T) = \begin{vmatrix}
f(X) & g(X) & 1 \\
f(Y) & g(Y) & 1 \\
f(T) & g(T) & 1 \\
\end{vmatrix}
\]

and apply the law of the mean for 2-place functions at \( X \) and \( Y \).

**Theorem 1.** Let \( A: (a_1, a_2) \in E^2 \) and let \( f \) and \( g \) be functions whose domains include a set \( S \subseteq E^2 \) which is starlike with respect to \( A \). Suppose that on \( S \) the functions are differentiable and that \( g(A) = 0 \), the directional derivative of \( g \) with respect to \( A \), is never zero. With the understanding that all limits are taken from within \( S \) at \( A \), there are two cases:

(i) \( f(A) = g(A) = 0 \) or

(ii) \( |g| \to \infty \) and both \( f \) and \( g \) are externally bounded with respect to \( A \).

In either case we have that if
(iii) \[ \lim_{\Delta} \frac{f_\Delta(X)}{g_\Delta(X)} = L \quad (i.e. \lim_{\Delta} \sum_{\Delta} \frac{(x_i - a_i) f_\Delta(x_i, a_i)}{(x_i - a_i) g_\Delta(x_i, a_i)} = L), \]
then
(iv) \[ \lim_{\Delta} \frac{f(X)}{g(X)} = L. \]

Proof. We first note that if only \( \lim f = \lim g = 0 \) is assumed, then the hypotheses of case (i) are fulfilled if we define \( f(\Delta) = g(\Delta) = 0 \).

Case (i). Let \( \varepsilon > 0 \) be given. Then there exists a neighborhood \( N(A, \delta) \) such that for every point \( U : (u_1, u_2) \) in \( N(A, \delta) \)

\[ L - \varepsilon < \frac{\sum_{\Delta}(u_i - a_i)f_\Delta(U)}{\sum_{\Delta}(u_i - a_i)g_\Delta(U)} < L + \varepsilon. \]

Let \( X \) be any point \((\neq \Delta)\) in \( N(A, \delta) \) and consider the quotient

\[ \frac{f(X) - f(\Delta)}{g(X) - g(\Delta)} = \frac{f(X)}{g(X)}. \]

By the ordinary law of the mean, \( g(X) \neq 0 \) because \( g_\Delta(U) \neq 0 \) on \( N(A, \delta) \). \( N(A, \delta) \) is starlike with respect to \( A \); therefore \([AX] \subseteq N(A, \delta) \). Hence, by the lemma there exists a point \( P \in \{AX\} \) for which

\[ \sum_{\Delta}(x_i - a_i) f_\Delta(P) \quad \sum_{\Delta}(x_i - a_i) g_\Delta(P) \]

Now (1) certainly holds for \( U = P \); therefore

\[ L - \varepsilon < \frac{\sum_{\Delta}(p_i - a_i)f_\Delta(P)}{\sum_{\Delta}(p_i - a_i)g_\Delta(P)} < L + \varepsilon. \]

By \((*)\) and \((2)\), however

\[ \frac{\sum_{\Delta}(p_i - a_i)f_\Delta(P)}{\sum_{\Delta}(p_i - a_i)g_\Delta(P)} = \frac{\sum_{\Delta}(x_i - a_i)f_\Delta(P)}{\sum_{\Delta}(x_i - a_i)g_\Delta(P)} = \frac{f(X)}{g(X)}. \]

Thus \( L - \varepsilon < f(X)/g(X) < L + \varepsilon \) for all \( X \in N(A, \delta) \), whence \( \lim f(X)/g(X) = L \).

Case (ii). \( |g| \to \infty \). Let \( X \) be any point \((\neq \Delta)\) in \( N(A, \delta) \) and \( Y \) any point in \( (AX) \). We repeat the argument in case (i), with \( A \) replaced by \( Y \), and find that

\[ L - \varepsilon < \frac{f(X) - f(Y)}{g(X) - g(Y)} < L + \varepsilon, \]
for every \( X, Y \in N(A, \delta) \), where \( Y \in (AX) \). As before, \( g(X) - g(Y) \neq 0 \). We can assume \( g \neq 0 \) on \( N(A, \delta) \) and so may divide in \( (3) \) to get

\[ L - \varepsilon < \frac{f(Y) - f(X)}{g(Y) - g(X)} < L + \varepsilon. \]

(4) \[ L - \varepsilon < \frac{f(Y) - f(X)}{g(Y)} - \frac{f(X)}{g(Y)} < L + \varepsilon. \]
Now consider any sequence of points \( Q_i \) of \( N(A, \delta) \) which converges to \( A \). By hypothesis there exists an external sequence of points \( Q_i \) of \( N(A, \delta) \) on which both \( f \) and \( g \) are bounded. (4) is satisfied when \( X \) and \( Y \) are replaced by \( Q_i \) and \( Q_i \), respectively. Thus

\[
L - \varepsilon < \frac{f(Q_i)}{g(Q_i)} < \frac{f(Q_i)}{g(Q_i)} < L + \varepsilon.
\]

Since the \( g(Q_i) \) are bounded, for large enough \( i \), say in the neighborhood \( N(A, \delta') \), we have that \( 1 - \frac{g(Q_i)}{g(Q_i)} > 0 \) for all \( Q_i \in N(A, \delta') \). Thus

\[
(L - \varepsilon) \left[ 1 - \frac{g(Q_i)}{g(Q_i)} \right] + \frac{f(Q_i)}{g(Q_i)} < \frac{f(Q_i)}{g(Q_i)} < (L + \varepsilon) \left[ 1 - \frac{g(Q_i)}{g(Q_i)} \right] + \frac{f(Q_i)}{g(Q_i)}
\]

for the \( Q_i, Q_i \in N(A, \delta) \), where \( \delta = \min(\delta, \delta') \). If \( i \to \infty \), then \( 1 - \frac{g(Q_i)}{g(Q_i)} \to 1 \)
and \( \frac{f(Q_i)}{g(Q_i)} \to 0 \), from which it follows that

\[
L - 2\varepsilon < \frac{f(Q_i)}{g(Q_i)} < L + 2\varepsilon
\]

for all \( Q_i \) in some subneighborhood of \( N(A, \delta) \). Since the sequence \( Q_i \) was chosen arbitrarily, \( \lim_{A} (f/g) = L \).

**Corollary 1.** In the theorem, \( L \) may be replaced by either of the symbols \( \infty, -\infty \).

**Proof.** We consider only \( \infty \). If \( \lim_{A} (f/g) = \infty \), then for given \( M > 0 \) there exists \( N(A, \delta) \) such that for all points \( P : (p_1, p_2) \in N(A, \delta) \) we have

\[
\sum (p_1 - a_1) f_i(P) > \sum (p_1 - a_1) g_i(P).
\]

As before, this gives

\[
\frac{f(X) - f(Y)}{g(X) - g(Y)} > M
\]

for all points \( X, Y \in N(A, \delta) \) such that \( Y \in [AX] \) in case (i) and \( Y \in (AX) \) in case (ii).

Setting \( Y = A \) in case (i), we obtain immediately that \( f(X) / g(X) > M \) for all \( X \in N(A, \delta) \).

In case (ii), for any sequence \( \{Q_i\} \) converging to \( A \) we have for sufficiently large \( i \)

\[
\frac{f(Q_i)}{g(Q_i)} > \left( 1 - \frac{g(Q_i)}{g(Q_i)} \right) M + \frac{f(Q_i)}{g(Q_i)}
\]
where \( \{Q_i\} \) is a sequence external to \( \{Q'_i\} \) on which \( f \) and \( g \) are bounded. It follows that as \( i \to \infty \), \( \frac{f(Q'_i)}{g(Q'_i)} \to \infty \).

**Corollary 2.** If \( A \) has coordinates \((a_1, a_2)\) where either \( a_1 \) or \( a_2 \) is one of the symbols \( \infty, -\infty \) then sufficient conditions that \( f(X)/g(X) \to L \) as \( X \to A \) are provided by using Theorem 1 to show that \( f(X)/g(X) \to L \) as \( X' \to A' \), where \( X' \) and \( A' \) are defined as follows. With \( X: (x_1, x_2) \), let \( X': (x'_1, x'_2) \) and \( A': (a'_1, a'_2) \) be given by having, for each \( i \), \( x'_i = a_i \) and \( a'_i = a_i \) iff \( a_i \) is finite, while \( x'_i = 1/x_i \) and \( a'_i = 0 \) iff \( a_i \) is either \( \infty \) or \( -\infty \).

**Proof.** It suffices to show that for a 2-place function \( F \), if \( \lim_{X \to A'} F(X) = L \), then \( \lim_{X \to A} F(X) = L \). There are \( 3^2 - 1 = 8 \) possibilities for \( A \). Since \( X \to A \) the argument is similar for each, we shall just illustrate it for the case \( A: (a, -\infty) \). Then, for given \( \varepsilon > 0 \), we have \( \delta_1, \delta_2 > 0 \) such that \( |F(x_1, x_2) - L| < \varepsilon \) provided \( 0 < |x'_2 - a| < \delta_1 \) and \( 0 < |x'_2| < \delta_2 \). It follows, since \( x'_1 = x_1 \) and \( x'_2 = 1/x_2 \), that if \( 0 < |x_1 - a| < \delta \), and \( x_2 < -1/\delta \) then \( |F(x_1, x_2) - L| < \varepsilon \). Hence

\[
\lim_{x_2 \to -\infty} F(x_1, x_2) = L.
\]

**§ 4. Remarks.** (i) It can be shown that if \( f/g \) has no unique value as \( X \to A \) from within \( S \), then \( L_1 \subseteq L_2 \), where \( L_1 \) is the set of all limit points of \( f/g \) and \( L_2 \) is the set of all limit points of \( f/g \).

(ii) The operator \( D \) defined by \( Df(x_1, x_2) = x_1f_1(x_1, x_2) + x_2f_2(x_1, x_2) \) is a derivation. When \( f \) is homogeneous of degree \( k \) then \( Df = kf \). Thus if \( f \) and \( g \) are homogeneous functions of distinct (non-zero) degrees that meet the conditions of Theorem 1 with \( A \) at the origin, then either \( \lim_{A} (f/g) = 0 \) or else the limit does not exist.

(iii) The theorem takes on a particularly useful and simple form when reformulated in terms of polar coordinates \( r, \theta_1, \ldots, \theta_{n-1} \) with point \( A \) set at the origin. Then \( f_n = rF_r, \ g_n = rG_r \) and \( f_n/g_n = F_r/G_r \). Here \( F \) and \( G \) are the functions in polar coordinates corresponding to \( f \) and \( g \) respectively, \( F_r \) and \( G_r \) have their usual meanings with respect to the polar coordinate \( r \), and limits are taken as \( r \to 0 \) uniformly in the remaining polar coordinates \( \theta_i \).

(iv) If one knows that the limit of \( f/g \) exists, one can readily find it by applying the familiar one-dimensional form of the theorem along an appropriate path of approach. Thus in practice the theorem is useful not so much as a device in computing \( \lim_{A} (f/g) \) but to guarantee the existence of the limit within a given set \( S \), which is starlike with respect to \( A \). In some cases, one may actually find the largest set \( S \) for which
the limit of \( f/g \) exists, as well as \( L_1 \) of (i), which is always an interval of the real line.

We give an example. Consider the quotient \( (xy^3+2y^2)/(x^4+y^2) \) at the origin. This seems to fall under case (i) of our theorem, but the computation there yields an even more complicated quotient. Divide numerator and denominator by \( y^4 \) to get \( f/g \) with
\[
 f(x, y) = x/y + 2/y^2 \quad \text{and} \quad g(x, y) = x^4/y^4 + 1/y^2.
\]
This quotient comes under case (ii) and one easily finds that \( f/g \to 2 \). Nevertheless, the quotient has no limit at \((0, 0)\), as one can verify by trying the paths \( y = x \) and \( y = x^2 \). One might suspect, however, that the limit is 2 if taken from within some “nice” region and indeed a routine working out of the hypotheses of Theorem 1 produces such a region: it is \( E^3 \) with certain open wedges excluded; viz., all points \((w, y)\) between \( y = \varepsilon_1 w, y = -\varepsilon_2 w\), for \( \varepsilon > 0 \), arbitrarily small.

References