ANNALES
POLONICI MATHEMATICI
XXXIII (1977)

On mixed inequalities between solutions of an almost linear partial differential equation of the first order with a retarded argument

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Abstract. In this paper we give theorems concerning mixed inequalities occurring between solutions of an almost linear differential equation of the first order with a retarded argument

$$\frac{\partial z(x,y)}{\partial x}+P(x,y)\frac{\partial z(x,y)}{\partial y} = R\big(x,y,z(x,y),\ z(x-\tau(x),y)\big).$$

The theorems contained in this paper are generalizations of theorems given in [4] concerning mixed inequalities between solutions of an almost linear partial differential equation of the first order.

Assume that the functions $u(x, Y)$ and $v(x, Y)$ are solutions of a partial differential equation of the first order

$$z_x = f(x, Y, z, z_Y), \tag{1}$$

where $Y = (y_1, \ldots, y_n)$, $z_Y = (z_{y_1}, \ldots, z_{y_n})$. Assume also that these solutions are generated by characteristics according to the definition given in paper [9], p. 179. In paper [9] (Theorem 59.2, p. 179) sufficient conditions are given for the strong initial inequality $u(x_0, Y) < v(x_0, Y)$ to imply the strong inequality $u(x, Y) < v(x, Y)$ in a certain set formed by projections of characteristics of equation (1) into the space x, Y (cf. also [8], Theorem 1). Papers [4]-[7] contain theorems concerning mixed inequalities between solutions $u(x, Y)$ and $v(x, Y)$ of equation (1). Certain generalizations of theorems from paper [9] concerning partial differential equations and inequalities to the case of partial differential equations and inequalities of the first order with a functional argument are given in papers [10] and [11] (cf. also [3]).

In the present paper we shall give theorems concerning mixed inequalities occurring between solutions of a partial differential equation of the first order with a functional argument. We shall quote theorems concerning the mutual situation of the solutions $u(x, y)$ and $v(x, y)$ of an almost linear equation with a retarded argument

$$\frac{\partial z(x,y)}{\partial x}+P(x,y)\frac{\partial z(x,y)}{\partial y}=R\big(x,y,z(x,y),z(x-\tau(x),y)\big). \tag{2}$$

The theorems contained in this paper are generalizations of theorems given in [4] concerning mixed inequalities between solutions of an almost linear partial differential equation of the first order.

We shall assume here that the solutions u and v of equation (2) are defined in the set $E\cup D$ and that

$$u(x,y)=v(x,y) \quad \text{for } (x,y)\in\tilde{E} \tag{3}$$

and

$$u(x,y)<v(x,y) \quad \text{for } (x,y)\in E-\tilde{E}, \tag{4}$$

where $\tilde{E}$ is a closed domain contained in E. We shall prove that in this case there exists a set $\tilde{D}\subset D$ formed by integral curves of an ordinary differential equation and such that

$$u(x,y)=v(x,y) \quad \text{for } (x,y)\in\tilde{D}$$

and

$$u(x,y)<v(x,y) \quad \text{for } (x,y)\in D-\tilde{D}.$$

We shall also consider the case where the solutions u and v of equation (2) satisfy the initial inequalities (3) and (4) and the strong inequality $u(x,y)<v(x,y)$ holds in D.

In Theorem 5 we shall consider the case where the solutions u and v of (2) are equal in $\tilde{E}$ and

$$u(x,y)<v(x,y) \quad \text{for } (x,y)\in E_1$$

and

$$u(x,y)>v(x,y) \quad \text{for } (x,y)\in E_2,$$

where $E_1, E_2\subset E$. We shall prove that there exists a set $\tilde{D}\subset D$ formed by integral curves of an ordinary differential equation and there exist sets $D_1, D_2\subset D$ such that

$$u(x,y)<v(x,y) \quad \text{for } (x,y)\in D_1,$$
$$u(x,y)=v(x,y) \quad \text{for } (x,y)\in\tilde{D},$$
$$u(x,y)>v(x,y) \quad \text{for } (x,y)\in D_2.$$

1. Assumptions and Lemmas. We make the following assumptions:

ASSUMPTION H.

1° *The function R of the variables (x,y,z,u) is continuous, satisfies the Lipschitz condition with respect to z and is strongly increasing with re-*

spect to u in the domain Ω of the space (x, y, z, u). The projection of Ω onto the plane (x, y) contains the domain Ω_0. The function P of the variables (x, y) is continuous and satisfies the Lipschitz condition with respect to y for $(x, y)\in\Omega_0$.

2° $$E = \{(x, y)\colon x\in\langle x_0-\tau_0, x_0\rangle\ y\in\langle y_0-b, y_0+b\rangle\}, \qquad \tau_0>0, b>0,$$
$$D = \{(x, y)\colon x\in\langle x_0, a), \ \alpha(x)\leqslant y\leqslant\beta(x)\},$$

where α and β are functions of the class C^1 in the interval $\langle x_0, a)$ and

(5) $$\alpha(x_0) = y_0-b, \quad \beta(x_0) = y_0+b, \quad \alpha'(x)\geqslant 0, \quad \beta'(x)\leqslant 0$$

for $x\in\langle x_0, a)$ and

(6) $$\alpha'(x) > P\big(x, \alpha(x)\big), \quad \beta'(x) < P\big(x, \beta(x)\big), \quad x\in\langle x_0, a).$$

Assume that $E\cup D\subset\Omega_0$.

3° *The function τ is continuous in the interval $\langle x_0, a)$ and $\inf\limits_{x\in\langle x_0,a)}[x-\tau(x)] = x_0-\tau_0$. There exists a constant $\delta>0$ such that $\tau(x)\geqslant\delta$ for $x\in\langle x_0, a)$.*

4° *The functions u and v are solutions of equation (2) defined in $D\cup E$. These solutions are of class C^1 in D and they fulfil the initial conditions*

(7) $$u(x, y) = \varphi(x, y), \quad v(x, y) = \psi(x, y) \quad \text{for } (x, y)\in E,$$

where the functions φ and ψ are continuous in E. Assume that $\big(x, y, u(x, y), u\big(x-\tau(x), y\big)\big)\in\Omega$ and

$$\big(x, y, v(x, y), v\big(x-\tau(x), y\big)\big)\in\Omega \quad \text{for } (x, y)\in D.$$

5° *The functions g and h are continuous in the interval $\langle x_0-\tau_0, x_0\rangle$ and $y_0-b\leqslant g(x)\leqslant h(x)\leqslant y_0+b$ for $x\in\langle x_0-\tau_0, x_0\rangle$,*

$$\tilde{E} = \{(x, y)\colon x\in\langle x_0-\tau_0, x_0\rangle, g(x)\leqslant y\leqslant h(x)\}.$$

6° *Let $I^* = \{x^*\colon x_0<x^*\leqslant a$ and $x-\tau(x)\leqslant x_0$ for $x\in\langle x_0, x^*)\}$. Denote by a_1 the upper bound of I^*. (It follows from assumption 3° that I^* is non-void and $a_1\geqslant\delta+x_0$.) Let*

$$c = \max\big[g(x_0), g\big(x_0-\tau(x_0)\big)\big], \quad d = \min\big[h(x_0), h\big(x_0-\tau(x_0)\big)\big].$$

Assume that

(8) $$c\leqslant d.$$

Let

$$K = \{(x, y)\colon x = x_0, \ y_0-b\leqslant y\leqslant y_0+b\},$$
$$\tilde{K} = \{(x, y)\colon x = x_0, \ c\leqslant y\leqslant d\}.$$

7° *Assume that $y = y(x)$ is a solution of the differential equation*

(9) $$\frac{dy}{dx} = P(x, y)$$

and $y(x_0) = \tilde{y}$, *where* $(x_0, \tilde{y}) \in \tilde{K}$. *Let* $\tilde{I}$ *be the biggest interval contained in* $\langle x_0, a_1)$ *such that* $g(x-\tau(x)) \leqslant y(x) \leqslant h(x-\tau(x))$ *for* $x \in \tilde{I}$. ($\tilde{I} = \langle x_0, \tilde{a}\rangle$, *where* $\tilde{a} < a_1$ *or* $\tilde{I} = \langle x_0, a_1)$. *In the first case the inequality* $g(x-\tau(x)) \leqslant y(x) \leqslant h(x-\tau(x))$ *is satisfied for* $x \in \langle x_0, \tilde{a}\rangle$ *and for each* $\varepsilon > 0$ *there exists an* $x_\varepsilon \in (\tilde{a}, \tilde{a}+\varepsilon)$ *such that* $y(x_\varepsilon) < g(x_\varepsilon - \tau(x_\varepsilon))$ *or* $y(x_\varepsilon) > h(x_\varepsilon - \tau(x_\varepsilon))$, *whereas in the second case the inequality* $g(x-\tau(x)) \leqslant y(x) \leqslant h(x-\tau(x))$ *is satisfied for* $x \in \langle x_0, a_1)$.) *We shall denote the curve* $y = y(x)$ *for* $x \in \tilde{I}$ *by* $\tilde{C}$. *Let* $\tilde{\Delta}$ *denote the plane set formed by all curves* $\tilde{C}$ *issuing from the segment* $\tilde{K}$, *and* $\Delta = \{(x, y)\colon x \in \langle x_0, a_1), \alpha(x) \leqslant y \leqslant \beta(x)\}$.

8° *There exists a finite sequence of interval* $I_0, I_1, \ldots, I_n$, *where* $I_0 = \langle x_0 - \tau_0, x_0\rangle$, $I_1 = \langle x_0, a_1\rangle$, $I_k = \langle a_{k-1}, a_k\rangle$ *for* $k = 2, 3, \ldots, n-1$, $I_n = \langle a_{n-1}, a)$, *satisfying the following condition: there exists for each* $k \in \{1, 2, \ldots, n\}$ *an* $l \in \{0, 1, \ldots, k-1\}$ *such that if* $x \in I_k$, *then* $x - \tau(x) \in I_l$.

Remark 1. If $|P(x, y)| < M$ for $(x, y) \in \Omega_0$, then the functions $\alpha(x) = y_0 - b + M(x - x_0)$, $\beta(x) = y_0 + b - M(x - x_0)$ satisfy condition 2° of Assumption H.

Remark 2. Assumption 8° is satisfied if the function $\eta(x) = x - \tau(x)$ is e.g. monotone by intervals. The function

$$\eta(x) = \begin{cases} x_0 + \dfrac{\delta}{a_1 - x_0}(x - a_1) & \text{for } x \in \langle x_0, a_1\rangle, \\[2ex] x_0 + (x - a_1)\sin\dfrac{1}{x - a_1} & \text{for } x > a_1, \end{cases}$$

where $\delta > 0$, $x_0 > 0$, $a_1 > x_0$, satisfies condition 3°, whereas condition 8° is not satisfied for $x = a_1$.

Adopt the following definitions:

1. The solutions u and v of equation (2) satisfy in D mixed inequalities of the first type if there exists a set

$$\tilde{D} = \{(x, y)\colon x_0 \leqslant x < \tilde{a}, \tilde{\alpha}(x) \leqslant y \leqslant \tilde{\beta}(x)\},$$

$\tilde{D} \subset D$, $\tilde{D} \neq D$, such that $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{D}$, and $u(x, y) < v(x, y)$ for $(x, y) \in D - \tilde{D}$.

2. The solutions u and v of equation (2) satisfy in D mixed inequalities of the second type if there exist non-empty sets

$$D_1 = \{(x, y)\colon x_0 \leqslant x < \tilde{a}_1,\ \alpha(x) \leqslant y < \tilde{\alpha}(x)\},$$
$$D_2 = \{(x, y)\colon x_0 \leqslant x < \tilde{a}_2,\ \tilde{\beta}(x) < y \leqslant \beta(x)\},$$
$$\tilde{D} = \{(x, y)\colon x_0 \leqslant x < \tilde{a},\ \tilde{\alpha}(x) \leqslant y \leqslant \tilde{\beta}(x)\},$$

$D_1, D_2, \tilde{D} \subset D$, such that $u(x, y) < v(x, y)$ for $(x, y) \in D_1$, $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{D}$ and $u(x, y) > v(x, y)$ for $(x, y) \in D_2$.

In the sequel we shall use the following lemmas:

LEMMA 1. *Assume that:*

1° *The function f of the variables $(x, y) \in G$ is continuous in G and satisfies the Lipschitz condition with respect to y.*

2° *The functions u and v of one variable are of class C^1 for $x \in \langle x_0, a_0)$ and $(x, u(x)) \in G$, $(x, v(x)) \in G$ for $x \in \langle x_0, a_0)$.*

3°
$$u(x_0) < v(x_0),$$
$$v'(x) = f(x, v(x)) \quad \text{for } x \in \langle x_0, a_0),$$
$$u'(x) \leqslant f(x, u(x)) \quad \text{for } x \in \langle x_0, a_0).$$

Under these assumptions the inequality $u(x) < v(x)$ is satisfied for $x \in \langle x_0, a_0)$.

LEMMA 2. *Assume that conditions 1° and 2° of Lemma 1 are satisfied and that besides*
$$u(x_0) \leqslant v(x_0),$$
$$v'(x) = f(x, v(x)) \quad \text{for } x \in \langle x_0, a_0),$$
$$u'(x) < f(x, u(x)) \quad \text{for } x \in (x_0, a_0).$$
Under these assumptions the inequality $u(x) < v(x)$ is satisfied for $x \in (x_0, a_0)$.

Both these lemmas follow in a simple way from theorems concerning ordinary differential equations and inequalities ([9], Chapter III, cf. also Lemma 1 in [2] and [1]).

LEMMA 3. *If conditions 1°–4° of Assumption* H *are satisfied and if*

$$\varphi(x, y) \leqslant \psi(x, y) \quad \text{for } (x, y) \in E \tag{10}$$

and

$$\varphi(x_0, y) < \psi(x_0, y) \quad \text{for } y \in (y_0 - b, y_0 + b), \tag{11}$$

then the inequality

$$u(x, y) < v(x, y) \tag{12}$$

is satisfied for $(x, y) \in \{(x, y)\colon x > x_0, (x, y) \in D\}$.

Proof. Let $I_1 = \{x^*\colon x_0 < x^* \leqslant a$ and $x - \tau(x) \leqslant x_0$ for $x \in \langle x_0, x^*)\}$. Denote by a_1 the upper bound of I_1. (It follows from condition 3° of Assumption H that I_1 is non-empty and $a_1 \geqslant \delta + x_0$.) Let
$$\Delta_1 = \{(x, y)\colon x \in (x_0, a_1),\ a(x) \leqslant y \leqslant \beta(x)\}.$$

I. In the first place we shall prove that inequality (12) is satisfied in Δ_1.

Let $y = y(x)$ be a solution of equation (9) satisfying the initial condition $y(x_0) = \tilde{y}$, where $\tilde{y} \in (y_0 - b, y_0 + b)$. Assume furthermore that

$(x, y(x)) \in \Delta_1$ for $x \in \tilde{I}_0$. $\tilde{I}_0 = \langle x_0, a_1)$ or there exists an $\tilde{a}_0 \in (x_0, a_1)$ such that $\tilde{I}_0 = \langle x_0, \tilde{a}_0 \rangle$. We shall prove that

$$(13) \qquad u(x) < v(x) \quad \text{for } x \in \tilde{I}_0,$$

where $u(x) = u(x, y(x))$, $v(x) = v(x, y(x))$, $x \in \tilde{I}_0$.

It follows from (10), (11) and from conditions 1°–4° of Assumption H that

$$(14) \qquad u(x_0) < v(x_0),$$

$$(15) \qquad u'(x) \leqslant R_1(x, u(x)), \quad v'(x) = R_1(x, v(x)) \quad \text{for } x \in \tilde{I}_0,$$

where $R_1(x, z) = R(x, y(x), z, \psi(x - \tau(x), y(x)))$.

From Lemma 1 we obtain $u(x) < v(x)$ for $x \in \operatorname{Int} \tilde{I}_0$.

If $\tilde{I}_0 = \langle x_0, a_1)$, then the proof of inequality (13) is completed. Le tus therefore consider the case where $\tilde{I}_0 = \langle x_0, \tilde{a}_0 \rangle$, $\tilde{a}_0 < a_1$. Since

$$(16) \qquad u(x) < v(x) \quad \text{for } x \in \langle x_0, \tilde{a}_0)$$

we have $u(\tilde{a}_0) \leqslant v(\tilde{a}_0)$. Suppose that $u(\tilde{a}_0) = v(\tilde{a}_0)$. Then it follows from condition 1° of Assumption H and from (15) and Theorem 9.6 of [9] (p. 27) that $u(x) \geqslant v(x)$ for $x \in (x_0, \tilde{a}_0)$, which contradicts (16). Therefore $u(\tilde{a}_0) < v(\tilde{a}_0)$ and the proof of (13) is completed.

It follows from (13) that inequality (12) is satisfied along an arbitrary integral curve of equation (9) situated in Δ_1 and issuing from the point $(x_0, \tilde{y})$, where $\tilde{y} \in (y_0 - b, y_0 + b)$.

To complete the proof of inequality (12) for $(x, y) \in \Delta_1$ it is sufficient to show that every point $\bar{P}(\bar{x}, \bar{y})$ of the set Δ_1 can be joined by means of an integral curve $y = y(x)$ of equation (9) with some point $(x_0, \tilde{y})$, where $\tilde{y} \in (y_0 - b, y_0 + b)$ and $(x, y(x)) \in \Delta_1$ for $x \in \langle x_0, \bar{x} \rangle$.

Suppose that there exists a point $(\bar{x}, \bar{y}) \in \Delta_1$ and a curve $y = \bar{y}(x)$, where $\bar{y}(x)$ satisfies equation (9) and is such that $\bar{y}(\bar{x}) = \bar{y}$, $\beta(x') = \bar{y}(x')$, where $x_0 \leqslant x' < \bar{x}$, and that for $x \in (x', \bar{x})$ the inequality $\bar{y}(x) < \beta(x)$ holds. (We proceed in a similar way in the case where the curve $y = \bar{y}(x)$ possesses a common point with the curve $y = a(x)$.)

Since

$$\frac{d\bar{y}(x)}{dx} = P(x, \bar{y}(x)) \quad \text{for } x \in \langle x', \bar{x} \rangle,$$

$$\frac{d\beta(x)}{dx} < P(x, \beta(x)) \quad \text{for } x \in \langle x', \bar{x} \rangle$$

and $\beta(x') = \bar{y}(x')$, it follows from Lemma 2 that $\beta(x) < \bar{y}(x)$ for $x \in (x', \bar{x})$, which contradicts the assumption that $\bar{y}(x) < \beta(x)$ for $x \in (x', \bar{x})$.

For $a = a_1$ the proof of Lemma 3 is completed.

II. Assume that $a_1 < a$. It is easy to prove that in this case $u(x, y) < v(x, y)$ for $x_0 < x \leqslant a_1$ and $(x, y) \in D$. Let

$$I_2 = \{x^*\colon\, a_1 < x^* \leqslant a \text{ and } x - \tau(x) \leqslant a_1 \text{ for } x \in \langle a_1, x^*)\}$$

and let us denote by a_2 the upper bound of I_2. The set I_2 is non-void and $a_2 \geqslant \delta + a_1$. Just as in I we can show that $u(x, y) < v(x, y)$ for

$$(x, y) \in \Delta_2 = \{(x, y)\colon\, a_1 \leqslant x < a_2,\ a(x) \leqslant y \leqslant \beta(x)\}.$$

In an analogous manner we define the sets $\Delta_3, \ldots, \Delta_n \subset D$ and show that $u(x, y) < v(x, y)$ for $(x, y) \in \Delta_i$ and $i = 3, \ldots, n$. It follows from condition 3° of Assumption H that there exists an index n such that $D = \bigcup_{i=1}^{n} \Delta_i$.

The proof of the lemma is finished.

2. Mixed inequalities of the first type.

THEOREM 1. *Assume that conditions* 1°–7° *of Assumption* H *are satisfied and that*

$$\varphi(x, y) = \psi(x, y) \quad \textit{for } (x, y) \in \tilde{E}, \tag{17}$$

$$\varphi(x, y) < \psi(x, y) \quad \textit{for } (x, y) \in E - \tilde{E}. \tag{18}$$

Under these assumptions

$$u(x, y) = v(x, y) \quad \textit{for } (x, y) \in \tilde{\Delta}, \tag{19}$$

$$u(x, y) < v(x, y) \quad \textit{for } (x, y) \in \Delta - \tilde{\Delta}. \tag{20}$$

Proof. I. We shall demonstrate that $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{\Delta}$. Let $y = \tilde{y}(x)$ be a solution of equation (9) and $\tilde{y}(x_0) = \tilde{y}$, where $(x_0, \tilde{y}) \in \tilde{K}$. Suppose that the curve $y = \tilde{y}(x)$ is situated in $\tilde{\Delta}$ for $x \in \tilde{I} \subset \langle x_0, a_1)$, where $\tilde{I} = \langle x_0, \tilde{a}\rangle$, $\tilde{a} < a_1$ or $\tilde{I} = \langle x_0, a_1)$. Thus we obtain

$$g\big(x - \tau(x)\big) \leqslant \tilde{y}(x) \leqslant h\big(x - \tau(x)\big) \quad \text{for } x \in \tilde{I}. \tag{21}$$

It is easy to verify that the functions $u(x) = u\big(x, \tilde{y}(x)\big)$, $v(x) = v\big(x, \tilde{y}(x)\big)$; $x \in \tilde{I}$, satisfy respectively the differential equations

$$\frac{dz}{dx} = R_1(x, z), \qquad \frac{dz}{dx} = R_2(x, z),$$

where

$$R_1(x, z) = R\Big(x, \tilde{y}(x), z, \varphi\big(x - \tau(x), \tilde{y}(x)\big)\Big),$$
$$R_2(x, z) = R\Big(x, \tilde{y}(x), z, \psi\big(x - \tau(x), \tilde{y}(x)\big)\Big).$$

It follows from (7) and (17) that $u(x_0) = v(x_0)$. From (21) we obtain $\big(x - \tau(x), \tilde{y}(x)\big) \in \tilde{E}$ for $x \in \tilde{I}$ and hence and also by (17) we come to the conclusion that $R_1(x, z) = R_2(x, z)$ for $x \in \tilde{I}$. By condition 1° of Assumption H we get $u(x) = v(x)$ for $x \in \tilde{I}$. The integrals u and v of equation (2) are therefore equal along an arbitrary curve $\tilde{C}$ issuing from the segment $\tilde{K}$.

The proof of statement (19) is completed.

II. We shall now prove inequality (20).

(a) Assume that $y = y(x)$ is a solution of equation (9) issuing from the segment $\tilde{K}$ and that $(x, y(x)) \in \tilde{\Delta}$ for $x \in \langle x_0, \tilde{a}\rangle$ and $(x, y(x)) \in \Delta - \tilde{\Delta}$ for $x \in I$, where $I = (\tilde{a}, a_0\rangle$, $a_0 < a_1$ or $I = (\tilde{a}, a_1)$. We shall prove that $u(x, y) < v(x, y)$ along the curve $y = y(x)$ for $x \in \operatorname{Int} I$.

1° Assume that there exists a constant $\tilde{\delta}$, $\tilde{\delta} > 0$, such that $(x - \tau(x), y(x)) \in E - \tilde{E}$ for $x \in (\tilde{a}, \tilde{a} + \tilde{\delta})$. (This means that $y(x) < g(x - \tau(x))$ or $y(x) > h(x - \tau(x))$ for $x \in (\tilde{a}, \tilde{a} + \tilde{\delta})$.) It follows from (18) that $\varphi(x - \tau(x), y(x)) < \psi(x - \tau(x), y(x))$ for $x \in (\tilde{a}, \tilde{a} + \tilde{\delta})$, and hence, using also condition 1° from Assumption H, we conclude that the functions $u(x) = u(x, y(x))$ and $v(x) = v(x, y(x))$ satisfy the conditions

$$u(\tilde{a}) = v(\tilde{a}), \tag{22}$$

$$\begin{gathered} \frac{dv(x)}{dx} = \tilde{R}(x, v(x)), \quad x \in \langle \tilde{a}, \tilde{a} + \tilde{\delta}), \\ \frac{du(x)}{dx} < \tilde{R}(x, u(x)), \quad x \in (\tilde{a}, \tilde{a} + \tilde{\delta}), \end{gathered} \tag{23}$$

where $\tilde{R}(x, z) = R(x, y(x), z, \psi(x - \tau(x), y(x)))$. It follows from Lemma 2 and from condition 1° of Assumption H and also from (22) and (23) that $u(x) < v(x)$ for $x \in (\tilde{a}, \tilde{a} + \tilde{\delta})$.

If $(\tilde{a}, \tilde{a} + \tilde{\delta}) = \operatorname{Int} I$, then $u(x, y) < v(x, y)$ along the curve $y = y(x)$ for $x \in \operatorname{Int} I$.

Assume that the set $\operatorname{Int} I - (\tilde{a}, \tilde{a} + \tilde{\delta})$ is non-empty. We shall prove that $u(x) < v(x)$ for $x \in \{\operatorname{Int} I - (\tilde{a}, \tilde{a} + \tilde{\delta})\}$.

Let $\operatorname{Int} I - (\tilde{a}, \tilde{a} + \tilde{\delta}) = \langle \tilde{a} + \tilde{\delta}, a')$ and $\tilde{a} < x' < \tilde{a} + \tilde{\delta}$. We then have

$$u(x') < v(x'), \tag{24}$$

$$\frac{dv(x)}{dx} = \tilde{R}(x, v(x)), \qquad \frac{du(x)}{dx} \leqslant \tilde{R}(x, u(x)) \quad \text{for } x \in \langle x', a'). \tag{25}$$

It follows from Lemma 1 and from condition 1° of Assumption H and also from conditions (24) and (25) that $u(x) < v(x)$ for $x \in \langle x', a')$, which completes the proof of the inequality $u(x) < v(x)$ for $x \in \operatorname{Int} I$.

2° Assume that in an arbitrary right-hand side neighbourhood of the point $\tilde{a}$ there exist numbers x such that $(x - \tau(x), y(x)) \in E - \tilde{E}$ and also numbers x such that $(x - \tau(x), y(x)) \in \tilde{E}$. The functions $u(x) = u(x, y(x))$

and $v(x) = v(x, y(x))$ then satisfy the following conditions:

$$u(\tilde{a}) = v(\tilde{a}),$$

$$\frac{dv(x)}{dx} = \tilde{R}(x, v(x)), \quad \frac{du(x)}{dx} \leqslant \tilde{R}(x, u(x)) \quad \text{for } x \in I.$$

It follows from Theorem 11.1 from [9] (p. 35) that $u(x) \leqslant v(x)$ for $x \in I$. We shall now prove that for $x \in \operatorname{Int} I$ the strong inequality $u(x) < v(x)$ holds.

Suppose that there exists an $\tilde{x}$, $\tilde{x} \in \operatorname{Int} I$, such that

$$(26) \qquad u(\tilde{x}) = v(\tilde{x}).$$

Since in any right-hand side neighbourhood of the point $\tilde{a}$ there exist such numbers x that $(x - \tau(x), y(x)) \in E - \tilde{E}$ and $y(x), \tau(x)$ are continuous functions, there exists an interval $(\bar{a}, \bar{\bar{a}}) \subset \operatorname{Int} I$ such that $\bar{a} < \bar{\bar{a}} < \tilde{x}$ and $(x - \tau(x), y(x)) \in E - \tilde{E}$ for $x \in (\bar{a}, \bar{\bar{a}})$. Thus we have

$$u(\bar{a}) \leqslant v(\bar{a}),$$

$$\frac{dv(x)}{dx} = \tilde{R}(x, v(x)), \quad x \in \langle \bar{a}, \bar{\bar{a}}), \quad \frac{du(x)}{dx} < \tilde{R}(x, u(x)), \quad x \in (\bar{a}, \bar{\bar{a}}).$$

From Lemma 2 we obtain the inequality $u(x) < v(x)$ for $x \in (\bar{a}, \bar{\bar{a}})$. Let $\bar{x} \in (\bar{a}, \bar{\bar{a}})$. Then we have

$$u(\bar{x}) < v(\bar{x}),$$

$$\frac{dv(x)}{dx} = \tilde{R}(x, v(x)), \quad \frac{du(x)}{dx} \leqslant R(x, u(x)) \quad \text{for } x \in \langle \bar{x}, a').$$

It follows from Lemma 1 that $u(x) < v(x)$ for $x \in \langle \bar{x}, a')$. Since $\tilde{x} \in \langle \bar{x}, a')$, we have in particular $u(\tilde{x}) < v(\tilde{x})$, which contradicts condition (26). Therefore $u(x, y) < v(x, y)$ holds along the curve $y = y(x)$ for $x \in \operatorname{Int} I$.

We shall now prove that inequality (20) holds along the solutions of equation (9) issuing from the set $K - \tilde{K}$.

Let

$$L_1 = \{(x, y)\colon (x, y) \in K - \tilde{K},\ u(x_0, y) < v(x_0, y)\},$$

$$L_2 = \{(x, y)\colon (x, y) \in K - \tilde{K},\ u(x_0, y) = v(x_0, y)\}.$$

(b) We shall prove that $u(x, y) < v(x, y)$ holds along the curve $y = y_1(x)$, where $y_1(x)$ is a solution of equation (9) and $y_1(x_0) = y_1$, $(x_0, y_1) \in L_1$. Suppose that $(x, y_1(x)) \in \Delta - \tilde{\Delta}$ for $x \in \langle x_0, \tilde{a}_1)$, $(\tilde{a}_1, y_1(\tilde{a}_1)) \in \operatorname{Fr}(\Delta - \tilde{\Delta})$, $\tilde{a}_1 \leqslant a_1$. The functions $u(x) = u(x, y_1(x))$, $v(x) = v(x, y_1(x))$ satisfy the conditions:

$$u(x_0) < v(x_0),$$

$$\frac{dv(x)}{dx} = \tilde{R}_1(x, v(x)), \quad \frac{du(x)}{dx} \leqslant \tilde{R}_1(x, u(x)) \quad \text{for } x \in \langle x_0, \tilde{a}_1),$$

where $\tilde{R}_1(x, z) = R\big(x, y_1(x), z, \psi(x-\tau(x), y_1(x))\big)$. It follows from Lemma 1 that $u(x) < v(x)$ for $x\in\langle x_0, \tilde{a}_1)$. It follows hencefrom that inequality (20) holds along any integral curve $y = y_1(x)$ of equation (9), where $\big(x_0, y_1(x_0)\big) \in L_1$ and $x\in\langle x_0, \tilde{a}_1)$.

(c) Suppose that $y = y_2(x)$ is a solution of equation (9) satisfying the initial condition $y_2(x_0) = \dot{y}_2$, where $(x_0, y_2)\in L_2$. Assume that $\big(x, y_2(x)\big) \in \Delta - \tilde{\Delta}$ for $x\in\langle x_0, \tilde{a}_2)$, where $\tilde{a}_2 \leqslant a_1$ and $\big(\tilde{a}_2, y_2(\tilde{a}_2)\big)\in \mathrm{Fr}(\Delta - \tilde{\Delta})$. It follows from (18) and from the definition of the set $\tilde{\Delta}$ that there exists an interval $\langle x_0, x')$, $x_0 < x' \leqslant \tilde{a}_2$, such that $\big(x-\tau(x), y_2(x)\big)\in E - \tilde{E}$ for $x\in\langle x_0, x')$. Therefore $\varphi\big(x-\tau(x), y_2(x)\big) < \psi\big(x-\tau(x), y_2(x)\big)$ for $x\in(x_0, x')$. The functions $u(x) = u\big(x, y_2(x)\big)$ and $v(x) = v\big(x, y_2(x)\big)$ satisfy the conditions

$$u(x_0) = \dot{v}(x_0),$$

$$\frac{dv(x)}{dx} = \tilde{R}_2\big(x, v(x)\big), \quad x\in\langle x_0, x'), \qquad \frac{du(x)}{dx} < \tilde{R}_2\big(x, u(x)\big), \quad x\in(x_0, x'),$$

where $\tilde{R}_2(x, z) = R\big(x, y_2(x), z, \psi(x-\tau(x), y_2(x))\big)$. It follows from Lemma 2 that $u(x) < v(x)$ for $x\in(x_0, x')$. If $x' = \tilde{a}_2$, then $u(x, y) < v(x, y)$ along the curve $y = y_2(x)$ for $x\in(x_0, \tilde{a}_2)$.

Assume that $x' < a_2$. The proof of the inequality $u(x) < v(x)$ for $x\in\langle x', \tilde{a}_2)$ is analogous to the proof of the similar inequality given in 1(a).

(d) Each point $\bar{P}(\bar{x}, \bar{y})$, $\bar{x} > x_0$, belonging to Δ can be connected by the integral curve $y = y(x)$ of equation (9) with some point $\big(x_0, y(x_0)\big)\in K$ and $\big(x, y(x)\big)\in\Delta$ for $x\in\langle x_0, \bar{x}\rangle$. The proof of this property of the set Δ is analogous to the proof of a similar property of the set Δ_1 in Lemma 3.

(e) Let $Z = \{(x, y)\colon (x, y)\in\Delta - \tilde{\Delta},\ y = a(x) \text{ or } y = \beta(x)\}$. We shall now prove that $u(x, y) < v(x, y)$ for $(x, y)\in Z$.

It follows from I and II(a)–(d) that $u(x, y) \leqslant v(x, y)$ for $(x, y)\in Z$. Suppose that there exists a point $(x^*, y^*)\in Z$ such that

$$u(x^*, y^*) = v(x^*, y^*). \tag{27}$$

It follows from I and II(a)–(d) that there exists a set $D^* = E^*\cup\Delta^*$, where

$$E^* = \{(x, y)\colon \bar{x}-\bar{\tau} \leqslant x \leqslant \bar{x},\ \bar{y}-\bar{b} \leqslant y \leqslant \bar{y}+\bar{b}\},$$
$$\Delta^* = \{(x, y)\colon \bar{x} \leqslant x < \bar{a},\ \bar{a}(x) \leqslant y \leqslant \bar{\beta}(x)\},$$

such that

1) $\quad E^*\cup\Delta^* \subset E\cup D, \quad (x^*, y^*)\in\Delta^*, \quad x^* > \bar{x},$

2) $\quad x-\tau(x)\in E^* \quad \text{for } x\in\langle\bar{x}, \bar{a}),$

3) $\quad u(x, y) \leqslant v(x, y) \quad \text{for } (x, y)\in E^*$

$\quad\quad$ and $\quad u(\bar{x}, y) < v(\bar{x}, y) \quad \text{for } y\in(\bar{y}-\bar{b}, \bar{y}+\bar{b}).$

It follows from Lemma 3 that $u(x, y) < v(x, y)$ for $(x, y) \in \Delta^*$ and $x > \bar{x}$. Particularily $u(x^*, y^*) < v(x^*, y^*)$, which contradicts assumption (27).

The proof of Theorem 1 is completed.

If we accept additional assumptions for the functions g and h, we shall be able to obtain the set $\tilde{\Delta}$ in a simple way.

EXAMPLES. 1. If

$$D_-\big(h(x-\tau(x))\big) \geqslant P\big(x, h(x-\tau(x))\big), \qquad x \in \langle x_0, a_1),$$
$$D_-\big(g(x-\tau(x))\big) \leqslant P\big(x, g(x-\tau(x))\big), \qquad x \in \langle x_0, a_1),$$

then $\tilde{\Delta}$ is the set formed by integral curves of equation (9) issuing from the segment $\tilde{K}$ for $x \in \langle x_0, a_1)$. ($D_-f(x)$ denotes the left-hand lower Dini derivative of the function f at the point x.)

2. If

(28) $$D_-\big(g(x-\tau(x))\big) \geqslant P\big(x, g(x-\tau(x))\big), \qquad x \in \langle x_0, a_1),$$

(29) $$D_-\big(h(x-\tau(x))\big) \leqslant P\big(x, h(x-\tau(x))\big), \qquad x \in \langle x_0, a_1),$$

and

$$\max\big[g(x_0-\tau(x_0)), g(x_0)\big] = g\big(x_0-\tau(x_0)\big),$$
$$\min\big[h(x_0-\tau(x_0)), h(x_0)\big] = h\big(x_0-\tau(x_0)\big),$$

then

$$\tilde{\Delta} = \big\{(x, y):\ x \in \langle x_0, a_1),\ g\big(x-\tau(x)\big) \leqslant y \leqslant h\big(x-\tau(x)\big)\big\}.$$

3. Assume that inequalities (28) and (29) are satisfied and that

$$\max\big[g(x_0-\tau(x_0)), g(x_0)\big] = g(x_0),$$
$$\min\big[h(x_0-\tau(x_0)), h(x_0)\big] = h(x_0).$$

Denote by $y_1(x)$ and $y_2(x)$ solutions of equation (9) satisfying the initial conditions $y_1(x_0) = g(x_0)$, $y_2(x_0) = h(x_0)$. Let

$$\tilde{I}_1 = \big\{x \in \langle x_0, a_1):\ y_1(x) \geqslant g\big(x-\tau(x)\big)\big\},$$
$$\tilde{I}_2 = \big\{x \in \langle x_0, a_1):\ y_2(x) \leqslant h\big(x-\tau(x)\big)\big\}.$$

Denote by $\tilde{g}$ and $\tilde{h}$ the functions

$$\tilde{g}(x) = \begin{cases} y_1(x) & \text{for } x \in \tilde{I}_1, \\ g\big(x-\tau(x)\big) & \text{for } x \in \langle x_0, a_1) - \tilde{I}_1, \end{cases}$$

$$\tilde{h}(x) = \begin{cases} y_2(x) & \text{for } x \in \tilde{I}_2, \\ h\big(x-\tau(x)\big) & \text{for } x \in \langle x_0, a_1) - \tilde{I}_2. \end{cases}$$

Under these assumptions $\tilde{\Delta} = \{(x, y):\ x_0 \leqslant x < a_1,\ \tilde{g}(x) \leqslant y \leqslant \tilde{h}(x)\}$.

4. Assume that conditions (28) and (29) are satisfied and

$$\max\left[g\left(x_0-\tau(x_0)\right), g(x_0)\right] = g\left(x_0-\tau(x_0)\right), \tag{30}$$

$$\min\left[h\left(x_0-\tau(x_0)\right), h(x_0)\right] = h(x_0). \tag{31}$$

Denote by $y = y_2(x)$ the solution of equation (9) satisfying the initial condition $y_2(x_0) = h(x_0)$ and assume that $y_2(x) > g\left(x-\tau(x)\right)$ for $x \in \langle x_0, a_1)$. Under these assumptions

$$\tilde{\Delta} = \left\{(x, y):\ x_0 \leqslant x < a_1,\ g\left(x-\tau(x)\right) \leqslant y \leqslant \tilde{h}(x)\right\},$$

where $\tilde{h}(x) = \min\left[y_2(x), h\left(x-\tau(x)\right)\right]$.

5. Assume that conditions (28)–(31) hold. Denote by $y = y_2(x)$ the solution of equation (9) satisfying the initial condition $y_2(x_0) = h(x_0)$ and assume that $y_2(x) > g\left(x-\tau(x)\right)$ for $x \in \langle x_0, \bar{x}\rangle$, $x_0 < \bar{x} < a_1$ and $y_2(\bar{x}) = g\left(\bar{x}-\tau(\bar{x})\right)$. Under these assumptions

$$\tilde{\Delta} = \left\{(x, y):\ x_0 \leqslant x \leqslant \bar{x},\ g\left(x-\tau(x)\right) \leqslant y \leqslant \tilde{h}(x)\right\},$$

where $\tilde{h}(x) = \min\left[y_2(x), h\left(x-\tau(x)\right)\right]$.

6. As in examples 4 and 5, the set $\tilde{\Delta}$ can be determined in the case where conditions (28), (29) and also the conditions

$$\max\left[g\left(x_0-\tau(x_0)\right), g(x_0)\right] = g(x_0),$$

$$\min\left[h\left(x_0-\tau(x_0)\right), h(x_0)\right] = h\left(x_0-\tau(x_0)\right)$$

are satisfied.

The proof of the construction of the sets $\tilde{\Delta}$ in examples 1–6 is quite simple. It is based on the fact that each point $\bar{P}(\bar{x}, \bar{y}) \in \tilde{\Delta}$ can be joined by means of an integral curve $y = y(x)$ of equation (9) with some point $\tilde{Q}(x_0, \tilde{y}) \in \tilde{K}$ and $\left(x, y(x)\right) \in \tilde{\Delta}$ for $x \in \langle x_0, \bar{x}\rangle$.

THEOREM 2. *Assume that conditions* 1°–6° *of Assumption* H — *with the exception of inequality* (8) — *are satisfied. Assume now that* $c > d$ *and that the initial functions* φ *and* ψ *satisfy the conditions*

$$\varphi(x, y) = \psi(x, y) \quad \textit{for } (x, y) \in \tilde{E}, \tag{32}$$

$$\varphi(x, y) < \psi(x, y) \quad \textit{for } (x, y) \in E-\tilde{E}. \tag{33}$$

Under these assumptions the inequality

$$u(x, y) < v(x, y) \tag{34}$$

is fulfilled for $(x, y) \in \Delta - K = \{(x, y):\ x_0 < x < a_1,\ a(x) \leqslant y \leqslant \beta(x)\}$.

Proof. Denote by L_1 and L_2 the sets:

$$L_1 = \{(x, y):\ (x, y) \in K,\ u(x_0, y) < v(x_0, y)\},$$

$$L_2 = \{(x, y):\ (x, y) \in K,\ u(u_0, y) = v(x_0, y)\}.$$

Assume that $(x_0, y_1) \in L_1$ and that $y = y_1(x)$ is the solution of equation (9) satisfying the initial condition $y_1(x_0) = y_1$. Assume also that $(x, y_1(x)) \in \Delta$ for $x \in \tilde{I}_1$, where $\tilde{I}_1 = \langle x_0, \tilde{a}_1)$, $\tilde{a}_1 \leqslant a_1$ and $(a_1, y_1(a_1)) \in \operatorname{Fr} \Delta$. Inequality (34) is satisfied along the curve $y = y_1(x)$ for $x \in \tilde{I}_1$, which follows from the conditions

$$u(x_0) < v(x_0),$$

$$\frac{du(x)}{dx} \leqslant \tilde{R}_1(x, u(x)), \quad x \in \tilde{I}_1, \quad \frac{dv(x)}{dx} = \tilde{R}_1(x, v(x)), \quad x \in \tilde{I}_1,$$

where $u(x) = u(x, y_1(x))$, $v(x) = v(x, y_1(x))$, $\tilde{R}_1(x, z) = R(x, y_1(x), z, \psi(x-\tau(x), y_1(x)))$ and also from Lemma 1.

Assume that $(x_0, y_2) \in L_2$ and that $y = y_2(x)$ is the solution of equation (9) satisfying the initial condition $y_2(x_0) = y_2$. Assume also that $(x, y_2(x)) \in \Delta$ for $x \in \tilde{I}_2$, where $\tilde{I}_2 = \langle x_0, \tilde{a}_2)$, $\tilde{a}_2 \leqslant a_1$ and $(\tilde{a}_2, y_2(\tilde{a}_2)) \in \operatorname{Fr} \Delta$. Inequality (34) is satisfied along the curve $y = y_2(x)$ for $x \in \tilde{I}_2$, which can be proved with the help of Lemma 2 (cf. the proof of Theorem 1, II(c)).

Since $K = L_1 \cup L_2$ and every point $(\bar{x}, \bar{y}) \in \Delta$ can be joined by means of the integral curve $y = y(x)$ of equation (9) with some point of the segment K and $(x, y(x)) \in \Delta$ for $x \in \langle x_0, \bar{x} \rangle$, it follows from what was said above that $u(x, y) < v(x, y)$ for $(x, y) \in \{(x, y): x_0 < x < a_1, \alpha(x) < y < \beta(x)\}$.

Inequality (34) for $(x, y) \in Z = \{(x, y): x_0 < x < a_1, y = \alpha(x) \text{ or } y = \beta(x)\}$ follows in a simple way from Lemma 3 (cf. the proof of Theorem 1, II(e)).

Remark 3. If $a_1 < a$, then it is easy to prove by Lemma 3 that inequality (34) is satisfied for $(x, y) \in \{(x, y): x_0 < x \leqslant a_1, \alpha(x) \leqslant y \leqslant \beta(x)\}$.

Theorems 1 and 2 concerned the mutual situation of solutions of equation (2) in that part of the set D, where $x \in \langle x_0, a_1)$. From these theorems and from Lemma 3 we shall obtain Theorems 3 and 4 concerning mixed inequalities between solutions of equation (2) in the entire set D.

Suppose that Assumption H holds and that the initial functions φ and ψ satisfy conditions (17) and (18). Denote by D_k $(k = 1, 2, \ldots, n)$ the sets

$$D_k = \{(x, y): x \in I_k, \alpha(x) \leqslant y \leqslant \beta(x)\},$$

where I_k is the interval defined in Assumption 8°. We shall now define a sequence of sets

$$\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_n \tag{35}$$

in the following way:

Consider the differential equation (2) in the set $E\cup D_1$. It follows from Theorem 1 (cf. also Remark 3) that there exists a set $\tilde{D}_1 \subset D_1$ such that

$$u(x,y) = v(x,y) \quad \text{for } (x,y)\in\tilde{D}_1,$$
$$u(x,y) < v(x,y) \quad \text{for } (x,y)\in D_1-\tilde{D}_1.$$

It follows from Assumption H that there exist continuous functions g_1 and h_1 such that

$$\tilde{D}_1 = \{(x,y)\colon x_0 \leqslant x \leqslant \tilde{a}_1,\ g_1(x)\leqslant y\leqslant h_1(x)\},$$

where $\tilde{a}_1 \leqslant a_1$.

Suppose now that the sets $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_k$ have already been constructed. We define $\tilde{D}_{k+1}$ as follows:

Consider the differential equation (2) in the set $E\cup D_1\cup D_2\cup\ldots\cup D_{k+1}$ and take $E\cup D_1\cup D_2\cup\ldots\cup D_k$ as the initial set and

$$\varphi_k(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y)\in E,\\ u(x,y) & \text{for } (x,y)\in D_1\cup D_2\cup\ldots\cup D_k,\end{cases}$$

$$\psi_k(x,y) = \begin{cases} \psi(x,y) & \text{for } (x,y)\in E,\\ v(x,y) & \text{for } (x,y)\in D_1\cup D_2\cup\ldots\cup D_k\end{cases}$$

as the initial functions. It follows from assumptions 2° and 8° that there exists an $l\in\{0,1,\ldots,k\}$ such that if $(x,y)\in D_{k+1}$, then $\big(x-\tau(x),y\big)\in D_l$. The set $\tilde{D}_l\subset D_l$ can have the form

$$\tilde{D}_l = \{(x,y)\colon x\in\tilde{I}_l,\ g_l(x)\leqslant y\leqslant h_l(x)\},$$

where $\tilde{I}_l\subset I_l$ and where g_l and h_l are continuous functions and $\alpha(x)\leqslant g_l(x)\leqslant h_l(x)\leqslant\beta(x)$ for $x\in\tilde{I}_l$.

1° If $\varphi_k(a_k,y)<\psi_k(a_k,y)$ for $y\in\big(\alpha(a^k),\ \beta(a^k)\big)$, then we easily find from Lemma 3 that $u(x,y)<v(x,y)$ for $(x,y)\in\{(x,y)\in D_{k+1}\colon a_k<x\leqslant a_{k+1}\}$. In this case $\tilde{D}_{k+1}$ is an empty set.

2° If $\max\big[g_l\big(a_k-\tau(a_k)\big),\ g_l(a_k)\big] > \min\big[h_l\big(a_k-\tau(a_k)\big),\ h_l(a_k)\big]$, then we easily find from assumption 8° and Theorem 2 that $u(x,y)<v(x,y)$ for $(x,y)\in\{(x,y)\in D_{k+1}\colon a_k<x\leqslant a_{k+1}\}$. In this case $\tilde{D}_{k+1}$ is an empty set.

3° If $\max\big[g_l\big(a_k-\tau(a_k)\big),\ g_l(a_k)\big] \leqslant \min\big[h_l\big(a_k-\tau(a_k)\big),\ h_l(a_k)\big]$, then Theorem 1 implies the existence of the set $\tilde{D}_{k+1}\subset D_{k+1}$ formed by integral curves of equation (9) such that

$$u(x,y) = v(x,y) \quad \text{for } (x,y)\in\tilde{D}_{k+1},$$
$$u(x,y) < v(x,y) \quad \text{for } (x,y)\in D_{k+1}-\tilde{D}_{k+1}.$$

$\tilde{D}_{k+1}$ can be presented in the form

$$\tilde{D}_{k+1} = \{(x,y)\colon x\in\tilde{I}_{k+1},\ g_{k+1}(x)\leqslant y\leqslant h_{k+1}(x)\},$$

where $\tilde{I}_{k+1} \subset I_{k+1}$ and g_{k+1}, h_{k+1} are continuous functions and $a(x) \leqslant g_{k+1}(x) \leqslant h_{k+1}(x) \leqslant \beta(x)$ for $x \in \tilde{I}_{k+1}$. We therefore have the following

THEOREM 3. *Assume that Assumption* H *holds and that the initial functions* φ *and* ψ *satisfy the conditions*

$$\varphi(x, y) = \psi(x, y) \quad \text{for } (x, y) \in \tilde{E},$$

$$\varphi(x, y) < \psi(x, y) \quad \text{for } (x, y) \in E - \tilde{E}.$$

Let $\tilde{D} = \bigcup_{i=1}^{n} \tilde{D}_i$. *Under these assumptions*

$$u(x, y) = v(x, y) \quad \text{for } (x, y) \in \tilde{D},$$

$$u(x, y) < v(x, y) \quad \text{for } (x, y) \in D - \tilde{D}.$$

We shall now prove the following

THEOREM 4. *Assume that conditions* 1°–6° — *with the exception of inequality* (8) — *hold. Assume that* $c > d$ *and that the initial functions* φ *and* ψ *satisfy the conditions*

$$\varphi(x, y) = \varphi(x, y) \quad \text{for } (x, y) \in \tilde{E}, \tag{36}$$

$$\varphi(x, y) < \psi(x, y) \quad \text{for } (x, y) \in E - \tilde{E}. \tag{37}$$

Under these assumptions the inequality

$$u(x, y) < v(x, y) \tag{38}$$

is satisfied for $(x, y) \in D - K$.

Proof. We define the sequence of numbers $a_1, a_2, \ldots, a_n$, where $x_0 < a_1 < a_2 < \ldots < a_n = a$, as follows:

a_1 is the constant defined in assumption 6°.

Assuming that the numbers $a_1, a_2, \ldots, a_k$ have already been defined, we define a_{k+1} as follows: let

$$I_k^* = \{x^*:\ a_k < x^* \leqslant a,\ x - \tau(x) \leqslant a_k \quad \text{for } x \in \langle a_k, x^*)\}.$$

We denote by a_{k+1} the upper bound of I_k^*. It follows from assumption 3° that I_k^* is non-void and that $a_{k+1} \geqslant a_k + \delta$.

Let $I_1 = \langle x_0, a_1\rangle$, $I_k = \langle a_{k-1}, a_k\rangle$ for $k = 2, 3, \ldots, n-1$, and $I_n = \langle a_{n-1}, a)$. By D_k $(k = 1, 2, \ldots, n)$ we denote the sets

$$D_k = \{(x, y):\ x \in I_k,\ a(x) \leqslant y \leqslant \beta(x)\}.$$

We shall now prove that inequality (38) is satisfied in each of the sets $D_1 - K, D_2, \ldots, D_n$.

It follows from Theorem 2 and Lemma 3 that $u(x, y) < v(x, y)$ for $(x, y) \in D_1 - K$.

Assume that (38) holds in the sets $D_1-K, D_2, \ldots, D_k$. We shall show that the same inequality is satisfied also in D_{k+1}. Consider the differential equation (2) in $E\cup D_1\cup D_2\cup \ldots \cup D_{k+1}$ and take $E\cup D_1\cup D_2\cup \ldots \cup D_k$ as the initial set and

$$\varphi_k(x, y) = \begin{cases}\varphi(x, y) & \text{for } (x, y)\in E,\\ u(x, y) & \text{for } (x, y)\in D_1\cup D_2\cup \ldots \cup D_k,\end{cases}$$

$$\psi_k(x, y) = \begin{cases}\psi(x, y) & \text{for } (x, y)\in E,\\ v(x, y) & \text{for } (x, y)\in D_1\cup D_2\cup \ldots \cup D_k\end{cases}$$

as the initial functions. Then we have

$$\varphi_k(x, y) \leqslant \psi_k(x, y) \quad \text{for } (x, y)\in E\cup D_1\cup D_2\cup \ldots \cup D_k,$$

$$\varphi_k(a_k, y) < \psi_k(a_k, y) \quad \text{for } y\in\big(\alpha(a_k), \beta(a_k)\big).$$

Thus we infer from Lemma 3 that $u(x, y) < v(x, y)$ for $(x, y)\in D_{k+1}$. Since $D-K = D_1-K\cup \bigcup_{k=2}^{n} D_k$, the proof of Theorem 4 is finished.

3. Mixed inequalities of the second type.

THEOREM 5. *Assume that:*

1° *Conditions* 1°–6° *of Assumption* H *hold.*

2° *The initial functions* φ *and* ψ *satisfy the following conditions:*

$$\varphi(x, y) < \psi(x, y) \quad \text{for } (x, y)\in E_1, \tag{39}$$

$$\varphi(x, y) = \psi(x, y) \quad \text{for } (x, y)\in \tilde{E}, \tag{40}$$

$$\varphi(x, y) > \psi(x, y) \quad \text{for } (x, y)\in E_2, \tag{41}$$

where

$$E_1 = \{(x, y):\ x\in\langle x_0-\tau_0, x_0\rangle,\ y_0-b\leqslant y < g(x)\},$$

$$E_2 = \{(x, y):\ x\in\langle x_0-\tau_0, x_0\rangle,\ h(x) < y\leqslant y_0+b\}.$$

2° *The functions* $\tilde{h}(x) = h\big(x-\tau(x)\big)$, $\tilde{g}(x) = g\big(x-\tau(x)\big)$, $x\in\langle x_0, a_1)$, *satisfy the differential inequalities*

$$D_-\tilde{h}(x) \geqslant P\big(x, \tilde{h}(x)\big), \tag{42}$$

$$D_-\tilde{g}(x) \leqslant P\big(x, \tilde{g}(x)\big) \tag{43}$$

for $x\in\langle x_0, a_1)$.

Under these assumptions the following assertions hold:

$$u(x, y) = v(x, y) \quad \text{for } (x, y)\in \tilde{\Delta}, \tag{44}$$

$$u(x, y) < v(x, y) \quad \text{for } (x, y)\in \tilde{\Delta}_1, \tag{45}$$

$$u(x, y) > v(x, y) \quad \text{for } (x, y)\in \tilde{\Delta}_2, \tag{46}$$

where $\tilde{\Delta}$ *is the set formed by integral curves of equation* (9) *issuing from the segment* $\tilde{K}$ *and* $\tilde{\Delta}_1, \tilde{\Delta}_2$ *are sets formed by integral curves of equation* (9) *issuing from the segments*

$$K_1 = \{(x, y):\ x = x_0,\ y \in \langle y_0 - b, c)\},$$

$$K_2 = \{(x, y):\ x = x_0,\ y \in (d, y_0 + b\rangle\},$$

respectively, where

$$c = \max\left[g\big(x_0 - \tau(x_0)\big), g(x_0)\right], \qquad d = \min\left[h\big(x_0 - \tau(x_0)\big), h(x_0)\right].$$

Furthermore $\tilde{\Delta}, \tilde{\Delta}_1, \tilde{\Delta}_2$ *satisfy the condition*

$$\Delta = \tilde{\Delta}_1 \cup \tilde{\Delta}_2 \cup \tilde{\Delta}, \tag{47}$$

where $\Delta = \{(x, y):\ x \in \langle x_0, a_1),\ \alpha(x) \leqslant y \leqslant \beta(x)\}$.

Proof. We shall demonstrate equality (44) in the first place. Assume that the curves $y = \tilde{g}(x)$ and $y = \tilde{h}(x)$ are in Δ for $x \in \tilde{I}_1$ and $x \in \tilde{I}_2$, respectively, where $\tilde{I}_1, \tilde{I}_2 \subset I_1 = \langle x_0, a_1)$. Assume that $y = y(x)$ is a solution of equation (9) issuing from segment $\tilde{K}$ and that $\big(x, y(x)\big) \in \Delta$ for $x \in \tilde{I} = \langle x_0, a_0)$, where $a_0 \leqslant a_1$. The functions $y(x), \tilde{g}(x), \tilde{h}(x)$ satisfy the initial inequalities

$$\tilde{g}(x_0) \leqslant y(x_0) \leqslant \tilde{h}(x_0).$$

Since $y = y(x)$ satisfies (9) and the functions $\tilde{g}, \tilde{h}$ satisfy the differential inequalities (42) and (43), we have

$$g\big(x - \tau(x)\big) = \tilde{g}(x) \leqslant y(x) \quad \text{for } x \in \tilde{I}_1 \cap \tilde{I},$$

$$y(x) \leqslant \tilde{h}(x) = h\big(x - \tau(x)\big) \quad \text{for } x \in \tilde{I}_2 \cap \tilde{I}.$$

It follows from these inequalities and from the definition of $\tilde{\Delta}$ that the points $\big(x - \tau(x), y(x)\big)$ are in $\tilde{E}$ for $x \in \tilde{I}$. Therefore the functions $u(x) = u\big(x, y(x)\big)$ and $v(x) = v\big(x, y(x)\big)$ satisfy by (40) the differential equation

$$\frac{dz}{dx} = \tilde{R}(x, z),$$

where $\tilde{R}(x, z) = R\Big(x, y(x), z, \varphi\big(x - \tau(x), y(x)\big)\Big)$. Since $u(x_0) = v(x_0)$, we have $u(x) = v(x)$ for $x \in \tilde{I}$. The integrals u and v of (2) are therefore equal along an arbitrary solution of (9) issuing from $\tilde{K}$ and situated in Δ and are therefore equal in $\tilde{\Delta}$.

We shall now prove (45). Let

$$K_1^{(1)} = \{(x, y):\ x = x_0,\ y \in \langle y_0 - b, c),\ u(x_0, y) < v(x_0, y)\},$$

$$K_1^{(2)} = \{(x, y):\ x = x_0,\ y \in \langle y_0 - b, c),\ u(x_0, y) = v(x_0, y)\}.$$

Let $y = y_1(x)$ be a solution of equation (9) satisfying the initial condition $y_1(x_0) = y_1$, where $(x_0, y_1) \in K_1^{(1)}$ and $\big(x, y_1(x)\big) \in \Delta$ for $x \in \bar{I}_1$. It follows from (42) and (43) that $(x-\tau(x), y_1(x))\in E_1$ for $x\in \bar{I}_1$. Inequality (45) is fulfilled along the curve $y = y_1(x)$ for $x\in \mathrm{Int}\,\bar{I}_1$. The proof of this property proceeds in the same manner as the proof of the analogous inequality given in II(b) of Theorem 1.

Let $y = y_2(x)$ be a solution of (9) satisfying the initial condition $y_2(x_0) = y_2$, where $(x_0, y_2)\in K_1^{(2)}$ and $(x, y_2(x))\in \Delta$ for $x\in \bar{I}_2$. It follows from (42) and (43) that $(x-\tau(x), y_2(x))\in E_2$ for $x\in \bar{I}_2$.

Inequality (45) holds along the curve $y = y_2(x)$ for $x\in \mathrm{Int}\,\bar{I}_2$, which can be shown in the same way as for the the analogous inequality of II(c) of Theorem 1.

In the same way as in II(e) we can show that (45) holds in the set

$$Z_1 = \{(x, y)\in \tilde{\Delta}_1 \colon\ y = \alpha(x) \ \text{or}\ y = \beta(x)\}.$$

The proof of (46) is analogous. We shall not quote here the simple proof of (47).

Remark 4. Assume that conditions 1°–6° and 8° of Assumption H hold and that

$$\tilde{\Delta} = \{(x, y)\colon\ x\in \langle x_0, a_1), a_1 < a, g_1(x) \leqslant y \leqslant h_1(x)\},$$

where the functions $\tilde{g}_1(x) = g_1(x-\tau(x))$, $\tilde{h}_1(x) = h_1(x-\tau(x))$ satisfy the differential inequalities (42) and (43). Then we infer from Theorem 5 that there exist sets $\tilde{\tilde{\Delta}}, \tilde{\tilde{\Delta}}_1, \tilde{\tilde{\Delta}}_2 \subset D_2 = \{(x, y)\colon\ x\in I_2,\ \alpha(x) \leqslant y \leqslant \beta(x)\}$ (the interval I_2 is defined in 8°), such that

$$D_2 = \tilde{\tilde{\Delta}} \cup \tilde{\tilde{\Delta}}_1 \cup \tilde{\tilde{\Delta}}_2,$$

$$u(x, y) < v(x, y) \quad \text{for } (x, y)\in \tilde{\tilde{\Delta}}_1,$$

$$u(x, y) = v(x, y) \quad \text{for } (x, y)\in \tilde{\tilde{\Delta}},$$

$$u(x, y) > v(x, y) \quad \text{for } (x, y)\in \tilde{\tilde{\Delta}}_2.$$

If $\tilde{\tilde{\Delta}}$ satisfies the conditions contained in the assumptions of Theorem 5, then the initial inequalities (39)–(41) hold also in further subsets of D contained in $D_3 = \{(x, y)\colon\ x\in I_3, \alpha(x) \leqslant y \leqslant \beta(x)\}$.

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Reçu par la Rédaction le 30. 5. 1974