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APPLICATIONES MATHEMATICAE
XIX, 2 (1987), pp. 181–188

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ON MOMENTS FOR BRANCHING PROCESSES

1. Introduction. Let $\{Z(t), t \geqslant 0\}$ be a stochastic process defined on the probability space $(\Omega, \mathscr{F}, P)$. The problem is to state the conditions under which $E\Phi(Z(t))$ is finite $(t > 0)$ for a wide class of functions Φ and for various types of stochastic processes.

For a point process $\{Z(t), t \geqslant 0\}$ and a convex non-negative function $\Phi(x)$, $x \geqslant 0$, Fieger [5] gave a necessary condition for the existence of $E\Phi(Z(t))$, $t \geqslant 0$.

For a one-dimensional continuous-time Markov branching process Harris [6] showed that $E\{Z^r(t)\}$ is finite for every positive integer r and every $t > 0$, provided $(d^r f(s)/ds^r)|_{s=1}$ is finite, where $f(s)$ is the offspring probability generating function associated with this process. This result was improved by Sevast'janov [9] who for a wide class of measurable functions $\mathcal{M}$ found the sufficient condition for the finiteness of $E\Phi(Z(t))$, $t > 0$, $\Phi \in \mathcal{M}$. Let us define the class $\mathcal{M}$.

Definition. A measurable function Φ: $R_+ = [0, \infty) \to R_+$ *belongs to the class* $\mathcal{M}$ if there exist constants $c \geqslant 0$ and $K > 0$ such that

(i) Φ is convex on $[c, \infty)$,

(ii) $\Phi(xy) \leqslant K\Phi(x)\Phi(y)$ for every x, y in $[c, \infty)$,

(iii) Φ is bounded on $[0, c]$.

We assume throughout the paper that $\Phi \in \mathcal{M}$.

For the Bellman–Harris age-dependent branching process Athreya [1] gave the necessary and sufficient condition for the finiteness of $E\Phi(Z(t))$, $t > 0$. For some class of branching processes with immigration an analogue of Athreya's result was proved by Kaplan [7].

A simple method of proof using renewal theory was obtained by Athreya [1]. This is the method which is used here to establish further results for more general branching processes.

The paper is arranged as follows: In Section 2 for the Sevast'janov process we establish a sufficient condition for the finiteness of $E\Phi(Z(t))$, $t > 0$. Next, in Section 3, we formulate and solve this problem for age-dependent branching processes with generation dependence. Section 4 concentrates on the study of finiteness of $E\Phi(Z(t))$, $t > 0$, for inhomogeneous Markov branching processes. Finally, in Section 5, a certain result for branching processes with immigration is stated.

2. Sevast'janov processes. Consider the population of particles with the following mechanism of reproduction. Let $\{p_n, n \geq 0\}$, p_n: $R_+ \to [0, 1]$, be a sequence of measurable functions and assume that if the particle lifelength L is equal to u, then $p_n(u)$ is the probability that at the moment of its death (or split) the particle creates (splits into) n new particles. The joint probability of the number of new particles ξ and the particle lifelength L is given by the formula

$$P(\xi = n, L \in B) = \int_B p_n(u)\, dG(u),$$

where the set B belongs to the σ-field of Borel subsets of R_+ and G is the lifetime distribution function of a particle, i.e., $G(u) = P(L \leq u)$. Other assumptions are the same as those in the Bellman–Harris branching processes.

If the probabilities $\{p_n, n \geq 0\}$ do not depend on the particle lifelength, then we have an ordinary Bellman–Harris process. Such processes were introduced and studied in their own right by Sevast'janov (cf. [9], p. 282). Note that his definition was more general than ours, namely more than one type of particles was considered. For fixed Φ let

$$a(u) = \sum_{n=0}^{\infty} \Phi(n)\, p_n(u).$$

For the Sevast'janov process we prove an analogue of Athreya's theorem (cf. [2], Theorem 4, p. 153).

THEOREM 1. *Let $\{Z(t), t \geq 0\}$ be a Sevast'janov branching process. If* $E\Phi(\xi) < \infty$ *and* $G(0+) = 0$, *then* $E\Phi(Z(t)) < \infty$ *for* $t > 0$.

Proof. By Theorem 1 ([9], p. 265) we know that the number of split times in any finite interval is finite with probability one, and hence those times can be ordered. (If it is possible for more than one particle to die at a given time with positive probability, we give any fixed order to such particles.) Let τ_n be a moment of the n-th split, and set $\tau_n = \infty$ for $n > N$, where N is the total number of splits in $[0, \infty)$. Define

$$Y_n(t) = \begin{cases} Z(t) & \text{if } \tau_n > t, \\ 0 & \text{if } \tau_n \leq t. \end{cases}$$

It is clear that

$$Y_n(t) \leq 1 + \sum_{j=1}^{n} \xi_j, \tag{1}$$

where ξ_j ($j = 1, 2, \ldots$) are the numbers of particles produced in the j-th split. Let $m_n(t) = E\Phi(Y_n(t))$. From (1) and the properties of Φ we infer that $m_n(t)$ is finite for every integer $n \geqslant 1$ and every $t > 0$. Now we prove that for every $n \geqslant 1$ and constants c_1 and c_2 the following inequality holds:

$$m_{n+1}(t) \leqslant c_1\left(1-G(t)\right)+c_2 \int_0^t a(u)\, m_n(t-u)\, dG(u), \tag{2}$$

where $0 < c_1,\ c_2 < \infty$. We have

$$\mathrm{E}\Phi(Y_{n+1}(t)) = \mathrm{E}\left[\Phi(Y_{n+1}(t));\ \tau_1 > t\right]+\mathrm{E}\left[\Phi(Y_{n+1}(t));\ \tau_1 \leqslant t\right]. \tag{3}$$

By the definition of $Y_{n+1}(t)$ we can write

$$\mathrm{E}\left[\Phi(Y_{n+1}(t));\ \tau_1 > t\right] = \Phi(1)\left(1-G(t)\right). \tag{4}$$

Since on the set $\{\tau_1 \leqslant t\}$ the inequality

$$Y_{n+1}(t) \leqslant \sum_{j=1}^{\xi} \tilde{Y}_n^{(j)}(t-\tau_1)$$

holds, where $\tilde{Y}_n^{(j)}(u)$, $j = 1, 2, \ldots$, are independent copies of $Y_n(u)$ and the sum is taken to be zero when $\xi = 0$, we obtain

$$\mathrm{E}\left[\Phi(Y_{n+1}(t));\ \tau_1 \leqslant t\right] \leqslant \int_0^t \left[p_0(u)\,\Phi(0)+K \sum_{k=1}^{\infty} p_k(u)\,\Phi(k)\, m_n(t-u)\right] dG(u). \tag{5}$$

By (5) and the assumption of the theorem, there exists a constant c_2 ($0 < c_2 < \infty$) such that

$$\mathrm{E}\left[\Phi(Y_{n+1}(t));\ \tau_1 \leqslant t\right] \leqslant c_2 \int_0^t a(u)\, m_n(t-u)\, dG(u). \tag{6}$$

Now (2) follows from (3), (4), and (6). The rest of the proof is quite similar to the proof of Athreya's theorem and will be only sketched.

Under the assumption of Theorem 1 the integral equation

$$m(t) = c_1\left(1-G(t)\right)+c_2 \int_0^t a(u)\, m(t-u)\, dG(u)$$

has a unique non-negative solution m bounded on finite intervals.

It is easy to prove that there exist constants c_1 and c_2 such that m satisfies the inequality $m_n(t) \leqslant m(t)$ for every integer $n \geqslant 1$ and for $t > 0$. Therefore

$$\sup_{n \in Z_+} \mathrm{E}\Phi(Y_n(t)) < \infty$$

(Z_+ – positive integers). Now, by the monotone convergence theorem, we have

$$\lim_{n \to \infty} \mathrm{E}\Phi(Y_n(t)) = \mathrm{E}\Phi(Z(t)),$$

which completes the proof.

3. Age-dependent branching processes with generation dependence. Consider another extension of Bellman–Harris age-dependent branching process with generation dependence (cf. [4]). If the process starts with one particle in the n-th generation, then this particle lives for a random length of time $T_{n,1}$ with distribution function G (it is assumed throughout that $G(0+) = 0$) and then splits into a random number $\zeta_{n,1}$ of particles in the $(n+1)$-st generation. These particles live random lengths of time $T_{n+1,1}, \ldots, T_{n+1,\zeta_{n,1}}$, respectively, in the $(n+2)$-nd generation. The process continues; $Z_n(t)$ is the number of particles alive at time t having started with one particle in the n-th generation. It is assumed that the random variables $T_{m,j}$ and $\zeta_{n,k}$, where $m, n \in Z_+^0$ (Z_+^0 – non-negative integers) and $j, k \in Z_+$, are mutually independent, that the random variables $T_{n,k}$ for $n \in Z_+^0$ and $k \in Z_+$ have the same distribution, and that for each fixed $n \in Z_+^0$ the random variables $\zeta_{n,k}$, $k \in Z_+$, have a probability distribution on Z_+^0 depending only on n.

Let $t_0 = \inf\{x\colon G(x+) > 0\}$. For this process we prove the following extension of Athreya's theorem (cf. [2]):

THEOREM 2. *Let* $\{Z_n(t), t \geqslant 0\}$, $n \geqslant 0$, *be a sequence of age-dependent branching processes with generation dependence. If* $\mathrm{E}\Phi(\zeta_{n,1}) < \infty$, $n \in Z_+^0$, *then* $\mathrm{E}\Phi(Z_n(t))$ *are uniformly bounded on finite intervals. Further, if* $\mathrm{E}\Phi(Z_n(t)) < \infty$ *for every* $t > t_0$ *and* $n \in Z_+^0$, *then* $\mathrm{E}\Phi(\zeta_{n,1}) < \infty$ *for every integer* $n \in Z_+^0$.

Proof. From [1] and [4] we know that the number of split times in any finite interval is finite with probability one, and hence these times can be ordered. Let τ_i be the moment of the i-th split and set $\tau_i = \infty$ for $i > N$, where N is the total number of splits in $[0, \infty)$. Define

$$Z_{n,i}(t) = \begin{cases} Z_n(t) & \text{if } \tau_i > t, \\ 0 & \text{if } \tau_i \leqslant t. \end{cases}$$

The same method as in Theorem 1 may be used to derive the inequality

$$m_{n,i+1}(t) \leqslant c_1(1 - G(t)) + c_2 \int_0^t m_{n+1,i}(t-u)\,dG(u),$$

where $m_{n,i}(t) = \mathrm{E}\Phi(Z_{n,i}(t))$ and c_1 and c_2 are some positive constants. Further we need the following fact. Let $\{Y_n(t), n \geqslant 0, t \geqslant 0\}$ be a sequence of functions uniformly bounded on finite intervals and let G be a distribution function on $[0, \infty)$ such that $(Y_n * G)(t)$ exists for all $t > 0$ and every integer $n \geqslant 0$. According to Fearn [4], if $G(0+) = 0$ and the m_n are a uniformly bounded sequence of real numbers, then the system of renewal equations

$$X_n(t) = Y_n(t) + m_n(X_{n+1} * G)(t), \qquad n = 0, 1, \ldots, t \geqslant 0,$$

has a unique non-negative solution $\{X_n(t), n \geqslant 0\}$ uniformly bounded on finite intervals. Consequently, there exists a sequence of functions $\bar{m}_n$

uniformly bounded on finite intervals which is the solution of the following equation:

$$\bar{m}_n(t) = c_1\left(1-G(t)\right)+c_2 \int_0^t \bar{m}_{n+1}(t-u)\,dG(u).$$

Now there exist positive constants c_1 and c_2 (cf. Theorem 1) such that $m_{n,i}(t) \leqslant \bar{m}_n(t)$ for every $t > 0$ and every integer i, $n \geqslant 0$, which proves the first part of the theorem.

The proof of the second part of Theorem 2 is omitted. It can be easily obtained by modifying the argument used in the proof of Athreya's theorem (cf. [1]).

4. Inhomogeneous Markov branching processes. Consider a branching process defined by Cohn and Hering [3]. Let $k(t)$ be a non-negative and measurable function on non-negative reals and let $\{p_t(n),\ n \geqslant 0,\ t \geqslant 0\}$ be a set of probability distributions on non-negative integers with $p_t(n)$ measurable as a function of t. Let us define

$$f_t(x) = \sum_{n=0}^{\infty} x^n p_t(n), \qquad |x| \leqslant 1,$$

and

$$m_t = f_t'(1-) = \sum_{n=1}^{\infty} np_t(n).$$

Assume that

$$\sup_{s\leqslant t} k(s)+\sup_{s\leqslant t} m_s < \infty, \qquad t \in R_+.$$

Then $\{k(t),\ p_t(n)\}$ determines uniquely a continuous-time Markov branching process constructed according to the following intuitive rules: All particles behave independently, the probability that a particle undergoes branching in the time interval $[t, t+\Delta]$ is $k(t)\Delta+o(\Delta)$, and the probability that it is replaced by exactly n new particles, given that it undergoes branching at t, is $p_t(n)$.

For fixed Φ, let

$$\Phi_t = \sum_{n=0}^{\infty} \Phi(n)\,p_t(n), \qquad t \in R_+.$$

THEOREM 3. *Let* $\{Z_{0,t},\ t \geqslant 0\}$ *be an inhomogeneous Markov branching process with* $Z_{0,0} = 1$. *If*

$$\sup_{u\leqslant t} \Phi_u < \infty \qquad \textit{for every } t > 0,$$

then $\mathrm{E}\Phi(Z_{0,t}) < \infty$ *for every* $t > 0$.

Proof. We prove first the following fact:

For every pair of positive constants c_1 and c_2 there exists a unique non-negative solution of the integral equation

(7) $$X_{0,t} = c_1 T_{0,t} + c_2 \int_0^t T_{0,u} k(u) \Phi_u X_{u,t} du \quad (t > 0),$$

where

$$T_{s,t} = \exp\left\{-\int_s^t k(u) du\right\}, \quad t \geqslant s.$$

Indeed, it is sufficient to check that

$$X_{s,t} = c_1 \exp\left\{\int_s^t k(u)(c_2 \Phi_u - 1) du\right\}, \quad t \geqslant s,$$

is the solution of (7) (cf. [1]).

Let us define

$$Z_{0,t}^{(n)} = \begin{cases} Z_{0,t} & \text{if } \tau_n > t, \\ 0 & \text{if } \tau_n \leqslant t, \end{cases}$$

where τ_n is the moment of the n-th split. Clearly, $\mathrm{E}\Phi(Z_{0,t}^{(n)}) < \infty$ for every $t > 0$ and every integer $n \geqslant 1$. It is easy to show that

$$M_{0,t}^{(n+1)} \leqslant \Phi(1) T_{0,t} + K \int_0^t T_{0,u} k(u) \Phi_u M_{u,t}^{(n)} du,$$

where $M_{u,t}^{(n)} = \mathrm{E}\Phi(Z_{u,t}^{(n)})$, $0 \leqslant u \leqslant t$. By the above considerations the integral equation

$$X_{0,t} = c_1 T_{0,t} + c_2 \int_0^t T_{0,u} k(u) \Phi_u X_{u,t} du$$

has a unique non-negative solution. There exist positive constants c_1 and c_2 such that $M_{0,t}^{(n)} \leqslant X_{0,t}$ for every $t > 0$ and every integer $n \geqslant 1$. This implies

$$M_{0,t} = \mathrm{E}\Phi(Z_{0,t}) = \lim_{n \to \infty} M_{0,t}^{(n)} \leqslant X_{0,t} < \infty,$$

which completes the proof of Theorem 3.

5. Branching processes with immigration. Consider a population which reproduces according to a Sevast'janov process $\{Y(t),\ t \geqslant 0\}$. Let $G_1(\cdot)$ be the common distribution function of the random lifelength of an object. The objects are assumed to develop independently of each other. Moreover, it is assumed that the population is being augmented by an independent immigration process defined below, where each immigrant generates a Sevast'janov process independently of others. The immigration process epochs occur in times according to a renewal process with distribution of interimmigration times given by $G_0(\cdot)$. Also, at the i-th immigration epoch η_i ($\eta_0 = 0$), ϑ_i ($i = 0, 1, \ldots$) immigrants enter the population. These immigrant numbers are independent of each other and of everything else. It is assumed that $G_0(0+) = G_1(0+) = 0$. Let $X(t)$ denote the population size at time t. It is easy to see that

$$X(t) = \sum_{i=1}^{n(t)} \sum_{j=1}^{\vartheta_i} Y_{ij}(t-\eta_i),$$

where $n(t) = \sup\{n \geq 1\colon\ \eta_n \leq t\}$ and $\{Y_{ij}(t)\}$, $1 \leq j \leq \vartheta_i$, $i \geq 1$, are independent copies of $\{Y(t),\ t \geq 0\}$.

If $\{Y(t),\ t \geq 0\}$ is the Bellman–Harris age-dependent branching process, then the process $\{X(t),\ t \geq 0\}$ as defined above is called a *Bellman–Harris process with immigration.* We assume that $X(0) = 0$.

Let $t_0 = \inf\{x\colon\ G_0(x+) > 0\}$ and $t_1 = \inf\{x\colon\ G_1(x+) > 0\}$. Now we prove the following theorem similar to Kaplan's result (cf. [7]).

THEOREM 4. *Let $\{X(t),\ t \geq 0\}$ be a branching process with immigration for which $\{Y(t),\ t \geq 0\}$ is a branching process with the function $m(t) = \mathrm{E}\Phi(Y(t))$ bounded on finite intervals. Then $\mathrm{E}\Phi(X(t)) < \infty$ for any $t > t_0 + t_1$ if and only if $\mathrm{E}\Phi(\vartheta_1) < \infty$.*

Remark. Under the assumption

$$\mathrm{E}\Phi(\xi) = \int_{R_+} a(u)\,dG_1(u) < \infty$$

Sevast'janov processes satisfy the conditions of Theorem 4.

Proof. For fixed $t > t_0 + t_1$ we have

$$X(t) = \sum_{i=1}^{n(t)} Y_i, \quad \text{where } Y_i = \sum_{j=1}^{\vartheta_i} Y_{ij}(t-\eta_i).$$

From the assumptions of the theorem it follows easily that

$$\sup_{i \in Z_+} \mathrm{E}\Phi(Y_i) < \infty. \tag{8}$$

Now we use the following fact. Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and non-negative random variables, let $N \geq 0$ be an integer-valued random variable independent of the X_i's, and put $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n \geq 1$. If

$$\mathrm{E}\Phi(N) < \infty \quad \text{and} \quad \sup_{i \in Z_+} \mathrm{E}\Phi(X_i) < \infty,$$

then $\mathrm{E}\Phi(S_N) < \infty$. This fact and (8) imply that $\mathrm{E}\Phi(X(t)) < \infty$, which proves the sufficiency. For the necessity, let $t > t_0 + t_1$ and assume that $\mathrm{E}\Phi(X(t))$

$< \infty$. Then on the set $\{\eta_1 \leqslant t\}$ we have

$$\infty > \mathrm{E}\Phi(X(t)) \geqslant \int_{\eta_1 \leqslant t} \Phi\Big(\sum_{i=1}^{n(t)} \sum_{j=1}^{\vartheta_i} Y_{ij}(t-\eta_i)\Big) dP \overset{\mathrm{df}}{=} R.$$

Now, by the monotonicity of the function Φ (cf. [1]) we obtain

$$R \geqslant \int_0^t \mathrm{E}\Phi\Big(\sum_{j=1}^{\vartheta_1} Y_{1j}(t-u)\Big) dG_0(u).$$

From the last inequality we infer that there exists $u_0 > t_0 + t_1$ such that

$$\mathrm{E}\Phi\Big(\sum_{j=1}^{\vartheta_1} Y_{1j}(u_0)\Big) < \infty.$$

Now the lemmas of Athreya (cf. [2], Lemmas 4 and 5, p. 156–157) give $\mathrm{E}\Phi(\vartheta_1) < \infty$, which completes the proof of Theorem 4.

Note that if $\{Y(t), t \geqslant 0\}$ is an ordinary Bellman–Harris process, then Theorems 2 and 4 yield the result of Kaplan [7].

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Received on 1984.01.16;
revised version on 1985.01.15