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INVARIANT FUNCTIONS FOR POSITIVE OPERATORS ON NORMED KÖTHE SPACES

BY

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The main result of this paper is as follows. Suppose T is a positive operator on an absolutely continuous normed Köthe space (for example, an L_p space $(1 \leqslant p < \infty)$ or an Orlicz space with Δ_2-property) such that $\sup_n \|T^n\| < \infty$, and f is a non-negative function in this space. Then a sufficient condition for the existence of a non-negative invariant function with positive values on the support of f is that

$$\inf_n \int_E T^n f\, dm > 0,$$

for any positive measurable set E contained in the support of f.

This condition is necessary if the invariant function has the same support as f.

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Let (X, Σ, m) be a finite measure space, τ be a non-singular measurable transformation on X. The problem of finding the necessary and sufficient conditions for the existence of non-trivial invariant measures for τ has been studied thoroughly. One of the well-known results ([1] and [3]) is as follows:

τ has a finite invariant measure equivalent to m iff $\underline{\lim}_n m(\tau^{-n}(A)) > 0$ for all $A \in \Sigma$ with $m(A) > 0$.

In 1966, Dean and Sucheston [2] and, independently, Neveu [10] have extended this result to the case of contractions in L_1-spaces. Dean and Sucheston show that a positive operator on L_1 with $\|T\| \leqslant 1$ (contraction) has a positive invariant function iff

$$\inf_n \int_A T^n 1\, dm > 0$$

for any $A \in \Sigma$ with $m(A) > 0$ (where 1 stands for the constant function of value 1). Equivalently, Neveu shows that T has a positive invariant function iff

$$\varliminf_{n} \int hT^n f dm > 0$$

for any non-negative function h in $L_\infty - \{0\}$, where f is any fixed element in L_1 such that $f > 0$ a. e.

Their result not only covers the case of measurable transformations, but also includes the case of invariant measures for Markov processes that has been studied by Ito [6].

In 1970, Fong [5] generalized this result to the case of "semi-Markovian" (or "power bounded") operators, i. e. positive operators in L_1 with $\sup_n \|T^n\| < \infty$.

While reading their papers, we found that the lattice properties play a very important role in their proofs. So we wonder whether a similar theory could also be set up in some other spaces that have the same sort of lattice structure as L_1. We shall discuss such kind of spaces in this paper.

In the proof of the following lemmas, we shall make use of the theory of Riesz spaces. By a *Riesz space* we mean a vector space over the real field with a partial ordering $\leqslant$ such that:

(1) $x \leqslant y \Rightarrow x+z \leqslant y+z$ for all z;

(2) $0 \leqslant x \Rightarrow 0 \leqslant ax$ for any non-negative real number a;

(3) the least upper bound $\sup\{x, y\}$ exists for any x and y.

We shall write $x_n \uparrow x$ (resp. $x_n \downarrow x$) to indicate an increasing (resp. decreasing) sequence of elements in the Riesz space with

$$\sup_n x_n = x \qquad (\text{resp. } \inf_n x_n = x).$$

We let $x^+ = \sup\{x, 0\}$, $x^- = \sup\{-x, 0\}$, $|x| = \sup\{x, -x\}$, and $M^+ = \{x \in M\colon x \geqslant 0\}$, where M is a subset of a Riesz space.

A linear subspace B of the Riesz space L is called a *band* if

(1) for $x \in B$, $y \in L$ and $|y| \leqslant |x|$, we have $y \in B$;

(2) for $A \subset B$, if $\sup A$ exists in L, then $\sup A \in B$.

A Riesz space L is said to be *Dedekind complete* (resp. *Dedekind σ-complete*) if every subset (resp. countable subset) of L which is bounded above has a least upper bound in L. A *positive functional* f on L is a linear functional with the additional property that $x \geqslant 0 \Rightarrow f(x) \geqslant 0$. A *regular functional* is a linear functional that can be expressed as a difference of two positive functionals. It is well-known (for example, chap. VIII of [11]) that the set $L^\sim$ of all regular functionals on L forms a Dedekind complete Riesz space and the greatest lower bound of two regular functionals f and g is determined by the formula

$$\inf\{f, g\}(y) = \inf\{f(x)+g(y-x)\colon 0 \leqslant x \leqslant y\} \quad \text{for any } y \in L^+.$$

A regular functional is called a *completely linear functional* (an *integral*) if $x_n \downarrow 0$ implies $f(x_n) \to 0$.

LEMMA 1. *Let L be a Dedekind σ-complete Riesz space, f be a positive completely linear functional, h be a positive functional such that* $\inf\{h, f\} = 0$. *Then for any $y \in L^+$ and any positive real number ε, there exists $x \in L^+$ with $0 \leqslant x \leqslant y$ and $h(x) = 0$, $f(y-x) < \varepsilon$.*

Proof. If $h(y) = 0$, the proof is trivial (we can choose $x = y$). If $h(y) > 0$, we let $y_0 = y$. Since $\inf\{h, f\}(y_0) = 0$, there is $y_1 \in L^+$ such that $0 \leqslant y_1 \leqslant y_0$ and $h(y_1)+f(y_0-y_1) < \varepsilon/2$. Suppose $y_0, y_1, \ldots, y_n$ are defined. We choose $y_{n+1} \in L^+$ such that $0 \leqslant y_{n+1} \leqslant y_n$ and $h(y_{n+1})+f(y_n-y_{n+1}) < \varepsilon/2^{n+1}$. So we get a decreasing sequence $\langle y_n \rangle$ satisfying these inequalities. Let $x = \inf\limits_n y_n$. Since f is completely linear, we have

$$\begin{aligned} f(y-x) &= f(y)-f(x) = f(y)-\lim_n f(y_n) = \lim_n \big(f(y)-f(y_n)\big) \\ &= \lim_n \big(f(y)-f(y_1)+f(y_1)-f(y_2)+ \ldots +f(y_{n-1})-f(y_n)\big) \\ &\leqslant \lim_n (\varepsilon/2+\varepsilon/2^2+ \ldots +\varepsilon/2^n) = \varepsilon. \end{aligned}$$

On the other hand, $h(x) \leqslant h(y_n) < \varepsilon/2^n$ for any n. So $h(x) = 0$.

For a σ-finite measure space (X, Σ, m), the set $L_\infty(X, \Sigma, m)$ of essentially bounded functions is a Dedekind complete Riesz space; a finitely additive set function λ on (X, Σ) that vanishes on m-null sets can be regarded as a regular functional on $L_\infty(X, \Sigma, m)$ through the formula $\lambda(h) = \int h d\lambda$ (cf. [4], p. 296). In particular, a measure on (X, Σ) that is absolutely continuous with respect to m is a completely linear functional on $L_\infty(X, \Sigma, m)$. A finitely additive positive set function ν is called a *pure charge* if $\inf\{\nu, \mu\} = 0$ for any measure μ on (X, Σ). Yosida and Hewitt [12] have shown that if μ is a measure and ν is a pure charge defined on (X, Σ), then for any $\varepsilon > 0$ and any $A \in \Sigma$, there is a measurable subset $B \subset A$ with $\nu(B) = 0$ and $\mu(A-B) < \varepsilon$. Our lemma 1 is a generalization of their result.

We let $\mathcal{M}$ to be the class of real-valued measurable functions on (X, Σ, m), where two functions which have the same values almost everywhere are considered identical. Suppose we can define a mapping $\varrho\colon \mathcal{M} \to [0, \infty]$ satisfying

(i) $\varrho(f) = 0$ iff $f = 0$;

(ii) $\varrho(f+f_1) \leqslant \varrho(f)+\varrho(f_1)$;

(iii) $\varrho(af) = |a|\varrho(f)$ for any real number a;

(iv) $$|f| \leqslant |f_1| \Rightarrow \varrho(f) \leqslant \varrho(f_1);$$

(v) $$\varrho(f_n) \to \varrho(f) \quad \text{for } f, f_n \in \mathscr{M}^+ \text{ and } f_n \uparrow f;$$

(vi) for $A \in \Sigma$ with $m(A) > 0$, there is a measurable set $E \subset A$ with positive measure such that $\varrho(\chi_E) < \infty$, where χ_E stands for the characteristic function of E.

Then the set $L_\varrho = \{f \in \mathscr{M}: \varrho(f) < \infty\}$ is called a *normed Köthe space.*

This kind of function spaces has been studied in detail by Luxemberg and Zaanen ([8] and [13]), but they did not assume conditions (v) and (vi) in the definition of a general Köthe space. They define the normed Köthe space as a function space satisfying the first four conditions, where (v) is considered as an extra condition — the Fatou property. Moreover, a space with condition (vi) is considered to be *saturated*. So in [8] a function space satisfying conditions (i)-(vi) is called a saturated Köthe space with Fatou property.

We define ϱ': $\mathscr{M} \to [0, \infty]$ by the formula

$$\varrho'(g) = \sup\left\{\left|\int fg\,dm\right|:\ \varrho(f) \leqslant 1\right\}.$$

From § 69 and § 71, theorem 4 of [13], we know that ϱ' also satisfies conditions (i)-(vi). Therefore the space $L_{\varrho'} = \{g \in \mathscr{M}:\ \varrho'(g) < \infty\}$ is also a normed Köthe space — the associate space of L_ϱ.

On the other hand, a normed Köthe space is also a Dedekind complete Riesz space (cf. [8]). For a fixed $g \in L_{\varrho'}$, the mapping $f \mapsto \int fg\,dm$ is a completely linear functional on L_ϱ. It turns out that the set of all completely linear functionals is precisely the associate space $L_{\varrho'}$ (cf. [7], theorem 1.4).

It is worth-while to note that a normed Köthe space L_ϱ is also a Banach space, since condition (v) implies the completeness of ϱ ([13], § 65). Again the completeness of ϱ implies that the (continuous) dual space L_ϱ^* and the space $L_\varrho^\sim$ of regular functionals are identical ([8], p. 348). So L_ϱ^* is also a Dedekind complete Riesz space.

For $f \in L_\varrho$ we define $\hat{f} \in L_{\varrho'}^*$ such that $\hat{f}(g) = \int fg\,dm$ for $g \in L_{\varrho'}$. The mapping $f \mapsto \hat{f}$ is then an injection of L_ϱ into $L_{\varrho'}^*$. This injection also preserves the lattice structure, and so L_ϱ can be considered as a subspace of $L_{\varrho'}^*$. We shall identify f and $\hat{f}$ in the following discussion.

LEMMA 2. *If L_ϱ is a normed Köthe space, then $L_{\varrho'}^* = L_\varrho \oplus L_\varrho^d$, where*

$$L_\varrho^d = \{\nu \in L_{\varrho'}^*:\ \inf\{|\nu|, f\} = 0 \ \textit{for any}\ f \in L_\varrho^+\}.$$

Moreover, if $\lambda \geqslant 0$ in $L_{\varrho'}^*$, we have $f \geqslant 0$ in L_ϱ, $\nu \geqslant 0$ in L_ϱ^d such that $\lambda = f + \nu$.

Proof. Since $L_{\varrho''}$ is the set of all completely linear functionals on $L_{\varrho'}$, and the set of completely linear functionals is a band of $L_{\varrho'}^\sim$ ([8],

p. 348), $L_{\varrho''}$ is a band of $L_{\varrho'}^{\sim}$. On the other hand, by theorem 1, § 71 of [13], we have $\varrho = \varrho''$. So L_ϱ is a band of $L_{\varrho'}^{\sim}$. Since $L_{\varrho'}^{\sim} = L_{\varrho'}^{*}$ in our case, L_ϱ is a band of the Dedekind complete Riesz space $L_{\varrho'}^{*}$.

From § 3, chapter IV of [11], we know that $L_{\varrho'}^{*}$ can be decomposed into direct sums of the bands L_ϱ and L_ϱ^d by two positive projections. Hence the result.

A normed Köthe space L_ϱ is said to be *absolutely continuous* if it satisfies the condition

(vii) If $f_n \in L_\varrho$ and $f_n \downarrow 0$, then $\varrho(f_n) \to 0$.

All the L_p-spaces ($1 \leqslant p < \infty$) are absolutely continuous normed Köthe spaces. More generally, if φ: $[0, \infty) \to [0, \infty]$ is a non-decreasing left continuous function with $\varphi(0) = 0$, define ψ: $[0, \infty) \to [0, \infty]$ by

$$\psi(x) = \begin{cases} \inf \varphi^{-1}(\{x\}) & \text{if } x \text{ in the range of } \varphi, \\ u & \text{if } \varphi(u) < x < \varphi(u+0), \\ \infty & \text{if } x \geqslant \sup\limits_{y>0} \varphi(y). \end{cases}$$

The functions Φ, Ψ on $[0, \infty)$ defined as

$$\Phi(x) = \int_0^x \varphi\, dm, \qquad \Psi(x) = \int_0^x \psi\, dm$$

are called the *complementary Young functions.*

If $\mathscr{M}$ is the set of measurable real functions on a σ-finite non-atomic measure space (X, Σ, m), the function ϱ_Φ: $\mathscr{M} \to [0, \infty]$ defined as

$$\varrho_\Phi(f) = \inf\left\{\frac{1}{a}: a \geqslant 0, \int \Phi(|af|) \leqslant 1\right\}$$

satisfies conditions (i)-(vi). The corresponding Köthe space $L_\Phi = \{f \in \mathscr{M}: \varrho_\Phi(f) < \infty\}$ is called the *Orlicz space with respect to Φ*. Similarly we can define ϱ_Ψ and L_Ψ. It can be shown that ϱ_Ψ is equivalent to ϱ_Φ' and L_Ψ is the associate space of L_Φ. Moreover, if the function Φ satisfies the so--called Δ_2-condition (i.e. $\Phi(x) > 0$ for all $x > 0$ and there exists a positive real number M such that $\Phi(2x) \leqslant M\Phi(x)$ for all $x \geqslant 0$), then L_Φ is absolutely continuous ([9], p. 47).

An operator T defined on a normed Köthe space is said to be *power bounded* if there is a positive real number A such that $\varrho(T^n f) \leqslant A\varrho(f)$ for any $f \in L_\varrho$. If we write

$$\|T^n\| = \sup_n \{\varrho(T^n f): \varrho(f) \leqslant 1\},$$

then T is power bounded iff $\sup\limits_n \|T^n\| < \infty$.

We shall use $S(f)$ to denote the set $\{x \in X: f(x) \neq 0\}$. Furthermore, the notation $A \subset B$ always means that almost all elements of A are in B.

LEMMA 3. *If L_ϱ is an absolutely continuous Köthe space, $T: L_\varrho \to L_\varrho$ is a power bounded positive operator, f and f_1 are two functions in L_ϱ^+ with $S(f_1) \supset S(f)$, and g is a function in $L_{\varrho'}^+$, then*

$$\inf_n \int gT^n f\,dm > 0 \text{ implies } \inf_n \int gT^n f_1\,dm > 0.$$

Proof. Let

$$\delta = \inf_n \int gT^n f\,dm, \qquad M = \sup_n \|T^n\|.$$

From the assumption $S(f_1) \supset S(f)$, we know that $(f - kf_1)^+ \downarrow 0$ as $k \to \infty$. So $\varrho\big((f-kf_1)^+\big) \downarrow 0$ as L_ϱ is absolutely continuous. We choose a positive integer k_1 such that $\varrho\big((f-k_1 f_1)^+\big) < \delta/2M\varrho'(g)$. On the other hand, since $f = k_1 f_1 + (f - k_1 f_1) \leqslant k_1 f_1 + (f-k_1 f_1)^+$, we have

$$\begin{aligned}
\delta &\leqslant \int gT^n f\,dm \leqslant k_1 \int gT^n f_1\,dm + \int gT^n (f-k_1 f_1)^+\,dm \\
&\leqslant k_1 \int gT^n f_1\,dm + \varrho\big(T^n (f-k_1 f_1)^+\big)\varrho'(g) \\
&\leqslant k_1 \int gT^n f_1\,dm + M\varrho\big((f-k_1 f_1)^+\big)\varrho'(g) \\
&\leqslant k_1 \int gT^n f_1\,dm + \delta/2 \quad \text{for any positive integer } n.
\end{aligned}$$

Therefore $\delta/2k_1 \leqslant \int gT^n f_1\,dm$ for any n. Hence the result.

We shall utilize the concept of a Banach limit in the proof of our theorem. A *Banach limit* LIM is a linear functional defined on the space of all bounded real sequences with the following properties:

(a) $\underline{\lim}\, x_n \leqslant \mathrm{LIM}(x_n) \leqslant \overline{\lim}(x_n)$,

(b) $\mathrm{LIM}(x_n) = \mathrm{LIM}(x_{n+1})$.

The existence of a Banach limit can be easily deduced from the Hahn-Banach theorem (see [4], p. 73).

THEOREM. *Let L_ϱ be an absolutely continuous Köthe space, $T: L_\varrho \to L_\varrho$ be a positive operator with $\sup_n \|T^n\| < \infty$. If there exists a function $f \in L_\varrho^+$ such that*

$$\inf_n \int gT^n f\,dm > 0 \quad \textit{for all } g \in L_{\varrho'}^+ \textit{ with } g\cdot f \neq 0,$$

then there is a non-negative function $h \in L_\varrho^+$ with $S(h) \supset S(f)$ and $T(h) = h$.

Conversely, if $h \neq 0$ is a non-negative fixed point of T, then for any $f \in L_\varrho^+$ with $S(f) = S(h)$, we have

$$\inf \int gT^n f\,dm > 0 \quad \textit{for all } g \in L_{\varrho'}^+ \textit{ with } g\cdot f \neq 0.$$

Proof. Assume that

$$\inf_n \int gT^n f\,dm > 0 \quad \text{for all } g \in L^+_{\varrho'} \text{ with } g\cdot f \neq 0.$$

We define a linear transformation $\lambda\colon L_{\varrho'} \to \boldsymbol{R}$ such that

$$\lambda(g) = \operatorname{LIM} \int gT^n f\,dm = \operatorname{LIM} \int fT^{*n} g\,dm,$$

where T^* is the conjugate of T. Since

$$\begin{aligned}|\lambda(g)| &= \Big|\operatorname{LIM} \int gT^n f\,dm\Big| \leqslant \operatorname{LIM}\Big|\int gT^n f\,dm\Big| \leqslant \varrho'(g)\sup_n \varrho(T^n f)\\ &\leqslant \varrho'(g)\varrho(f)\sup_n \|T^n\|,\end{aligned}$$

we have $\lambda \in L^*_{\varrho'}$.

It is also obvious that $\lambda \geqslant 0$. Therefore, by lemma 2, there exist $\mu \in L_\varrho$, $\nu \in L^d_\varrho$ such that $\mu \geqslant 0$, $\nu \geqslant 0$ and $\lambda = \mu + \nu$.

Define

$$B = \Big\{u \in L^+_\varrho\colon \int ug\,dm \leqslant \lambda(g) \text{ for any } g \in L^+_{\varrho'}\Big\}.$$

We claim that $u \leqslant \mu$ for all $u \in B$. Since $u - \mu \leqslant \lambda - \mu = \nu$ and $\nu \geqslant 0$, so $(u-\mu)^+ \leqslant \nu$. Therefore $(u-\mu)^+ = \inf\{(u-\mu)^+, \nu\} = 0$ as $\nu \in L^d_\varrho$, whence $u \leqslant \mu$. So we can conclude that $\mu = \sup B$.

Secondly, we claim that $T\mu \in B$. Since

$$\begin{aligned}\int gT\mu\,dm &= \int \mu T^* g\,dm = \mu(T^* g) \leqslant \lambda(T^* g) = \operatorname{LIM}\int f\cdot T^{*n+1} g\,dm\\ &= \operatorname{LIM}\int f\cdot T^{*n} g\,dm = \lambda(g) \quad \text{for all } g \in L^+_{\varrho'},\end{aligned}$$

we have $T\mu \leqslant \mu$.

Now we let $h = \inf_n T^n\mu$; so $(T^n\mu - h)\downarrow 0$. Since L_ϱ is absolutely continuous, we have $\varrho(T^n\mu - h) \to 0$ and, therefore,

$$\begin{aligned}\varrho(Th-h) &\leqslant \varrho(Th - T^n\mu) + \varrho(T^n\mu - h)\\ &\leqslant \|T\|\varrho(h - T^{n-1}\mu) + \varrho(T^n\mu - h) \to 0.\end{aligned}$$

So $Th = h$.

To complete our proof, we have to show that $S(h) \supset S(f)$. First we prove that $S(\mu) \supset S(f)$. Suppose to the contrary that there is a measurable subset A in $S(f)$ such that $m(A) > 0$ and $A \cap S(\mu) = \emptyset$. By condition (vi) in the definition of a normed Köthe space, there is a measurable subset E of A with $m(E) > 0$ and $\chi_E \in L_{\varrho'}$. Therefore $\int \mu\chi_E\,dm = 0$ and $\int f\chi_E\,dm > 0$.

We let $\varepsilon = \frac{1}{2}\int f\chi_E dm$. Since $\inf\{f, \nu\} = 0$, by lemma 1 there exists $g \in L_{\varrho'}$ such that $0 \leqslant g \leqslant \chi_E$ and $\nu(g) = 0$,

$$\int (\chi_E - g) f dm < \varepsilon.$$

Therefore $\int gfdm > \int f\chi_E dm - \varepsilon = \frac{1}{2}\int f\chi_E dm > 0$. That means $gf \neq 0$. So by the assumption we have

$$(*) \qquad 0 < \inf_n \int gT^n f dm \leqslant \mathrm{LIM} \int gT^n f dm = \lambda(g).$$

On the other hand, since $0 \leqslant g \leqslant \chi_E$ and $\int \mu\chi_E dm = 0$, we have $\int \mu g dm = 0$. It follows that $\lambda(g) = \mu(g) + \nu(g) = \int \mu g dm + \nu(g) = 0$, a contradiction to $(*)$. Hence $S(\mu) \supset S(f)$ holds.

Since $S(\mu) \supset S(f)$, by lemma 3 we have

$$\inf_n \int gT^n \mu dm > 0$$

for those $g \in L_{\varrho'}$ such that $g \cdot f \neq 0$. Since $gT^n\mu \downarrow gh$, we then have $\int ghdm > 0$ for those $g \in L_{\varrho'}$ with $gf \neq 0$. By means of condition (vi) in the definition of a normed Köthe space, we can easily draw the conclusion that $S(h) \supset S(f)$.

Conversely, let $h \neq 0$ be a non-negative fixed point of T, and $f \in L_\varrho^+$ be such that $S(f) = S(h)$. Since

$$\inf_n \int gT^n h dm = \int ghdm > 0$$

for any $g \in L_{\varrho'}^+$ with $gf \neq 0$, by lemma 3 we have

$$\inf_n \int gT^n f dm > 0$$

for all $g \in L_{\varrho'}^+$ with $gf \neq 0$.

In case where the structure of $L_{\varrho'}$ is not clear, the condition $\inf_n \int gT^n f dm > 0$ seems to be useless. But if we consult condition (vi) in the definition of a normed Köthe space, we can easily restate the theorem in the following form:

Let L_ϱ be an absolutely continuous Köthe space, $T\colon L_\varrho \to L_\varrho$ a positive operator with $\sup_n \|T^n\| < \infty$. If there exists a function $f \in L_\varrho^+$ such that

$$\inf_n \int_E T^n f dm > 0$$

for any measurable set E with $m(E \cap S(f)) > 0$, then there is a non-negative function $h \in L_\varrho^+$ with $S(h) \supset S(f)$ and $T(h) = h$.

Conversely, if $h \neq 0$ is a non-negative fixed point of T, then for any $f \in L_\varrho^+$ with $S(f) = S(h)$, we have

$$\inf_n \int_E T^n f dm > 0$$

for all measurable sets E such that $m(E \cap S(f)) > 0$.

Remark. The quantity $\inf_n \int_E T^n f dm$ is not necessary finite in the above theorem.

If there is a strictly positive element f in the normed Köthe space (i.e. $f \geqslant 0$ and $S(f) = X$), then T has a strictly positive fixed point iff $\inf_n \int_E T^n f dm > 0$ for any measurable set E with positive measure.

Since any Orlicz space with Δ_2-property is an absolutely continuous normed Köthe space, this theorem also holds for such an Orlicz space.

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