ANNALES
POLONICI MATHEMATICI
XXV (1972)

The mean surface area of the boxes circumscribed about a convex body

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A convex polytope is said to be *circumscribed about a convex body* if and only if each of its facets is contained in a support plane of the body.

Radziszewski [4] has proved the following theorem:

In the plane, let K be a convex body of area $A(K)$, and let $B(K, \varphi)$ denote the area of the rectangle circumscribed about K such that one of its sides forms an angle φ with a fixed direction. Then the inequality

$$(1)\qquad (2\pi)^{-1}\int_{0}^{2\pi} B(K, \varphi)\,d\varphi \geqslant 4\pi^{-1}A(K)$$

is valid, where equality holds if and only if K is a circular disc.

This inequality was recently rediscovered by Chernoff [2]; see also Heil [3] for some generalizations and consequences.

The left-hand side of (1) may be interpreted as the mean value of the areas of all the rectangles circumscribed about K. In the following we shall prove an analogous inequality in three-dimensional space. The method will be similar to that of Chernoff. It seems that Radziszewski's proof does not immediately extend to higher dimensions, since such an extension would require an, apparently unknown, improved version of a Minkowskian inequality.

Let $S^2 = \{u \in E^3 | \langle u, u\rangle = 1\}$ be the unit sphere of three-dimensional Euclidean space E^3 ($\langle\, ,\rangle$ denotes the scalar product in E^3); let $SO(3)$ denote the rotation group acting on S^2. If (u_1, u_2, u_3) is a fixed orthonormal frame in E^3 and if $\delta \in SO(3)$ is a rotation, then there is a unique box (= rectangular parallelepiped) circumscribed about a given convex body $K \subset E^3$ such that the exterior unit normal vectors of the box are $\pm\delta u_1$, $\pm\delta u_2$, $\pm\delta u_3$. Let $B(K, \delta)$ denote the surface area of this box. Then the mean value of the surface areas of the boxes circumscribed about K is defined in a natural way by

$$\bar{B}(K) = \int_{SO(3)} B(K, \delta)\, d\mu(\delta),$$

where μ denotes the normalized Haar measure on the group $SO(3)$.

THEOREM. *If $K \subset E^3$ is a convex body with surface area $A(K)$, then*

$$\bar{B}(K) \geqslant 6\pi^{-1} A(K). \tag{2}$$

Equality holds if and only if K is a spherical ball.

Proof. Let $p(K, u)$ denote the support function of K, restricted to S^2, that is $p(K, u) = \sup\{\langle x, u\rangle | x \in K\}$ for $u \in S^2$. Let $Y_{m1}, \ldots, Y_{m,2m+1}$ be an orthonormal basis of the real vector space (with the usual scalar product) of spherical surface harmonics of degree m on S^2 $(m = 0, 1, 2, \ldots)$. We will first assume that the support function of K is differentiable sufficiently often, so that it can be represented as an absolutely and uniformly convergent series of spherical harmonics

$$p(K, u) = \sum_{m=0}^{\infty} \sum_{i=1}^{2m+1} a_{mi} Y_{mi}(u),$$

where

$$a_{mi} = \int_{S^2} p(K, u)\, Y_{mi}(u)\, d\omega(u)$$

(ω denotes the Lebesgue measure on S^2). It is well known (see Blaschke [1], p. 108-110) that the surface area of K is now expressed by

$$A(K) = a_{01}^2 - \tfrac{1}{2} \sum_{m=2}^{\infty} (m-1)(m+2) \sum_{i=1}^{2m+1} a_{mi}^2. \tag{3}$$

The functional $B(K, \delta)$ is given by

$$B(K, \delta) = 2\,[b(\delta u_1) b(\delta u_2) + b(\delta u_2) b(\delta u_3) + b(\delta u_3) b(\delta u_1)],$$

where

$$b(u) = p(K, u) + p(K, -u).$$

By the invariance of the Haar measure we obtain

$$\bar{B}(K) = 6 \int_{SO(3)} b(\delta u_1) b(\delta u_2)\, d\mu(\delta),$$

which leads to

$$\bar{B}(K) = 24 \sum_{\substack{m=0\\ 2|m}}^{\infty} \sum_{\substack{k=0\\ 2|k}}^{\infty} \sum_{i=1}^{2m+1} \sum_{j=1}^{2k+1} a_{mi} a_{kj} \int_{SO(3)} Y_{mi}(\delta u_1)\, Y_{kj}(\delta u_2)\, d\mu(\delta)$$

(observe that $Y_{mi}(u) = (-1)^m Y_{mi}(-u)$). Now for any continuous function f on S^2 and any two vectors $u, v \in S^2$ the formula

$$\int_{SO(3)} f(\delta u)\, Y_{mi}(\delta v)\, d\mu(\delta) = (4\pi)^{-1} \int_{S^2} f Y_{mi}\, d\omega\, P_m(\langle u, v\rangle)$$

is valid, where P_m is the Legendre polynomial of degree m. This follows from the fact that the left-hand side is, as a function of v alone, a spherical harmonic of degree m, which, by the invariance of the Haar measure, turns out to depend only on $\langle u, v\rangle$; hence it must be a constant multiple of $P_m(\langle u, v\rangle)$. The factor is then easily determined. (The complete argument may be found in [5], (3.3).) Since $\langle u_1, u_2\rangle = 0$ and since the system $\{Y_{mi}\}$ is orthonormal, this yields

$$\bar{B}(K) = 6\pi^{-1} \sum_{m=0}^{\infty} P_m(0) \sum_{i=1}^{2m+1} a_{mi}^2$$

(we have $P_m(0) = 0$ for odd m). In view of (3) we arrive at

$$\text{(4)} \quad \bar{B}(K) - 6\pi^{-1} A(K) = 6\pi^{-1} \sum_{m=2}^{\infty} [P_m(0) + \tfrac{1}{2}(m-1)(m+2)] \sum_{i=1}^{2m+1} a_{mi}^2 \geqslant 0,$$

since $P_0(0) = 1$ and $|P_m(0)| \leqslant 1$. A standard approximation argument now shows that inequality (2) holds for arbitrary convex bodies K.

Suppose that for a convex body K the equality sign holds in (2). For sufficiently smooth K we have, according to (4),

$$\bar{B}(K) - 6\pi^{-1} A(K)$$
$$\geqslant 6\pi^{-1} [P_m(0) + \tfrac{1}{2}(m-1)(m+2)] \sum_{i=1}^{2m+1} \left\{\int_{S^2} p(K, u)\, Y_{mi}(u)\, d\omega(u)\right\}^2$$

for $m = 2, 3, \ldots$ By approximation, this holds for general K. Since we have assumed the left-hand side to be zero, it follows that

$$\int_{S^2} p(K, u)\, Y_{mi}(u)\, d\omega(u) = 0$$

for $i = 1, 2, \ldots, 2m+1$ and for $m = 2, 3, \ldots$ From the completeness of the system of spherical harmonics we conclude that

$$p(K, \cdot) = Y_0 + Y_1,$$

where Y_k is a spherical harmonic of degree k ($k = 0, 1$). But Y_0 is constant and $Y_1(u) = \langle c, u\rangle$ with some constant vector c; hence K is a spherical ball.

Remarks. The preceding theorem and its proof extend to arbitrary dimensions $n \geqslant 2$ provided the surface area functional is replaced by the quermass integral W_{n-2}. It is, however, an open question whether an analogous result holds for each of the quermass integrals W_k, $0 \leqslant k \leqslant n-3$, especially for the volume W_0. (For W_{n-1}, which is essentially the mean width, there is an identity instead of an inequality.)

Results analogous to the above theorem can be proved if the boxes are replaced by other suitable circumscribed polytopes, for instance by simplices which are similar to a given simplex.

References

[1] W. Blaschke, *Kreis und Kugel*, 2nd ed., Berlin 1956.

[2] P. R. Chernoff, *An area-width inequality for convex curves*, Amer. Math. Monthly 76 (1969), p. 34-35.

[3] E. Heil, *Eine Verschärfung der Bieberbachschen Ungleichung und einige andere Abschätzungen für ebene konvexe Bereiche*, Elem. Math. (to appear).

[4] K. Radziszewski, *Sur une fonctionnelle définie sur les ovales*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A., 10 (1956), p. 57-59.

[5] R. Schneider, *Gleitkörper in konvexen Polytopen*, J. reine angew. Math. 248 (1971), p. 193-200.

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Reçu par la Rédaction le 7. 10. 1970

ERRATUM

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193_{14}	$g_n = \frac{1}{2} f_n$	$g_n = 2f_n$