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# ON A $M/G/1$ QUEUEING MODEL WITH FEEDBACK

**1.** In my earlier paper [5] a queueing model has been studied with the Poisson input parameter $\lambda_n = \lambda[n(t)]$ depending at any moment $t$ on the actual state $n(t)$ of the system. A special case of this kind of feedback occurs e.g. in a model with impatient customers who arrive according to the Poisson process but join the queue only with a probability $\gamma_n$ depending on the state $n$ of the system at the moment of their arrival (see also [8], p. 24). Due to the balking of some customers, those remaining in the system form a Poisson process with parameter $\lambda_n = \gamma_n \lambda$. We considered a single channel model with an arbitrary service time distribution $G(x)$. Let us denote this model by the modified Kendall's symbol $M_N/G/1$ (see [3]) where the subscript $N$ indicates the feedback between the state of the system and the input rate. In this model the number $n(t)$ of units being in the system at any moment $t$ ($t$ is a continuous parameter) is not, in general, a Markov process but the sequence of random variables $N_r = n(t_r+0)$, where $t_r$ ($r = 1, 2, \ldots$) is the departure moment of the $r$-th unit from the service channel, is an imbedded Markov chain (see [4]).

Let us denote by $P_r(n) = P(N_r = n)$ the probability of the state $N_r = n$ at the moment $t_r+0$ and by $P(\xi(x) = k \mid n)$ the conditional probability of $k$ units to enter the system during the time interval of length $x$, given that $n$ units are in the system at the beginning of this interval and that no service terminates during its full length. The main result of paper [5] consists in finding the system of equations

$$(1) \qquad p_n = p_0 P(\xi = n \mid 1) + \sum_{i=1}^{n+1} p_i P(\xi = n-i+1 \mid i),$$

where

$$(2) \qquad P(\xi = k \mid n) = \int_0^\infty P(\xi(x) = k \mid n)\, dG(x)$$

and

$$(3) \qquad p_n = \lim_{r\to\infty} P_r(n).$$

The generating function of the probabilities $p_n$ is of the form

$$\Phi(s) = p_0\varphi(s \mid 1) + \sum_{i=1}^{\infty} s^{i-1}\varphi(s \mid i)p_i, \tag{4}$$

where

$$\varphi(s \mid i) = \sum_{j=0}^{\infty} s^j P(\xi = j \mid i) \quad \text{for} \quad i = 1, 2, \ldots \tag{5}$$

In particular, if $\lambda_i = \lambda$, then

$$\varphi(s \mid i) = \varphi(s) = \int_0^{\infty} \sum_{j=0}^{\infty} s^j \frac{(\lambda x)^j}{j!} e^{-\lambda x} dG(x) = \int_0^{\infty} e^{-\lambda(1-s)x} dG(x) \tag{6}$$

and

$$\Phi(s) = p_0\varphi(s) + \sum_{i=1}^{\infty} s^{i-1}\varphi(s)p_i = \frac{\varphi(s)}{s}[(s-1)p_0 + \Phi(s)]. \tag{7}$$

Therefore

$$\Phi(s) = \frac{p_0(1-s)\varphi(s)}{\varphi(s)-s}. \tag{8}$$

Differentiating formula (7) and setting $s = 1$ we obtain

$$p_0 = 1 - \lambda\int_0^{\infty} x dG(x) = 1 - \frac{\lambda}{\mu} = 1 - \varrho. \tag{9}$$

The explicit form of the probabilities $P(\xi(x) = k \mid n)$ for the case of all $\lambda_i$ being different will be given in theorem 2.

**2.** Different methods have been used in queueing theory to solve problems concerning specified queueing models (see [7]). One of them is the method of imbedded Markov chains though in some applications the knowledge of the limit distribution of the continuous time process describing the work of the system may be of more interest than this of the imbedded Markov chain. In this section it will be shown how the known limit distribution of the imbedded Markov chain may be used to calculate some distributions defined for a process in continuous time. In the case of the $M/G/1$ model one of the problems has been solved by Foster [2]. He has shown that for the $M/G/1$ model without feedback the limit distributions of the states for the continuous time process and for the imbedded Markov chain are identical.

If the initial state of the system (i.e. the state at the moment $\tau_0 = 0$) is a random variable with a distribution equal to the limit one, then the state of the system is a stationary stochastic process. For this kind of stochastic process the probability of a given state may be defined as the fraction of time for the system to remain in this state. This may be expressed by the formula(1)

$$p_j^* = \lim_{n\to\infty} \frac{ES_n^{(j)}}{Er_n}. \tag{10}$$

Here $S_n^{(j)}$ $(j = 0, 1, \ldots)$ stands for the length of time for the system to remain in the state $j$ during the time interval $[0, r_n)$, where $r_n$ is the moment of departure of the $n$-th unit from the system; $ES_n^{(j)}$ is its expected value and $Er_n$ is the expected length of the interval $[0, r_n)$.

Let $M(j \mid i, k, x)$ be the mean length of time in which the system remains in the state $j$ during the time interval between two consecutive exits from the service under the conditions that: 1° the length of the interval is $x$; 2° there are $i$ units in the system at the beginning of the interval, and 3° $k$ units enter the system during the interval in question. In a similar way, by $M(j \mid i)$ we denote the defined mean length of time under the condition 2° only; and by $M(j \mid i, x)$ the same value under the conditions 1° and 2°. We shall prove

THEOREM 1. *In an $M_N/G/1$ system the probabilities of the states (for a stationary process) are of the form*

$$p_0^* = \frac{p_0}{p_0 + \lambda_0/\mu}, \tag{11}$$

*and*

$$p_j^* = \frac{p_0 M(j \mid 1) + \sum_{i=1}^{j} p_i M(j \mid i)}{p_0/\lambda_0 + 1/\mu} \quad \textit{for} \quad j > 0, \tag{12}$$

*where*

$$M(j \mid i) = \int_0^\infty \sum_{k=j-i}^{\infty} P(\xi(x) = k \mid i) M(j \mid i, k, x)\, dG(x). \tag{13}$$

An explicit form of the mean $M(j \mid i, k, x)$, for the case of all $\lambda_i$ being different, will be given in lemma 1 and theorem 2.

Proof of theorem 1. If $X_k^{(j)}$ $(k = 0, 1, \ldots, n-1)$ denotes the length of time in which the system remains in the state $j$ during the time interval $[r_k, r_{k+1})$, then

$$S_n^{(j)} = X_0^{(j)} + X_1^{(j)} + \ldots + X_{n-1}^{(j)}. \tag{14}$$

(1) The probability $p_j^*$ is defined here in a manner somewhat similar to that of the definition of ergodic probability $\tilde{p}_j$ (see [9], p. 85).

We are interested in calculating the expected value

$$ES_n^{(j)} = E(X_0^{(j)}+X_1^{(j)}+\ldots+X_{n-1}^{(j)}) = EX_0^{(j)}+EX_1^{(j)}+\ldots+EX_{n-1}^{(j)}. \tag{15}$$

Now we calculate $EX_k^{(j)}$. For $x \geqslant 0$ we have

$$\begin{aligned} P(X_k^{(j)} \geqslant x) &= \sum_{i=0}^{\infty} P(X_k^{(j)} \geqslant x \mid N_{r_k} = i)P(N_{r_k} = i) \\ &= \sum_{i=0}^{j} P(X_k^{(j)} \geqslant x \mid N_{r_k} = i)P(N_{r_k} = i). \end{aligned} \tag{17}$$

The last equality is based on the fact that in the interval $[r_k, r_{k+1})$ the state of the system is a non-decreasing function of time (see Fig. 1), hence

$$P(X_k^{(j)} \geqslant x \mid N_{r_k} = i) = 0 \quad \text{for} \quad i > j.$$

The conditional mean $M(j \mid i)$ is equal to

$$M(j \mid i) = -\int_0^{\infty} x dP(X_k^{(j)} \geqslant x \mid N_{r_k} = i). \tag{18}$$

Since the chain $N_{r_k}$ is stationary, we have $P(N_{r_0} = i) = p_i$ and

$$P(N_{r_k} = i) = \lim_{k\to\infty} P(N_{r_k} = i) = p_i. \tag{19}$$

From (17), (18) and (19) we get

$$EX_k^{(j)} = \sum_{i=0}^{j} p_i M(j \mid i). \tag{20}$$

From (20) and from (14) follows

$$ES_n^{(j)} = n \sum_{i=0}^{j} p_i M(j \mid i), \tag{21}$$

and, in particular,

$$ES_n^{(0)} = np_0 M(0 \mid 0). \tag{22}$$

By (17), the distribution of the random variable $X_k^{(0)}$ for a Poisson input is

$$P(X_k^{(0)} \geqslant x) = P(X_k^{(0)} \geqslant x \mid N_{r_k} = 0)P(N_{r_k} = 0) = e^{-\lambda_0 x} p_0,$$

whence

$$ES_n^{(0)} = \frac{np_0}{\lambda_0}. \tag{23}$$

Now, let us notice that $M(j\mid 1) = M(j\mid 0)$ for $j \geqslant 1$. This results from the fact that we observe a continuous time process at the moments of output. If at a given moment $r_k$ there is the zero state, then the system is idle till the next input. At that moment the zero state changes into state 1 and a new service begins. Later on the system behaves as if there was no interruption in the service and the system was in the state 1 at the moment $r_k$ (see Fig. 1).

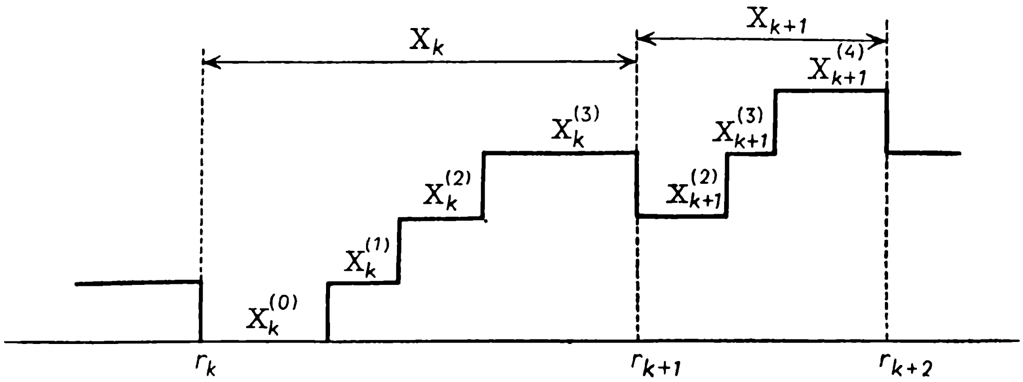


**Fig.1**

The length of the interval $[0, r_n)$ is a random variable:

$$r_n = \sum_{j=0}^{\infty} S_n^{(j)} = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1} X_k^{(j)}.$$

Therefrom and from (20) we get the expected value of $r_n$:

$$(24)\qquad Er_n = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\sum_{i=0}^{j} p_i M(j\mid i) = n\sum_{i=0}^{\infty} p_i \sum_{j=i}^{\infty} M(j\mid i)$$

$$= n\left[p_0\sum_{j=0}^{\infty} M(j\mid 0) + \sum_{j=1}^{\infty} p_i \sum_{j=i}^{\infty} M(j\mid i)\right] = n\left[\frac{p_0}{\lambda_0} + \frac{1}{\mu}\right].$$

The last passage was possible due to the equality $\sum_{j=i}^{\infty} M(j\mid i) = 1/\mu$, where $1/\mu$ is the mean service time. We can write

$$p_0^* = \frac{p_0/\lambda_0}{p_0/\lambda_0 + 1/\mu},$$

$$p_j^* = \frac{p_0 M(j\mid 1) + \sum_{i=1}^{j} p_i M(j\mid i)}{p_0/\lambda_0 + 1/\mu} \qquad \text{for} \quad j \geqslant 1.$$

To calculate $M(j \mid i)$ we notice that if $N_{r_k} = i \geqslant 1$, then the length of the interval $[r_k, r_{k+1})$ is a random variable with the distribution $G(x)$. Therefore we have

$$M(j \mid i) = \int_0^\infty M(j \mid i, x)\, dG(x), \tag{25}$$

where $M(j \mid i, x)$ is the mean length of time in which the system remains in state $j$ during the time interval $[r_k, r_{k+1})$ providing it was in state $i$ at the initial moment $r_k$ and the length of the interval $[r_k, r_{k+1})$ is $x$. The mean $M(j \mid i, x)$ is equal to the sum

$$M(j \mid i, x) = \sum_{k=j-i}^{\infty} P(\xi(x) = k \mid i) M(j \mid i, k, x), \tag{26}$$

where the summation starts with $k = j-i$, since obviously $M(j \mid i, k, x) = 0$ for the smaller values of $k$. Putting (22) to (25) we come to the end of the proof of Theorem 1.

LEMMA 1. *We have*

$$M(j \mid n, k, x) = \frac{M_j(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)}{P(\xi(x) = k \mid n)} \quad \textit{for} \quad n = 0, 1, \ldots, k, \tag{27}$$

*where*

$$M_j(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x) = \lambda_n \lambda_{n+1} \ldots \lambda_{n+k-1} \int_{\Omega} \ldots \int y_j e^{-(\lambda_n y_0 + \ldots + \lambda_{n+k} y_k)} dy_0\, dy_1 \ldots dy_{k-1}$$

*and* $\Omega = \{(y_0, y_1, \ldots, y_k) : y_0 + y_1 + \ldots + y_k = x, y_i \geqslant 0\}$.

Proof. For the Poisson input we have

$$\begin{aligned} M(j \mid n, k, x) &= E(Y_j \mid Y_0 + Y_1 + \ldots + Y_j + \ldots + Y_k = x) \\ &= \frac{\int_{\Omega} \ldots \int y_j e^{-(\lambda_n y_0 + \lambda_{n+1} y_1 + \ldots + \lambda_{n+k} y_k)} dy_0\, dy_1 \ldots dy_{k-1}}{\int_{\Omega} \ldots \int e^{-(\lambda_n y_0 + \lambda_{n+1} y_1 + \ldots + v_{n+k} y_k)} dy_0\, dy_1 \ldots dy_{k-1}} \\ &= \frac{M_j(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)}{P(\xi(x) = k \mid n)} \quad \text{for} \quad n = 0, 1, \ldots, k. \end{aligned}$$

Here $Y_0, Y_1, \ldots, Y_k$ are independent random variables with a common exponential distribution $P(Y_j < y) = 1 - e^{-\lambda_{n+j} y}$.

THEOREM 2. *If $\lambda_i \neq \lambda_j$ for $i \neq j$, then*

$$(28)\qquad P(\xi(x)=k\mid n)=\sum_{j=0}^{k}\frac{\lambda_{n+j}}{\lambda_{n+k}}\Big[\prod_{\substack{i=0\\ i\neq j}}^{k}\Big(1-\frac{\lambda_{n+j}}{\lambda_{n+i}}\Big)\Big]^{-1}e^{-\lambda_{n+j}x};$$

$$M_s(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)$$

$$=\sum_{\substack{j=0\\ j\neq s}}^{k}\frac{\lambda_{n+j}}{\lambda_{n+k}}\Big[\prod_{\substack{i=0\\ i\neq j}}^{k}\Big(1-\frac{\lambda_{n+j}}{\lambda_{n+i}}\Big)\Big]^{-1}\Big[\frac{1}{\lambda_{n+s}-\lambda_{n+j}}(e^{-\lambda_{n+j}x}-e^{-\lambda_{n+s}x})-xe^{-\lambda_{n+s}x}\Big]$$

*for $0 \leqslant s \leqslant k-1$ and*

$$xP(\xi(x)=k\mid n)=\sum_{j=0}^{k}M_j(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x).$$

Proof of theorem 2. It has been proved in [5] that

$$(29)\qquad P(\xi(x)=0\mid n)=e^{-\lambda_n x}$$

and

$$(30)\qquad P(\xi(x)=k\mid n)=\lambda_n\lambda_{n+1}\ldots\lambda_{n+k-1}\int\limits_{\Omega}\cdots\int e^{-(\lambda_n y_0+\lambda_{n+1}y_1+\ldots+\lambda_{n+k}y_k)}\,dy_0\,dy_1\ldots dy_{k-1}$$

for $k \geqslant 1$. Expression (30) is a convolution of $k+1$ functions:

$$\lambda_n e^{-\lambda_n v},\ \lambda_{n+1}e^{-\lambda_{n+1}v},\ \ldots,\ \lambda_{n+k-1}e^{-\lambda_{n+k-1}v},\ e^{-\lambda_{n+k}v}.$$

Its transform is thus equal to

$$\int_0^\infty P(\xi(x)=k\mid n)e^{-sx}dx=\frac{\lambda_n}{\lambda_n+s}\,\frac{\lambda_{n+1}}{\lambda_{n+1}+s}\cdots\frac{\lambda_{n+k-1}}{\lambda_{n+k-1}+s}\,\frac{1}{\lambda_{n+k}+s}=\frac{p}{q(s)}.$$

Decomposing this into partial fractions on the condition that $\lambda_i \neq \lambda_j$ for $i \neq j$ we get (see formula (136) in [1], p. 148)

$$(31)\qquad \frac{p}{q(s)}=\sum_{j=0}^{k}\frac{p}{q'(\lambda_{n+j})}\,\frac{1}{\lambda_{n+j}+s}$$

since

$$q'(\lambda_{n+j})=\prod_{\substack{i=0\\ i\neq j}}^{k}(\lambda_{n+i}-\lambda_{n+j}).$$

This completes the proof of formula (28). Another proof was published in paper [6].

Now, let us derive the explicit expression for $M_j(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)$. From lemma 1 for $j = 0$ we get

$$M_0(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x) = \lambda_n \lambda_{n+1} \ldots \lambda_{n+k-1} \int_{\Omega}\ldots\int y_0 e^{-(\lambda_n y_0 + \lambda_{n+1} y_1 + \ldots + \lambda_{n+k} y_k)} dy_0\, dy_1 \ldots dy_{k-1}$$

$$= \lambda_n \int_0^x y_0 e^{-\lambda_n y_0} P(\xi(x-y_0) = k-1 \mid n+1) dy_0$$

$$= \lambda_n \int_0^x \sum_{j=1}^{k} \frac{\lambda_{n+j}}{\lambda_{n+k}} \Big[\prod_{\substack{i=1\\ i\neq j}}^{k} \Big(1-\frac{\lambda_{n+j}}{\lambda_{n+i}}\Big)\Big]^{-1} e^{-\lambda_{n+j}x} y_0 e^{-(\lambda_n-\lambda_{n+j})y_0} dy_0$$

$$= \sum_{j=1}^{k} \frac{\lambda_{n+j}}{\lambda_{n+k}} \Big[\prod_{\substack{i=0\\ i\neq j}}^{k} \Big(1-\frac{\lambda_{n+j}}{\lambda_{n+i}}\Big)\Big]^{-1} \Big[\frac{1}{\lambda_n-\lambda_{n+j}} (e^{-\lambda_{n+j}x} - e^{-\lambda_n x}) - x e^{-\lambda_n x}\Big].$$

In general, for $0 \leqslant s \leqslant k-1$, it is easy to obtain

$$M_s(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)$$

$$= \sum_{\substack{j=0\\ j\neq s}}^{k} \frac{\lambda_{n+j}}{\lambda_{n+k}} \Big[\prod_{\substack{i=0\\ i\neq j}}^{k} \Big(1-\frac{\lambda_{n+j}}{\lambda_{n+i}}\Big)\Big]^{-1} \Big[\frac{1}{\lambda_{n+s}-\lambda_{n+j}} (e^{-\lambda_{n+j}x} - e^{-\lambda_{n+s} x}) - x e^{-\lambda_{n+s} x}\Big].$$

For $s = k$ we have

$$M_k(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x)$$

$$\stackrel{\text{df}}{=} \lambda_n \lambda_{n+1} \ldots \lambda_{n+k-1} \int_{\Omega}\ldots\int (x-y_0-y_1-\ldots-y_{k-1}) e^{-(\lambda_n y_0 + \lambda_{n+1} y_1 + \ldots + \lambda_{n+k} y_k)} dy_0\, dy_1 \ldots dy_{k-1}$$

$$= xP(\xi(x) = k \mid n) - \sum_{s=0}^{k-1} M_s(\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}; x).$$

This completes the proof of theorem 2.

**3.** Foster [2] has proved for the model $M/G/1$ without feedback the identity of the limit distributions of the imbedded Markov chain and of the continuous time process. In this section we shall prove that also the probabilities $p_j^*$ are of identical form.

THEOREM 3. *In the $M/G/1$ model the probabilities $p_j$ and $p_j^*$ are equal for any $j$* (²).

(²) We owe this theorem to C. Ryll-Nardzewski, who has kindly helped us to generalize the obtained partial results on the identity in question.

Proof. From Theorem 1 and from formula (9) we get

$$p_0^* = \frac{p_0}{p_0+\lambda/\mu} = 1-\frac{\lambda}{\mu} = p_0. \tag{32}$$

Now we shall show that the generating functions of the probabilities $p_j$ and $p_j^*$ are equal. In the case of Poisson input with a constant arrival rate $\lambda$ we infer, by Lemma 1, that

$$M(j \mid n, k, x) = \frac{\int\limits_{\Omega}\dots\int y_j e^{-\lambda x} dy_0\, dy_1 \dots dy_{k-1}}{\int\limits_{\Omega}\dots\int e^{-\lambda x} dy_0\, dy_1 \dots dy_{k-1}} = \frac{x^{k+1}}{(k+1)!} \Big/ \frac{x^k}{k!} = \frac{x}{k+1}.$$

Since

$$P(\xi(x) = k \mid i) = \frac{(\lambda x)^k}{k!} e^{-\lambda x}$$

by (13) in theorem 1, we have

$$M(j \mid i) = \int\limits_0^\infty \sum_{k=j-i}^{\infty} \frac{\lambda^k x^{k+1}}{(k+1)!} e^{-\lambda x} dG(x).$$

Now, the generating function of the sequence $M(j \mid i)$ is equal to

$$\sum_{j=i}^{\infty} s^j M(j \mid i) = \frac{s^i}{\lambda} \int\limits_0^\infty \sum_{j=0}^{\infty} s^j \sum_{k=j}^{\infty} \frac{(\lambda x)^{k+1}}{(k+1)!} e^{-\lambda x} dG(x)$$

$$= \frac{s^i}{\lambda} \int\limits_0^\infty \frac{1}{1-s} [e^{\lambda x} - e^{\lambda x s}] e^{-\lambda x} dG(x)$$

$$= \frac{s^i}{\lambda(1-s)} \left[1 - \int\limits_0^\infty e^{-\lambda x(1-s)} dG(x)\right] = \frac{s^i}{\lambda} \frac{1-\varphi(s)}{1-s}.$$

In view of (32) the generating function $\Phi^*(s)$ of the probabilities $p_j^*$ may be obtained as follows:

$$\Phi^*(s) = \sum_{j=0}^{\infty} s^j p_j^* = p_0 + \lambda \sum_{j=1}^{\infty} s^j \left[p_0\, M(j \mid 1) + \sum_{i=1}^{j} p_i\, M(j \mid i)\right]$$

$$= p_0 + \frac{1-\varphi(s)}{1-s} \left(s p_0 + \sum_{i=1}^{\infty} s^i p_i\right)$$

$$= \big[p_0 + \big(1 - \varphi(s)\big)\big] \left[\frac{\Phi(s)}{1-s} - p_0\right].$$

If we recall now formula (8), then we can write

$$\Phi^*(s) = [p_0 + (1-\varphi(s))]\left[\frac{p_0\varphi(s)}{\varphi(s)-s} - p_0\right] = \frac{p_0(1-s)\varphi(s)}{\varphi(s)-s} = \Phi(s).$$

We have thus obtained the equality $\Phi^*(s) = \Phi(s)$, which completes the proof of theorem 3.

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## MODEL $M/G/1$ ZE SPRZĘŻENIEM ZWROTNYM

### STRESZCZENIE


Rozpatrzmy system typu $M/G/1$ ze sprzężeniem zwrotnym, tj. system, w którym zakładamy (a) poissonowski strumień zgłoszeń z parametrem $\lambda_n = \lambda[n(t)]$ zależnym od aktualnego stanu $n(t)$ systemu w danym momencie $t$, (b) nieograniczoną kolejkę, oraz (c) pojedynczy kanał obsługi z dowolnym rozkładem czasu obsługi $G(x)$.

W pracy [5] opisaliśmy ten model przy użyciu włożonego łańcucha Markowa zdefiniowanego w momentach zakończenia obsługi w systemie. Znaleźliśmy wówczas graniczne prawdopodobieństwa $p_j$ tego łańcucha. W tej pracy zdefiniowane zostały prawdopodobieństwa stanów systemu $p_j^*$ rozumiane jako frakcja czasu przebywania systemu w danym stanie,

$$p_j^* = \lim_{n\to\infty} \frac{ES_n^{(j)}}{Er_n},$$

gdzie $ES_n^{(j)}$ dla $j = 0, 1, 2, \ldots$ oznacza średni czas przebywania systemu w stanie $j$ w przedziale czasowym $[0, r_n)$, natomiast $r_n$ oznacza moment wyjścia $n$-tej jednostki z systemu, a $Er_n$ — oczekiwaną długość przedziału $[0, r_n)$.

Udowodniono wzory (11) i (12) w których prawdopodobieństwa $p_j^*$ zostały przedstawione jako funkcje prawdopodobieństw stanów włożonego łańcucha Markowa oraz wyrażeń $P(\xi(x) = k \mid i)$ i $M(j \mid i, k, x)$, gdzie $P(\xi(x) = k \mid i)$ oznacza prawdopodobieństwo $k$ zgłoszeń w czasie trwania obsługi o długości $x$, gdy na początku tego przedziału znajduje się $i$ jednostek w systemie. $M(j \mid i, k, x)$ oznacza średni czas pobytu systemu w stanie $j$ w przedziale czasowym o długości $x$, gdy na początku tego przedziału jest $i$ jednostek w systemie oraz nastąpi $k$ zgłoszeń w tym przedziale. Wyrażenia te zostały obliczone w lemacie 1 i twierdzeniu 2.

W systemie $M/G/1$, w którym intensywność strumienia zgłoszeń jest stała, udowodniono, że prawdopodobieństwa $p_j$ są identyczne z uprzednio zdefiniowanymi wyrażeniami $p_j^*$.

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## МОДЕЛЬ $M/G/1$ С ЗАПАЗДЫВАЮЩЕЙ ОБРАТНОЙ СВЯЗЬЮ

### РЕЗЮМЕ

Рассматривается система типа $M/G/1$ с запаздывающей обратной связью, это есть система охарактеризованная (а) пуассоновским потоком заявок с параметром $\lambda_n = \lambda[n(t)]$ зависимым от актуального состояния $n(t)$ системы во время $t$, (б) неограниченной очередью, и (в) одним каналом обслуживания с произвольным распределением $G(x)$ времени обслуживания.

В работе [5] эта модель была исследована методом вложенной цепи Маркова, определенной на моментах окончания обслуживания в системе. В работе были найдены предельные вероятности состояний $p_j$. В настоящей работе определяются вероятности состояний системы $p_j^*$ как относительные времена пребывания системы в данном состоянии:

$$p_j^* = \lim_{n\to\infty} \frac{ES_n^{(j)}}{Er_n},$$

где $ES_n^{(j)}, j = 0, 1, 2, \ldots$ — среднее время пребывания системы в состоянии $j$ до момента $r_n, r_n$ — момент окончания обслуживания $n$-той заявки, $Er_n$ — математическое ожидание момента $r_n$. Автором доказаны формулы (11) и (12), в которых вероятности $p_j^*$ представлены в зависимости от вероятностей состояний вложенной цепи Маркова и величин $P(\xi(x) = k \mid i)$ и $M(j \mid i, k, x)$, где $P(\xi(x) = k \mid i)$ — вероятность $k$ заявок во время обслуживания длины $x$, если в начале времени обслуживания в системе было $i$ заявок, $M(j \mid i, k, x)$ — среднее время пребывания системы в состоянии $j$ в интервале времени длины $x$, если в начале этого интервала $i$ заявок находилось в системе и $k$ новых заявок появилось во время этого интервала. Эти величины выражены явно в лемме 1 и теореме 2.

Для системы $M/G/1$ с постоянной интенсивностью потока заявок доказано, что вероятности $p_j$ совпадают с выше определенными величинами $p_j^*$.