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On the uniqueness of the solution of the equation of transverse vibrations of a plate

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1. Introduction. We define a plate [1] as an elastic, material, flat surface in repose, which, while bent, gains a potential energy called the elastic energy, equal to the integral of a certain quadratic form of the main curvatures of the bent surface. This definition gives the model of a plate called the thin plate by the theory of elasticity. In the case of the isotropic plate, the elastic energy is expressed by the formula

$$U = \frac{1}{2}\iint\limits_{\Omega} D(x,y)\left(\frac{1}{\varrho_1^2}+\frac{1}{\varrho_2^2}+2\nu\frac{1}{\varrho_1\cdot\varrho_2}\right)dx\,dy \tag{1}$$

where ϱ_1 and ϱ_2 are the main radii of the curvature of the deflected plate, $D(x,y)$ is a non-negative function, defining the rigidity of the plate against bending, ν is the so-called Poisson ratio $(0<\nu<0.5)$ and Ω corresponds to the domain of the plate.

In the present paper we give the proof of the uniqueness of the solution of the problem of transverse vibrations of a plate—when applied to a given group of boundary conditions. The method of the proof is related to the method applied to hyperbolic equations by S. Zaremba, K. Friedrichs and others (cf. [2]) and to that used in the paper of A. Dawidowicz [3]. In this article we present the physical interpretation of the proof and point to the connection existing between the method of the proof and the derivation of the equation of the plate and of the boundary conditions from Hamilton's principle.

2. The differential equation of transverse vibrations of a plate and of boundary conditions as derived from Hamilton's principle. Let us consider the problem of transverse vibrations of a plate assuming that the vibrations are small enough to permit the assumption [1]

$$\frac{1}{\varrho_1}+\frac{1}{\varrho_2} = \Delta w, \quad \frac{1}{\varrho_1\varrho_2} = \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y^2}\right) \tag{2}$$

where $w(x,y,t)$ = the function of deflection of the plate.

By the said assumptions, the elastic energy of the plate is expressed by the formula

$$(3) \qquad U = \frac{1}{2}\iint\limits_{\Omega} D(x,y)\left\{(\Delta w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\,dy\,.$$

Yet, if a) the function $p(x, y, t)$ defines the strain of external forces acting along the vertical axis upon the surface of the plate, b) $q(s, t)$ is the function of the strain of normal forces acting upon the edge of the plate, and c) $m(s, t)$ corresponds to the strain of the bending moments acting upon the edge of the plate on planes normal to the edge—then the form of the potential energy of the external forces will be:

$$(4) \qquad U_1 = -\iint\limits_{\Omega} p(x,y,t)\,w\,dx\,dy - \int\limits_{F\Omega} q(s,t)\,w\,ds - \int\limits_{F\Omega} m(s,t)\frac{\partial w}{\partial n}\,ds\,.$$

The kinetic energy of the plate is expressed by

$$(5) \qquad E = \frac{1}{2}\iint\limits_{\Omega} \mu(x,y)\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy$$

where $\mu(x, y) =$ the surface density, being a function continuous and positive in Ω.

As we know the formulae E, U and U_1, we can use Hamilton's principle as a starting point when deriving the differential equation of the transverse vibrations of the plate:

$$(6) \qquad \delta\int\limits_{t_1}^{t_2}[E-(U+U_1)]\,dt = 0\,.$$

The variation of the first term leads to

$$(7) \qquad \delta\int\limits_{t_1}^{t_2} E\,dt = \int\limits_{t_1}^{t_2} dt\iint\limits_{\Omega}\mu\frac{\partial w}{\partial t}\cdot\frac{\partial\delta w}{\partial t}\,dx\,dy = -\int\limits_{t_1}^{t_2} dt\iint\limits_{\Omega}\mu\frac{\partial^2 w}{\partial t^2}\,\delta w\,dx\,dy$$

since, by Hamilton's principle, $\delta w = 0$ for $t = t_1$ and $t = t_2$. The variation of the second sub-integral term yields

$$(8) \qquad \delta U = \iint\limits_{\Omega} D\Big\{\Delta w\,\Delta\delta w + \\ + (1-\nu)\left[2\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2\delta w}{\partial x\partial y} - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2\delta w}{\partial y^2} - \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2\delta w}{\partial x^2}\right]\Big\}\,dx\,dy$$

and the variation of the third sub-integral term gives

$$(9) \qquad \delta U_1 = -\iint\limits_{\Omega} p\,\delta w\,dx\,dy - \int\limits_{F\Omega} \left(q\,\delta w + m\frac{\partial \delta w}{\partial n}\right) ds\,.$$

According to Hamilton's principle we therefore obtain

$$(10) \qquad \int\limits_{t_1}^{t_2} dt \iint\limits_{\Omega} \Bigg(\mu\frac{\partial^2 w}{\partial t^2}\,\delta w - p\,\delta w + {} \\ + D\left\{\Delta w\,\Delta\delta w + (1-\nu)\left[2\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2 \delta w}{\partial x\partial y} - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial y^2} - \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 \delta w}{\partial x^2}\right]\right\}\Bigg) dx\,dy - \\ - \int\limits_{t_1}^{t_2} dt \int\limits_{F\Omega} \left(q\,\delta w + m\frac{\partial \delta w}{\partial n}\right) ds = 0\,.$$

We convert the left side of equation (10) by applying Green's formula to expression (8). Therefore we assume that $D(x, y)$ is the function of class C^2 in Ω and Ω is a normal domain with respect to the two axes x and y.

Considering the apparent identities

$$D\,\Delta w\,\Delta\delta w \equiv \Delta(D\,\Delta w)\,\delta w + \\ + \frac{\partial}{\partial x}\left[D\,\Delta w\frac{\partial \delta w}{\partial x} - \frac{\partial}{\partial x}(D\,\Delta w)\,\delta w\right] + \\ + \frac{\partial}{\partial y}\left[D\,\Delta w\frac{\partial \delta w}{\partial y} - \frac{\partial}{\partial y}(Dw)\,\delta w\right],$$

$$(11) \qquad 2D\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2 \delta w}{\partial x\partial y} \equiv 2\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)\delta w + \\ + \frac{\partial}{\partial x}\left[D\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial \delta w}{\partial y} - \frac{\partial}{\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)\delta w\right] + \\ + \frac{\partial}{\partial y}\left[D\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial \delta w}{\partial x} - \frac{\partial}{\partial x}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)\delta w\right],$$

$$D\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial y^2} \equiv \frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)\delta w + \\ + \frac{\partial}{\partial y}\left[D\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial \delta w}{\partial y} - \frac{\partial}{\partial y}\left(D\frac{\partial^2 w}{\partial x^2}\right)\delta w\right],$$

$$D\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 \delta w}{\partial x^2} \equiv \frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)\delta w + \\ + \frac{\partial}{\partial x}\left[D\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial \delta w}{\partial x} - \frac{\partial}{\partial x}\left(D\frac{\partial^2 w}{\partial y^2}\right)\delta w\right]$$

we obtain from Green's expression the equality

$$(8')\qquad \delta U = \iint_{\Omega} \Big\{\Delta(D\Delta w) + (1-\nu)\Big[2\frac{\partial^2}{\partial x\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big) - \frac{\partial^2}{\partial x^2}\Big(D\frac{\partial^2 w}{\partial y^2}\Big) - \frac{\partial^2}{\partial y^2}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}\delta w\,dx\,dy +$$

$$+\int_{F\Omega}\Bigg(\Big\{D\Delta w\frac{\partial\delta w}{\partial x} - \frac{\partial}{\partial x}(D\Delta w)\delta w + (1-\nu)\Big[D\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial\delta w}{\partial y} - \frac{\partial}{\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big)\delta w - D\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial\delta w}{\partial x} + \frac{\partial}{\partial x}\Big(D\frac{\partial^2 w}{\partial y^2}\Big)\delta w\Big]\Big\}\cos(n,x) +$$

$$+\Big\{D\Delta w\frac{\partial\delta w}{\partial y} - \frac{\partial}{\partial y}(D\Delta w)\delta w + (1-\nu)\Big[D\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial\delta w}{\partial x} - \frac{\partial}{\partial x}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big)\delta w - D\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial\delta w}{\partial y} + \frac{\partial}{\partial y}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\delta w\Big]\Big\}\cos(n,y)\Bigg)ds$$

where $\bar{n}$ is the external normal and the direction of the boundary is positive (fig. 1). Substituting the operators $\partial/\partial n$ and $\partial/\partial s$ for the operators $\partial/\partial x$ and $\partial/\partial y$, we obtain a different form of expression (8):

$$(8'')\qquad \delta U = \iint_{\Omega} \Big\{(D\Delta w) + (1-\nu)\Big[2\frac{\partial^2}{\partial x\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big) - \frac{\partial^2}{\partial x^2}\Big(D\frac{\partial^2 w}{\partial y^2}\Big) - \frac{\partial^2}{\partial y^2}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}\delta w\,dx\,dy +$$

$$+\int_{F\Omega} D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big)\frac{\partial\delta w}{\partial n}ds -$$

$$-\int_{F\Omega}\Big\{\frac{\partial}{\partial n}\Big[D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big)\Big] + (1-\nu)\frac{\partial}{\partial s}\Big(D\frac{\partial^2 w}{\partial n\partial s}\Big)\Big\}\delta w\,ds +$$

$$+\int_{F\Omega} D(1-\nu)\frac{\partial^2 w}{\partial n\partial s}\cdot\frac{\partial\delta w}{\partial s}ds\,.$$

Since the last integral has been integrated along a closed curve, and the functions $D\dfrac{\partial^2 w}{\partial n\partial s}$ and δw are unique, consequently we have the equality

$$(12)\qquad \int_{F\Omega} D(1-\nu)\frac{\partial^2 w}{\partial n\partial s}\cdot\frac{\partial\delta w}{\partial s}ds = -\int_{F\Omega}(1-\nu)\frac{\partial}{\partial s}\Big(D\frac{\partial^2 w}{\partial n\partial s}\Big)\delta w\,ds$$

and accordingly the form of expression (8) becomes

$$(8''')\quad \delta U = \iint_{\Omega} \Big\{\Delta(D\Delta w) + (1-\nu)\Big[2\frac{\partial^2}{\partial x\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big) - \frac{\partial^2}{\partial x^2}\Big(D\frac{\partial^2 w}{\partial y^2}\Big) - \frac{\partial^2}{\partial y^2}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}\delta w\,dx\,dy + \int_{F\Omega} D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big)\frac{\partial \delta w}{\partial n}ds - \int_{F\Omega}\Big\{\frac{\partial}{\partial n}\Big[D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big)\Big] + 2(1-\nu)\frac{\partial}{\partial s}\Big(D\frac{\partial^2 w}{\partial n\partial s}\Big)\Big\}\delta w\,ds\,.$$

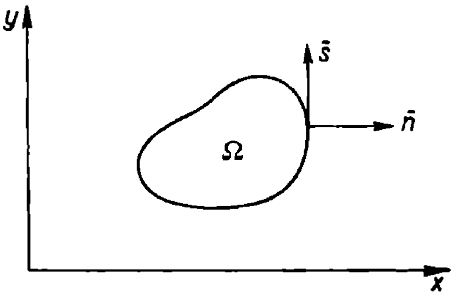

Fig. 1

By (8''') instead of equality (10) we obtain the equivalent (1) equality

$$(13)\quad \int_{t_1}^{t_2} dt \iint_{\Omega} \Big\{\mu\frac{\partial^2 w}{\partial t^2} - p + \Delta(D\Delta w) + (1-\nu)\Big[2\frac{\partial^2}{\partial x\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big) - \frac{\partial^2}{\partial x^2}\Big(D\frac{\partial^2 w}{\partial y^2}\Big) - \frac{\partial^2}{\partial y^2}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}\delta w\,dx\,dy + \int_{t_1}^{t_2} dt\int_{F\Omega}\Big[D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big) - m\Big]\frac{\partial \delta w}{\partial n}ds - \int_{t_1}^{t_2} dt\int_{F\Omega}\Big\{\frac{\partial}{\partial n}\Big[D\Big(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2}\Big)\Big] + 2(1-\nu)\frac{\partial}{\partial s}\Big(D\frac{\partial^2 w}{\partial n\partial s}\Big) + q\Big\}\delta w\,ds = 0\,.$$

Taking advantage of Lagrange and Du Bois-Reymond lemma, we obtain from (13) the following equalities:

(1) Hamilton's principle, assuming that $\delta w = 0$ for $t = t_1$ and $t = t_2$, does not interfere with the proof of the equivalence of (10) and (13).

$$(14)\qquad \Delta(D\Delta w)+(1-\nu)\left[2\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)-\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)-\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)\right]+$$

$$+\mu\frac{\partial^2 w}{\partial t^2}=p(x,y,t)\quad \text{for}\quad (x,y)\in\Omega\quad\text{and}\quad t\geqslant 0\,,$$

$$(15)\qquad \left|\begin{array}{l} D\left(\dfrac{\partial^2 w}{\partial n^2}+\nu\dfrac{\partial^2 w}{\partial s^2}\right)-m=0\quad \text{or}\quad \dfrac{\partial w}{\partial n}=0\,,\quad\text{and}\\ \dfrac{\partial}{\partial n}\left[D\left(\dfrac{\partial^2 w}{\partial n^2}+\nu\dfrac{\partial^2 w}{\partial s^2}\right)\right]+2(1-\nu)\dfrac{\partial}{\partial s}\left(D\dfrac{\partial^2 w}{\partial n\partial s}\right)+q=0\quad\text{or}\quad w=0\\ \qquad\qquad\qquad\text{for}\quad (x,y)\in F\Omega\quad\text{and}\quad t\geqslant 0\,(^2)\,. \end{array}\right.$$

Equation (14) is the differential equation of transverse vibrations of a plate of variable rigidity, whereas the dependencies (15) form the group of boundary conditions of our problem. E.g. the boundary conditions of a plate supported freely along its whole adge are as follows:

$$w(x,y,t)=0\,,$$
$$D(x,y)\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)-m(s,t)=0\quad\text{for}\quad (x,y)\in F\Omega\quad\text{and}\quad t\geqslant 0;$$

this means that the deflection on the contour of the plate is equal to 0; that the sum of the bending moment of the plate on its edge

$$m_n=-D(x,y)\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)$$

and of the moment acting opon the contour of the plate in planes normal to it with the strain of $m(s,t)$ equals 0.

Likewise, $w(x,t)=0$, and $\partial w(x,y,t)/\partial n=0$ for $(x,y)\in F\Omega$ and $t\geqslant 0$ are the boundary conditions of the plate fixed along its whole boundary; and

$$D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)-m=0\,,$$

$$\frac{\partial}{\partial n}\left[D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)\right]+2(1-\nu)\frac{\partial}{\partial s}\left(D\frac{\partial^2 w}{\partial n\partial s}\right)+q=0$$

for $(x,y)\in F\Omega$ and $t\geqslant 0$ represent the boundary conditions of a plate free along its whole edge, upon which act forces normal to the surface of the plate, their strain being $q(s,t)$, and moments with strains of $m(s,t)$.

([2]) Conditions (15) on the boundary $F\Omega$ can be dulfilled gradually, by several steps.

But

$$\frac{\partial w(x, y, t)}{\partial n} = 0 ,$$

$$\frac{\partial}{\partial n}\left[D\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2}\right)\right] + 2(1-\nu)\frac{\partial}{\partial s}\left(D\frac{\partial^2 w}{\partial n \partial s}\right) + q = 0$$

for $(x, y) \in F\Omega$ and $t \geqslant 0$ are the boundary conditions of a plate fixed on its edge, with free displacement transverse to the surface of the plate, the forces acting upon the edge having the strain of $q(s, t)$.

Besides the aforesaid four kinds of boundary conditions, the group of boundary conditions (15) comprises also the cases where one segment of the contour is free, another being fixed, etc.

When deducing equation (14) and conditions (15), besides the assumptions concerning the domain Ω and the functions $D(x, y)$ and $\mu(x, y)$, we have taken for granted that the function $w(x, t)$ belongs to class C^2 in the semi-cylinder Σ: $\{(x, y) \in \Omega,\ t \geqslant 0\}$ and that it has continuous derivatives of fourth order with respect to the variables x and y in Σ, and that the function δw belongs to class C^2 in Σ.

3. The uniqueness proof of the problem of transverse vibrations of a plate.

THEOREM 1. *The differential equation* (14) *when the initial conditions are given*

$$\text{16)}\qquad w(x, y, 0) = \varphi_0(x, y), \qquad \left.\frac{\partial w(x, y, t)}{\partial t}\right|_{t=0} = \varphi_1(x, y) \qquad \text{for} \qquad (x, y) \in \Omega$$

and the boundary conditions chosen from group (15)—*has at most one solution within the range of the functions of class C^2 with 4th order continuous derivatives with respect to spatial variables in the semi-cylinder Σ: $\{(x, y) \in \Omega, \Omega$ normal domain relative to axes x and y, $t \geqslant 0\}$* ([3]).

To verify this theorem, we only have to prove the following theorem:

THEOREM 2. *The only solution of the partial differential homogeneous equation*

$$\text{(17)}\qquad \Delta(D\Delta w) + (1-\nu)\left[2\frac{\partial^2}{\partial x \partial y}\left(D\frac{\partial^2 w}{\partial x \partial y}\right) - \frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial y^2}\right) - \frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)\right] + \mu\frac{\partial^2 w}{\partial t^2} = 0$$

by homogeneous initial conditions

$$\text{(18)}\qquad w(x, y, 0) = 0, \qquad \left.\frac{\partial w(x, y, t)}{\partial t}\right|_{t=0} = 0$$

([3]) The normally of the domain can be weakened to such an extent as to make possible the application of Green's theorem.

and established homogeneous boundary conditions chosen from the group

$$D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)=0 \quad \textit{or} \quad \frac{\partial w}{\partial n}=0 \quad \textit{and}$$

(19) $$\frac{\partial}{\partial n}\left[D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)\right]+2(1-\nu)\frac{\partial}{\partial s}\left(D\frac{\partial^2 w}{\partial n\partial s}\right)=0 \quad \textit{or} \quad w=0$$

$$\textit{for} \quad (x, y)\in F\Omega \quad \textit{and} \quad t\geqslant 0$$

within the range of functions of class C^3 with continuous derivatives of 4th order with respect to the spatial variables in the semi-cylinder is the function $w(x, y, t)=0$ in Σ.

Proof. We will integrate over the area (fig. 2) Θ: $\{(x, y)\in\Omega,$ $0\leqslant t\leqslant T\}$ (where T is an arbitrary number $0<T<\infty$) the expression

$$\frac{\partial w}{\partial t}\left\{\Delta(D\Delta w)+(1-\nu)\left[2\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)-\right.\right.$$
$$\left.\left.-\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)-\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)\right]+\mu\frac{\partial^2 w}{\partial t^2}\right\}$$

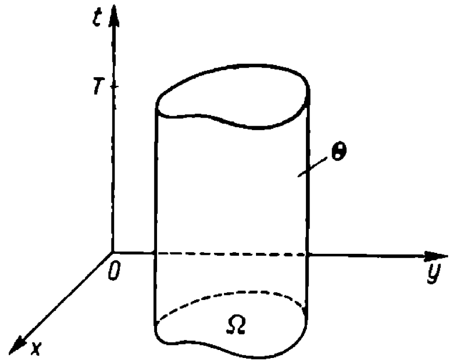

Fig. 2

and, as the function $w(x, y, t)$ is the solution of the homogeneous equation (17), the above expression is equal to 0 within the area Θ. Moreover, we will integrate over the surface Γ: $\{F\Omega\times[0, T]\}$ the expressions

$$\frac{\partial^2 w}{\partial n\partial t}D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right),$$

$$-\frac{\partial w}{\partial t}\left\{\frac{\partial}{\partial n}\left[D\left(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)\right]+2(1-\nu)\frac{\partial}{\partial s}\left(D\frac{\partial^2 w}{\partial n\partial s}\right)\right\},$$

which are also equal to 0 on the surface Γ, since the function $w(x, y, t)$ fulfils the boundary conditions chosen arbitrarily from (19) (e.g.: if $\partial w/\partial n=0$ for $(x, y)\in F\Omega$ and for $t\geqslant 0$, then $\partial^2 w/\partial n\partial t=0$ for $(x, y)\in F\Omega$ and $t\geqslant 0$).

Summing up the integrals thus constructed, we obtain the equality

$$(20)\quad \int_0^T dt \iint_\Omega \Big\{\mu\frac{\partial^2 w}{\partial t^2}+\Delta(D\Delta w)+(1-\nu)\Big[2\frac{\partial^2}{\partial x\partial y}\Big(D\frac{\partial^2 w}{\partial x\partial y}\Big)- -\frac{\partial^2}{\partial x^2}\Big(D\frac{\partial^2 w}{\partial y^2}\Big)-\frac{\partial^2}{\partial y^2}\Big(D\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}\frac{\partial w}{\partial t}\,dx\,dy+$$

$$+\int_0^T dt\int_{F\Omega} D\Big(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\Big)\frac{\partial^2 w}{\partial n\partial t}\,ds-$$

$$-\int_0^T dt\int_{F\Omega}\Big\{\frac{\partial}{\partial n}\Big[D\Big(\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}\Big)\Big]+2(1-\nu)\frac{\partial}{\partial s}\Big(D\frac{\partial^2 w}{\partial n\partial s}\Big)\Big\}\frac{\partial w}{\partial t}\,ds=0\,.$$

Equality (20) is a particular case of equality (13) (with $p\equiv 0$, $m\equiv 0$, $q\equiv 0$), obtained from Hamilton's principle, namely when $\delta w = \partial w/\partial t$ and $t_1=0$ and $t_2=T$. Since equality (13) by adequate assumptions of regularity is equivalent to equality (10), in our case the equality

$$(21)\quad \int_0^T dt\iint_\Omega\Big\langle\mu\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t}+D\Big\{\Delta w\,\Delta\frac{\partial w}{\partial t}+ +(1-\nu)\Big[2\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^3 w}{\partial x\partial y\partial t}-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^3 w}{\partial y^2\partial t}-\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^3 w}{\partial x^2\partial t}\Big]\Big\}\Big\rangle dx\,dy=0$$

is equivalent to equality (20).

We could obtain equality (21) without resorting to equalities (10) and (13), by applying Green's theorem and by taking into account identities similar to (11).

Considering the identities

$$(22)\quad \begin{aligned} &\frac{\partial^2 w}{\partial t^2}\cdot\frac{\partial w}{\partial t}=\frac{1}{2}\cdot\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial t}\Big)^2,\qquad \Delta w\,\Delta\frac{\partial w}{\partial t}=\frac{1}{2}\cdot\frac{\partial}{\partial t}(\Delta w)^2,\\ &\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^3 w}{\partial x\partial y\partial t}\equiv\frac{1}{2}\cdot\frac{\partial}{\partial t}\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2,\\ &-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^3 w}{\partial y^2\partial t}-\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^3 w}{\partial x^2\partial t}\equiv-\frac{\partial}{\partial t}\Big(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\Big)\\ &\qquad\equiv\frac{\partial}{\partial t}\Big[\frac{1}{2}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+\frac{1}{2}\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2-\frac{1}{2}(\Delta w)^2\Big],\end{aligned}$$

we obtain from equality (21)

$$\iint_\Omega dx\,dy\int_0^T\frac{\partial}{\partial t}\Big\langle\frac{1}{2}\mu\Big(\frac{\partial w}{\partial t}\Big)^2+\frac{1}{2}D\Big\{\nu(\Delta w)^2+ +(1-\nu)\Big[2\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2\Big]\Big\}\Big\rangle dt=0\,,$$

whence

$$(23)\quad \frac{1}{2}\iint\limits_{\Omega}\left\langle\mu\left(\frac{\partial w}{\partial t}\right)^2+D\left\{\nu(\Delta w)^2+ \right.\right.$$
$$\left.\left.+(1-\nu)\left[2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\right\}\right\rangle\Bigg|_0^T dx\,dy=0\,.$$

Since the function $w(x, y, t)$ fulfils the initial conditions (18), the subintegral expression (23) for $t = 0$ equals 0; moreover, T is an arbitrary number from the range $(0, \infty)$, and thus the equality

$$(24)\quad \frac{1}{2}\iint\limits_{\Omega}\left\langle\mu\left(\frac{\partial w}{\partial t}\right)^2+D\left\{(\Delta w)^2+ \right.\right.$$
$$\left.\left.+(1-\nu)\left[2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\right\}\right\rangle dx\,dy=0$$

is true for an arbitrary $t > 0$.

On the other hand, we know from our assumption of $\mu(x, y) > 0$ and $D(x, y) \geqslant 0$ in Ω that $0 < \nu < 0{,}5$; and, since the subintegral expression is continuous, the necessary condition for the fulfilment of equality (24) in the semi-cylinder Σ is the equality $\partial w/\partial t \equiv 0$ occurring in Σ. This equality shows that the functions is independent of the time-parameter, and thus $w(x, y, t) = v(x, y)$ in Σ; and, since the function is continuous and must fulfil the initial homogeneous conditions (18), it is identically equal to 0 $w(x, y, t) \equiv 0$ in Σ, qu.e.d.

The expression

$$\frac{1}{2}\iint\limits_{\Omega}\left\langle\mu\left(\frac{\partial w}{\partial t}\right)^2+D\left\{\nu(\Delta w)^2+(1-\nu)\left[2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\right\}\right\rangle dx\,dy$$

can be interpreted physically, since it corresponds to the entire energy $E+U$ of the freely vibrating plate. In case of free vibrations, we have to do with a conservative system, and so the entire energy is constant. As the homogeneous initial conditions imply, the entire energy is equal to 0, which is expressed by equality (24).

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